Nonlinear structure formation

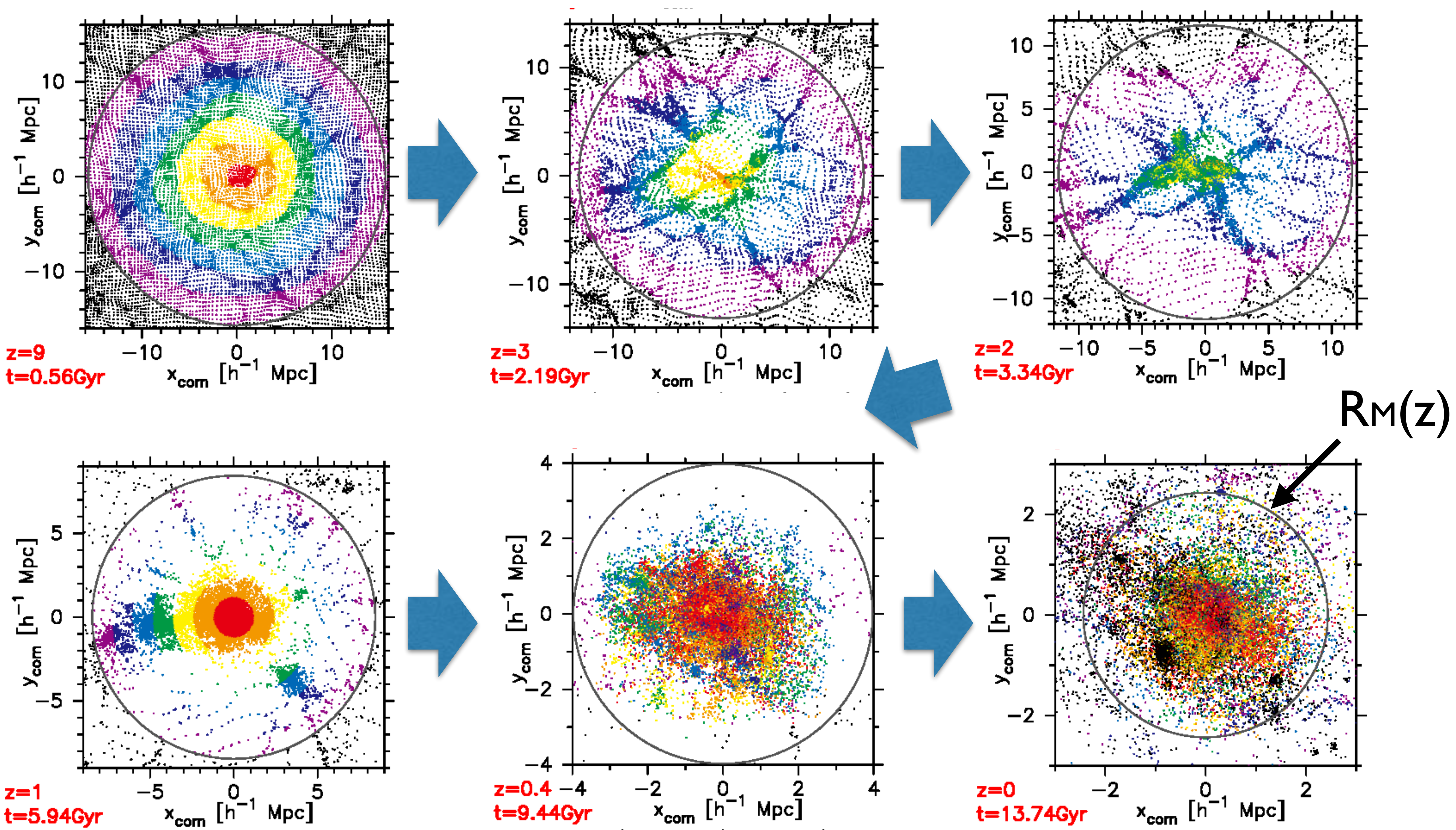
Contents

- Spherical collapse model
- Zel’dovich approximation
- Eulerian Perturbation theory

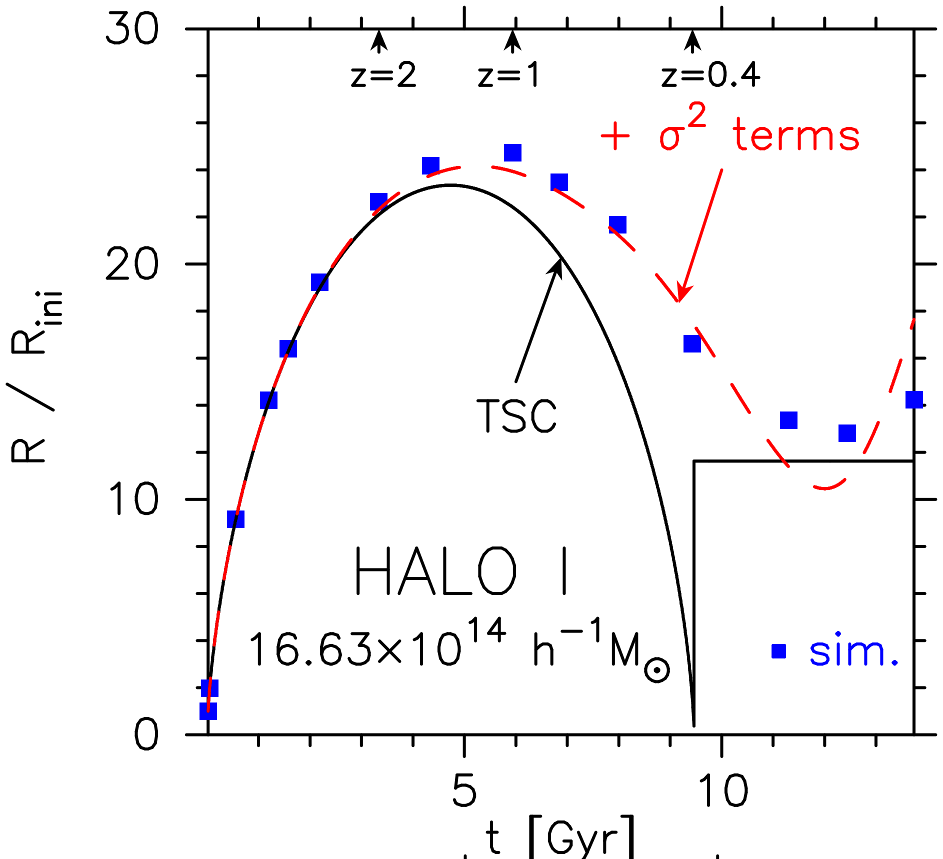
Spherical collapse model (SCM)

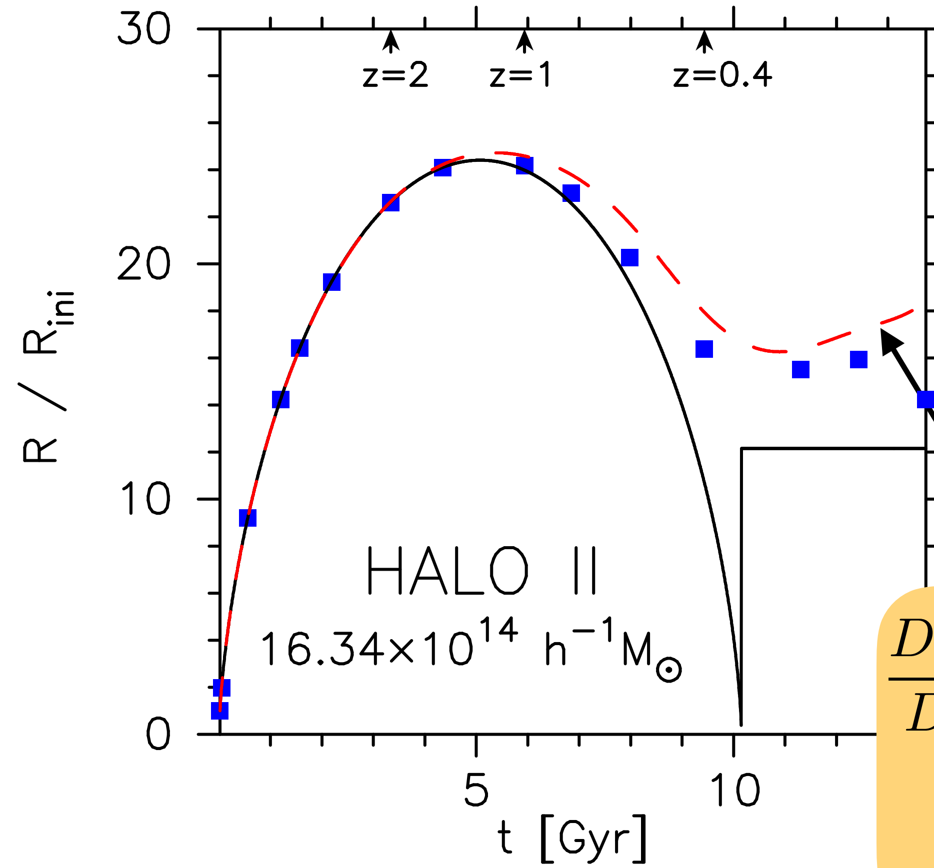
Halo formation

Halo 1 (M~10^15 M_sun/h) Suto, Kitayama, Osato, Sasaki & Suto (16)



Comparison with SCM

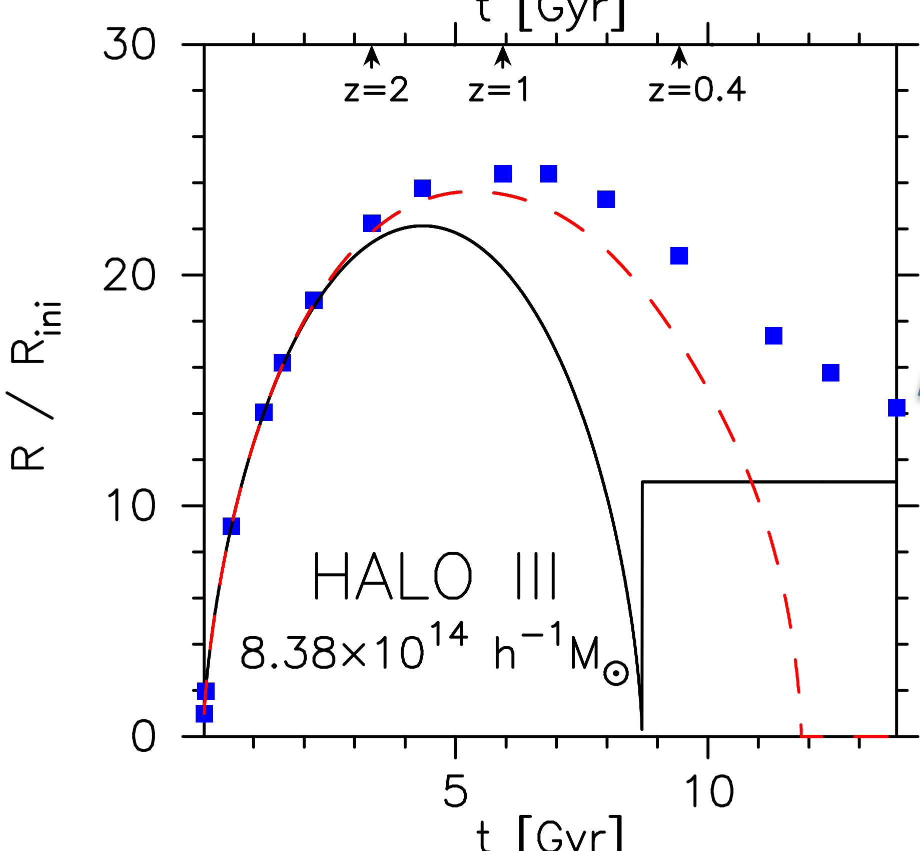


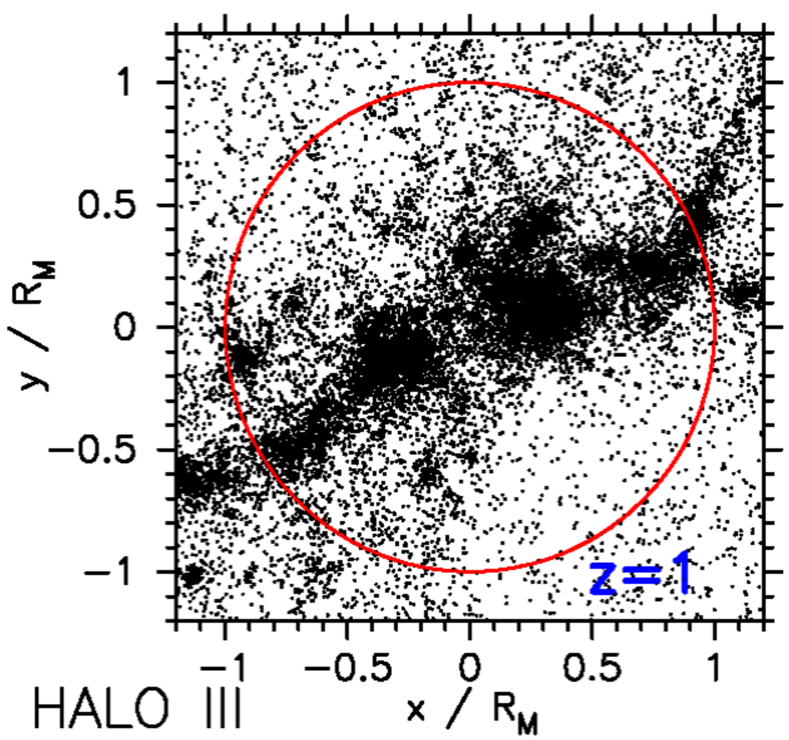


Evolution of

$R_M(z)$

$$\frac{Dv_r}{Dt} = -\frac{1}{\rho}\frac{\partial(\rho\sigma_r^2)}{\partial r} - \frac{2\sigma_r^2 - \sigma_{\tan}^2}{r} - \frac{GM}{r^2}$$ (Jeans eq.)



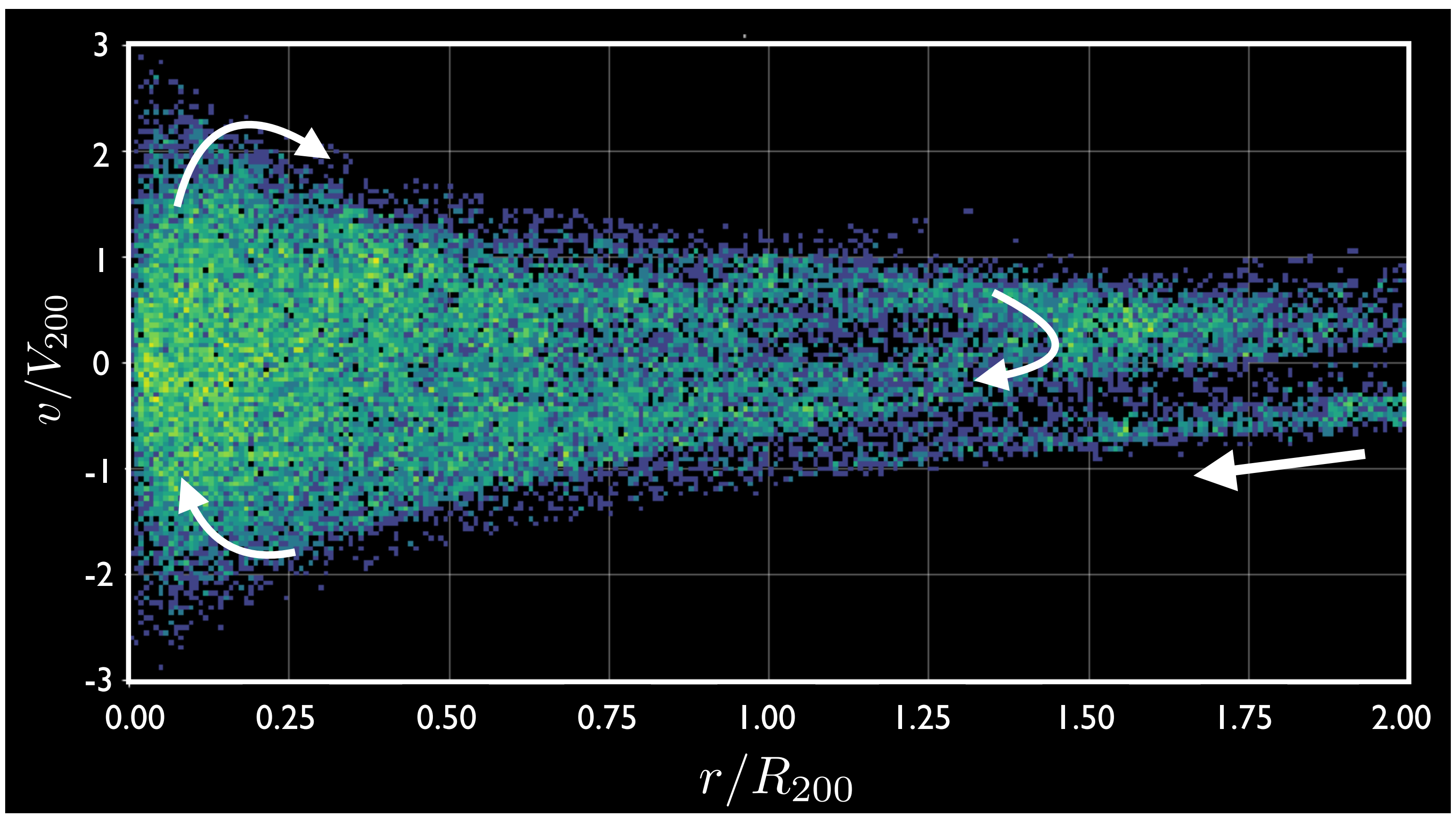


Suto, Kitayama, Osato, Sasaki & Suto (16)

Beyond spherical collapse model (SCM)

SCM only describes the onset of halo formation

In particular, SCM fails to describe phase-space structure of halo (continuous matter accretion)



Tractable analytic treatment

Self-Similar collapse

e.g., Fillmore & Goldreich ('84) Bertschinger ('85)

Self-similar collapse

Fillmore & Goldreich ('84)

- Einstein-de Sitter background, a(t) ∝ $t^{2/3}$
- Scale-free initial density perturbation, $\delta_{\rm i} \propto M_{\rm i}^{-\epsilon}$
- Motion of continuously infall shells at $r < r_*$

turn-around radius

Self-similar ansatz

$$r(t,t_*) = r_*(t_*)\lambda(t/t_*)$$

EoM

$$\Lambda(\tau) = \tau^{2/3+2/9\varepsilon}$$

$$\frac{d^2\lambda}{d\tau^2} = -\frac{\pi^2}{8}\frac{\tau^{2/3\varepsilon}}{\lambda^2}\mathscr{M}\left[\frac{\lambda}{\Lambda(\tau)}\right]\ ;\ \mathscr{M}(x) = \frac{2}{3\varepsilon}\int_1^\infty \frac{dy}{y^{1+2/3\varepsilon}}\underline{\mathscr{H}\left[x - \frac{\lambda(y)}{\Lambda(y)}\right]}$$

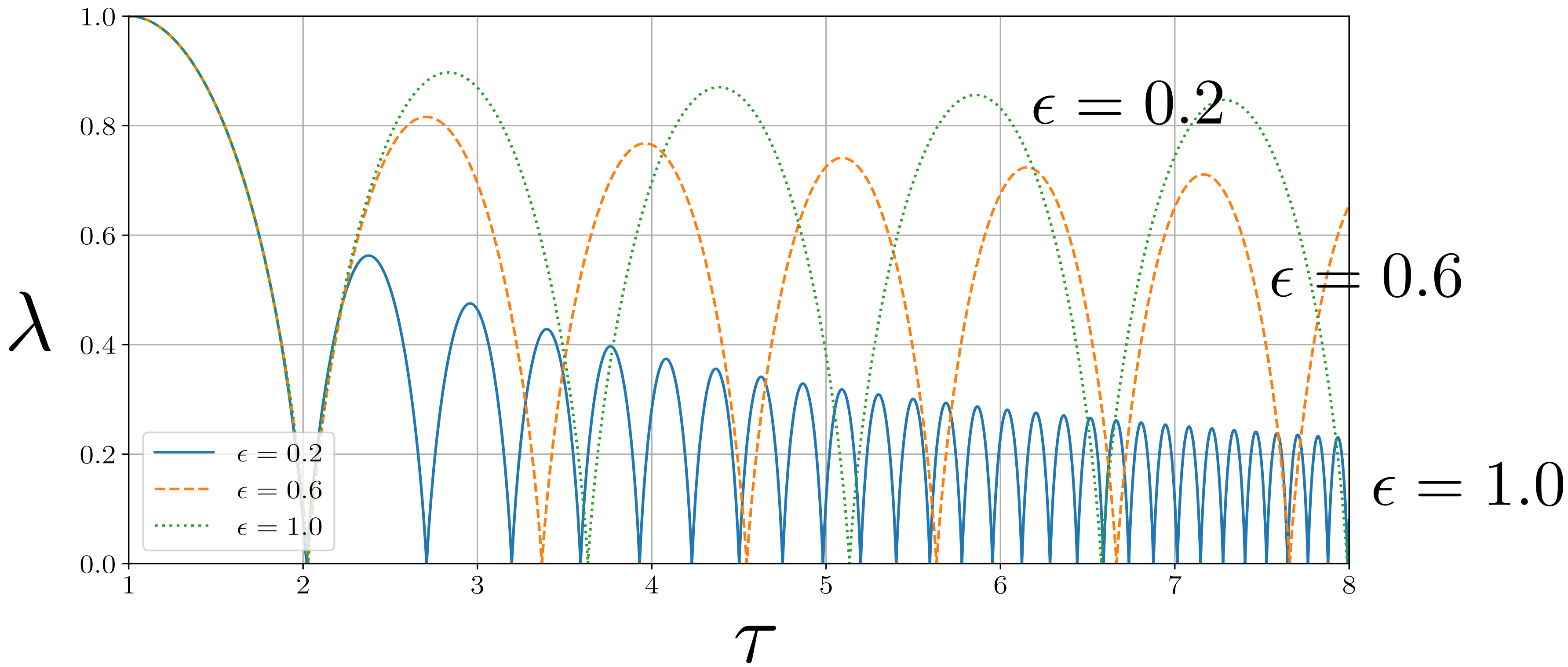
$\tau \equiv t/t_*$: time normalized by turn-around time

Heaviside step func.

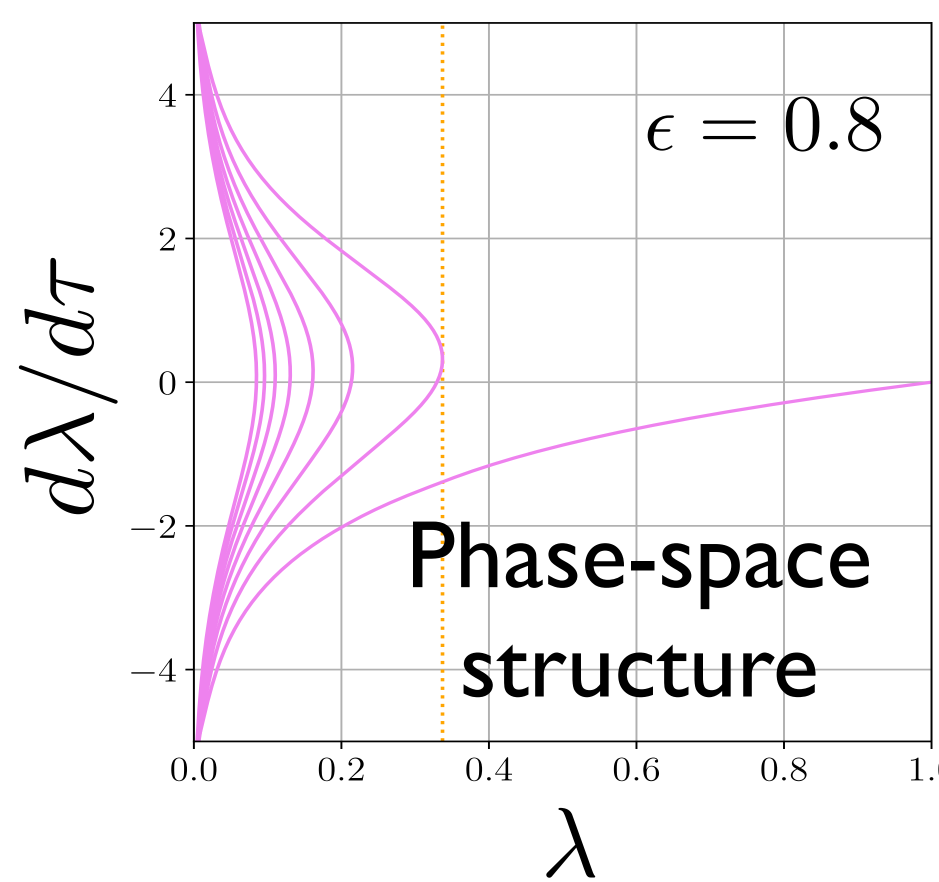
Mass: $M(r,t) = M_t\ \mathcal{M}\left(\frac{\lambda}{\Lambda(t)}\right)$; $M_t \propto a(t)^{1/\epsilon}$

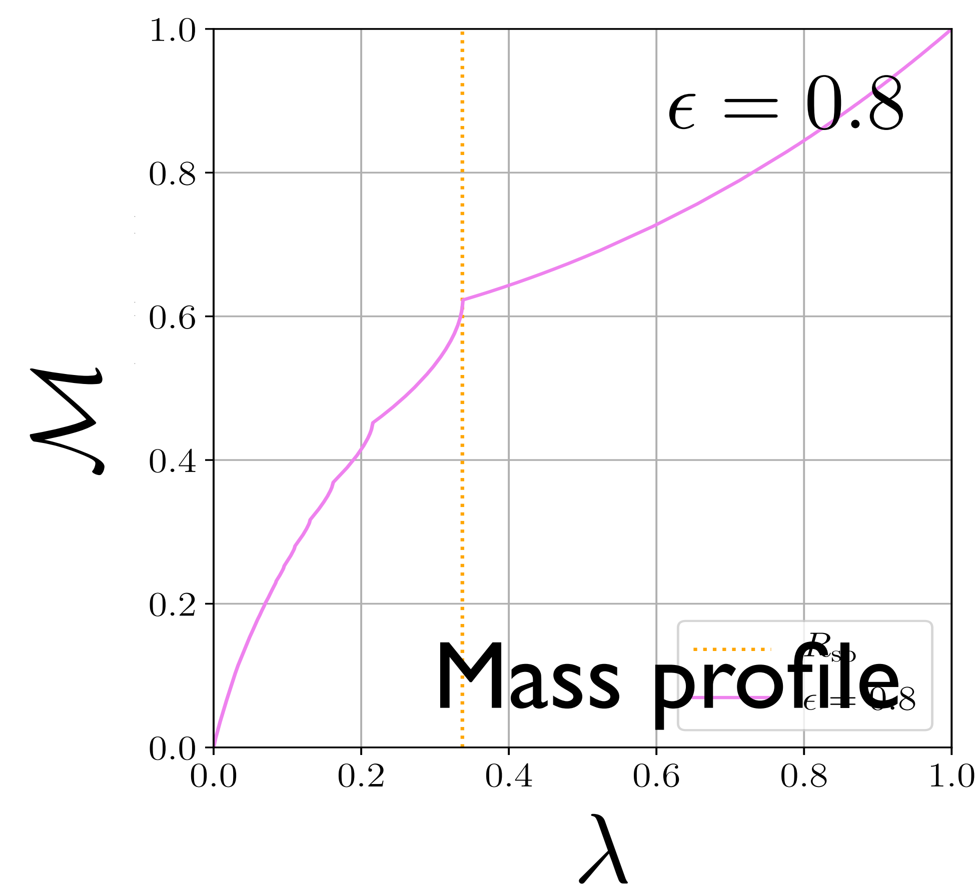
Solutions

修論 by 杉浦宏夢



trajectory of mass shell



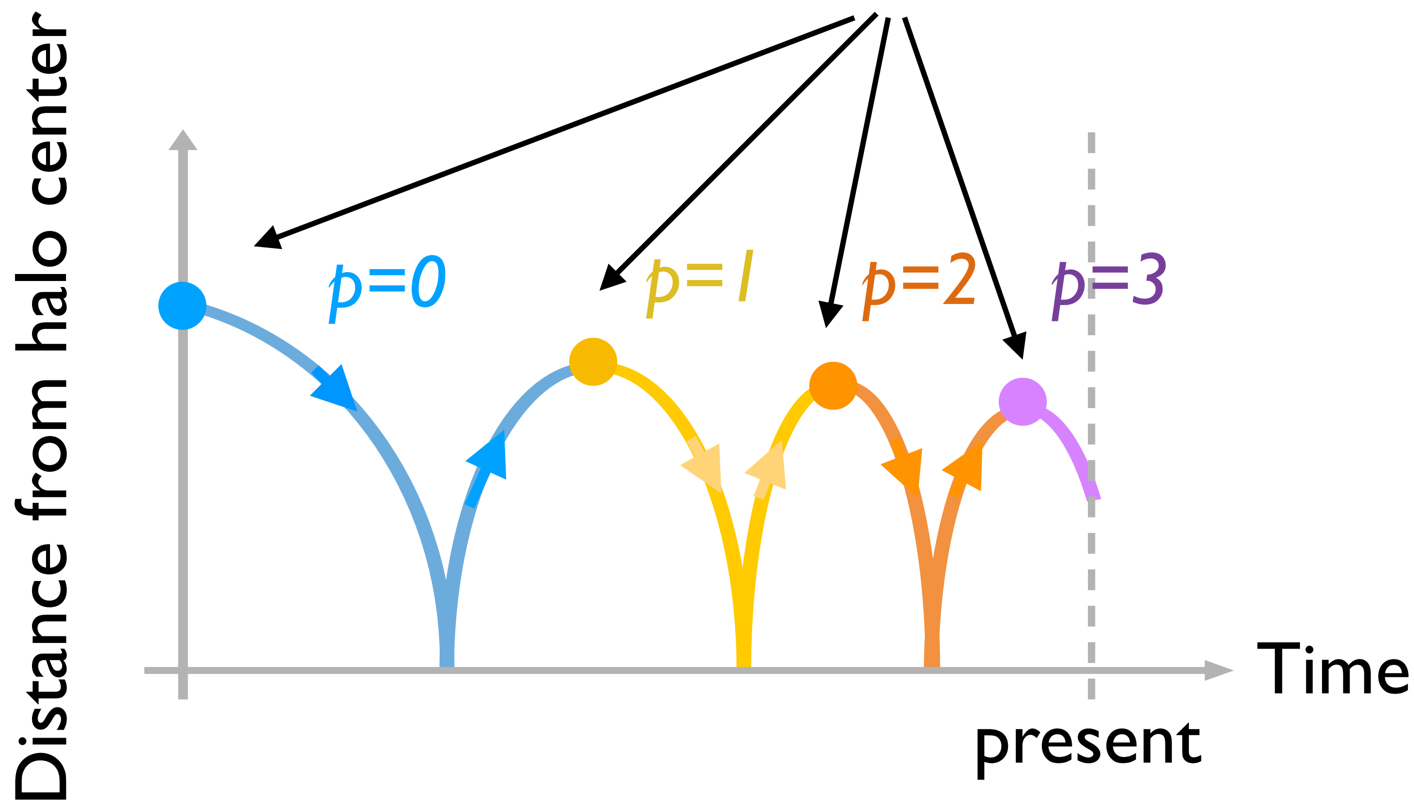


Tracing multi-stream flow with particle trajectories in *N*-body simulation

修論 by 杉浦宏夢

Keeping track of apocenter passage(s) for particle trajectories,

number of apocenter passages, *p*, is stored for each particle



= SPARTA algorithm + α

(Diemer'17; Diemer et al.'17)

Tiling phase-space streams with *p*

N-body simulation
Y. Rasera@
(Observatoire de Paris)

- L=316Mpc/h, N=512^3
- 60 snapshots at 0<z<1.43
- Einstein-de Sitter universe

11,000 halos
$(M_{200} \geq 10^{13} M_{\odot})$

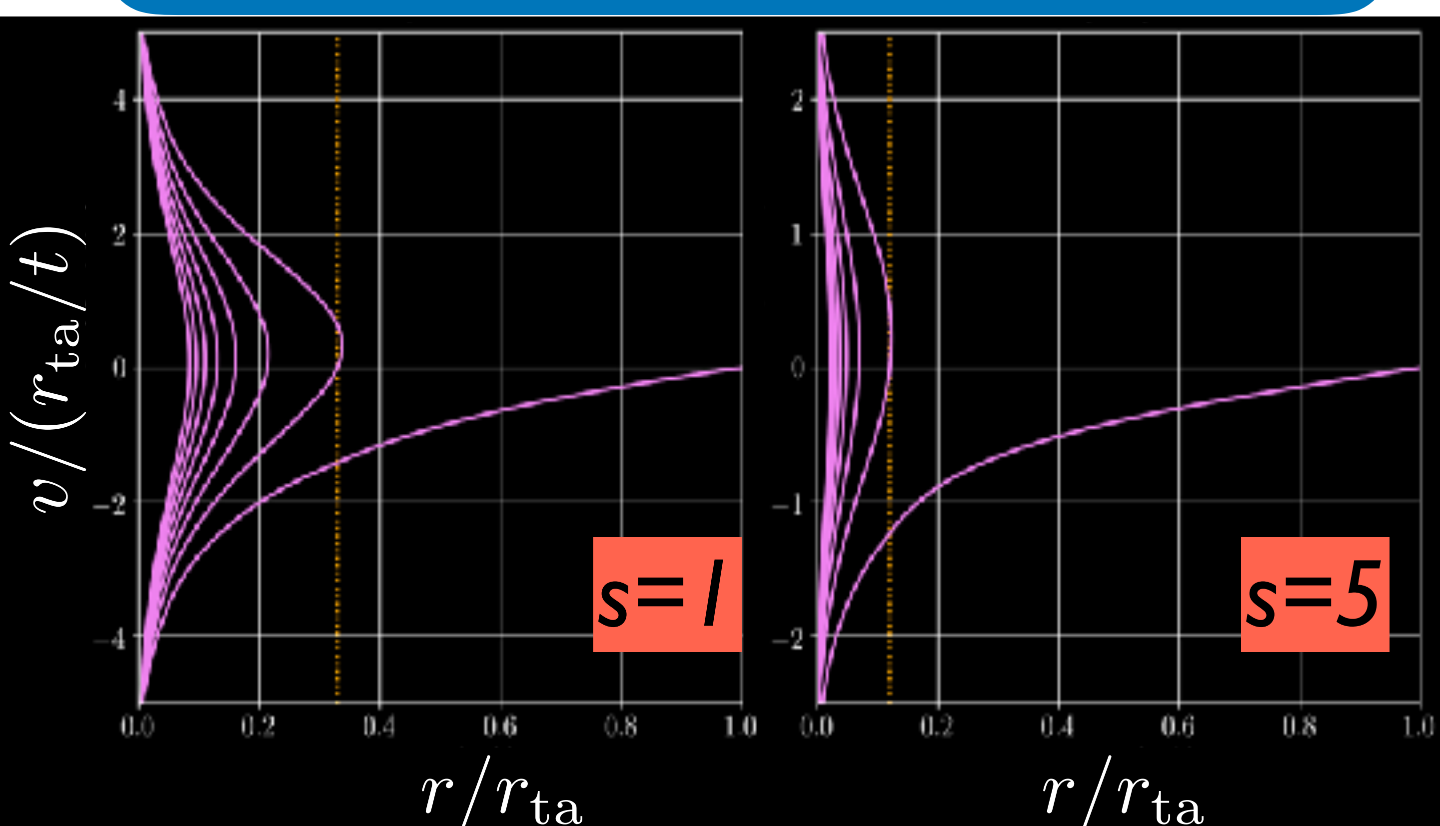
Comparison with self-similar solution

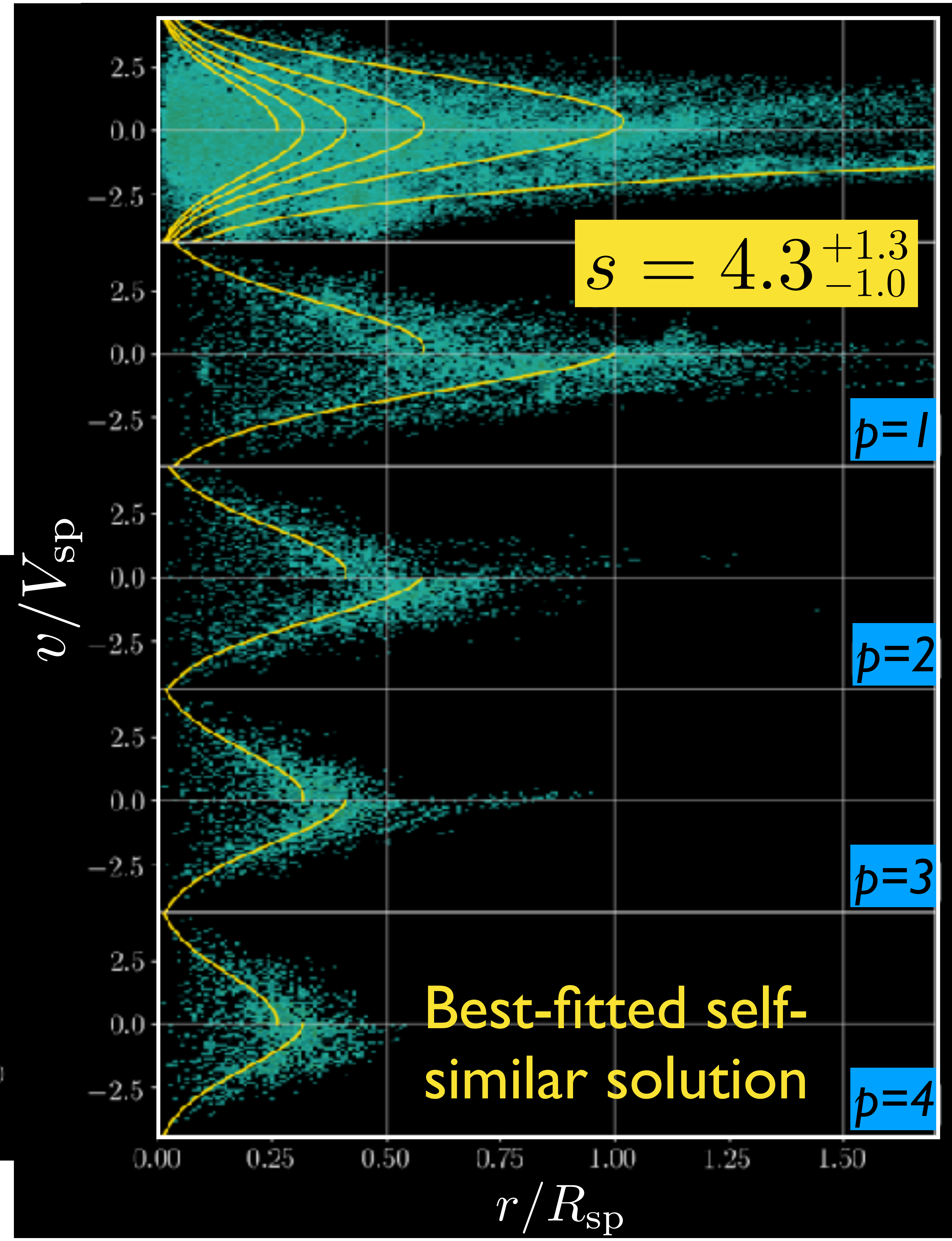
Use apocenter-passage positions
p=1~5 to fit to self-similar solution
by Fillmore & Goldreich ('84)

Fitting parameters:

$$s = (1/\epsilon), \quad r_{\rm ta}(t_{\rm ta})$$

accretion rate $(M \propto a^s)$ scale radius





(Master thesis by H. Sugiura)

Zel’dovich approximation

Lagrangian PT

Basic equations

$$L = \frac{1}{2}\, m\, a^2 \dot{\boldsymbol{x}}^2 - m\, \phi(\boldsymbol{x})$$

$$\ddot{\boldsymbol{x}} + 2H\dot{x} = -\frac{1}{a^2}\, \nabla_x\, \phi(\boldsymbol{x}),$$

$$\nabla_x^2\, \phi(\boldsymbol{x}) = 4\pi\, G a^2 \overline{\rho}_{\rm m}\, \delta(\boldsymbol{x}).$$

Lagrangian coordinate (***q***): $\boldsymbol{x}(\boldsymbol{q},t) = \boldsymbol{q} + \boldsymbol{\Psi}(\boldsymbol{q},t)$

In Lagrangian coordinate, mass density is assumed to be uniform:

$$\overline{\rho}_{\rm m}\, d^n\boldsymbol{q} = \rho_{\rm m}(\boldsymbol{x})\, d^n\boldsymbol{x} \longrightarrow \delta(\boldsymbol{x}) = \frac{\rho_{\rm m}(\boldsymbol{x})}{\overline{\rho}_{\rm m}} - 1 = \left|\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{q}}\right|^{-1} - 1$$

Rewriting quantities in Eulerian space with those in Lagrangian quantities

Lagrangian PT

Matsubara ('15)

$$\nabla_x \cdot \left[\ddot{\boldsymbol{x}} + 2H\dot{\boldsymbol{x}}\right] = -4\pi\, G\,\overline{\rho}_{\rm m}\,\delta,$$

$$\nabla_x \times \left[\ddot{\boldsymbol{x}} + 2H\dot{\boldsymbol{x}}\right] = \boldsymbol{0}.$$

$$\hat{\mathcal{T}} f(t) \equiv \ddot{f}(t) + 2H\dot{f}(t)$$

Longitudinal: $$\left(\hat{\mathcal{T}} - 4\pi\, G\,\overline{\rho}_{\rm m}\right)\Psi_{k,k} = -\epsilon_{ijk}\epsilon_{ipq}\,\Psi_{j,p}\left(\hat{\mathcal{T}} - 2\pi\, G\,\overline{\rho}_{\rm m}\right)\psi_{k,q} - \frac{1}{2}\epsilon_{ijk}\epsilon_{pqr}\,\Psi_{i,p}\Psi_{j,q}\left(\hat{\mathcal{T}} - \frac{4\pi\, G}{3}\,\overline{\rho}_{\rm m}\right)\Psi_{k,r},$$

Transverse: $$\epsilon_{ijk}\,\hat{\mathcal{T}}\,\Psi_{j,k} = -\epsilon_{ijk}\,\Psi_{p,j}\hat{\mathcal{T}}\Psi_{p,k}.$$

Levi-Civita symbol

PT expansion: $$\boldsymbol{\Psi}(\boldsymbol{q},\, t) = \boldsymbol{\Psi}^{(1)}(\boldsymbol{q},\, t) + \boldsymbol{\Psi}^{(2)}(\boldsymbol{q},\, t) + \boldsymbol{\Psi}^{(3)}(\boldsymbol{q},\, t) + \cdots$$

Zel'dovich solution: 1st-order LPT

$$\boldsymbol{\Psi}^{(1)} = \boldsymbol{\Psi}^{(1L)} + \boldsymbol{\Psi}^{(1T)} \ ;$$

$\eta \equiv \ln D_1(t)$

$$\left(\frac{\partial^2}{\partial\eta^2} + \frac{1}{2}\frac{\partial}{\partial\eta} - \frac{3}{2}\right)\Psi^{(1L)}_{k,k} = 0$$

$$\left(\frac{\partial^2}{\partial\eta^2} + \frac{1}{2}\frac{\partial}{\partial\eta}\right)\epsilon_{ijk}\Psi^{(1T)}_{j,k} = 0$$

Zel'dovich approximation : $\boldsymbol{\Psi}^{(1T)} = 0$ and take growing-mode only

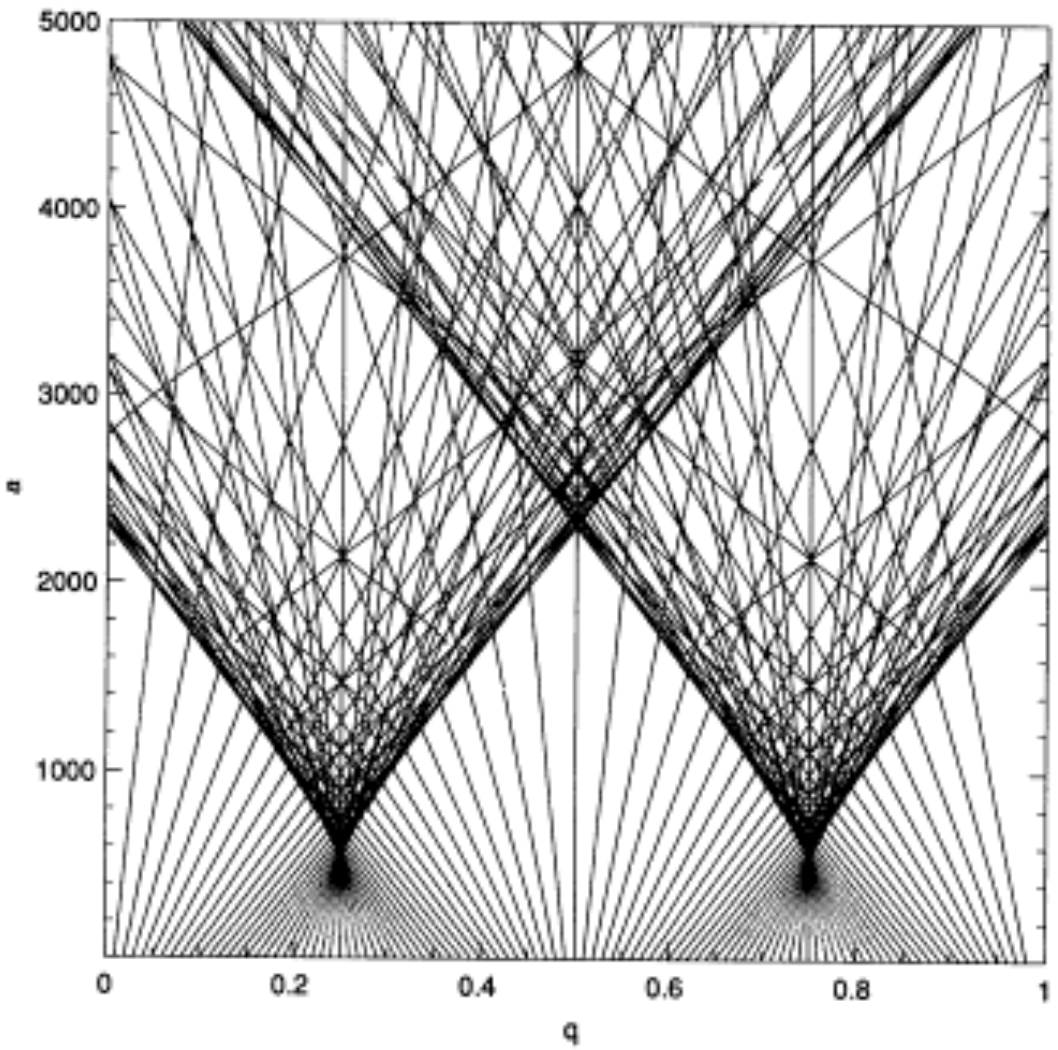
$$\boldsymbol{\Psi}^{(1)} = \boldsymbol{\Psi}^{(1L)} = -\,D_1(a)\,\nabla_q\varphi(\boldsymbol{q}), \qquad \nabla_q^2\varphi(\boldsymbol{q}) = \delta_0(\boldsymbol{q})$$

: initial density field

$$\because \quad 1+\delta_{\rm m}(\boldsymbol{x}) = \left|\frac{\partial\boldsymbol{x}}{\partial\boldsymbol{q}}\right|^{-1} \equiv \frac{1}{J} \simeq 1 - \nabla_q\cdot\boldsymbol{\psi}$$

Particle trajectories in ZA

ZA



2LPT

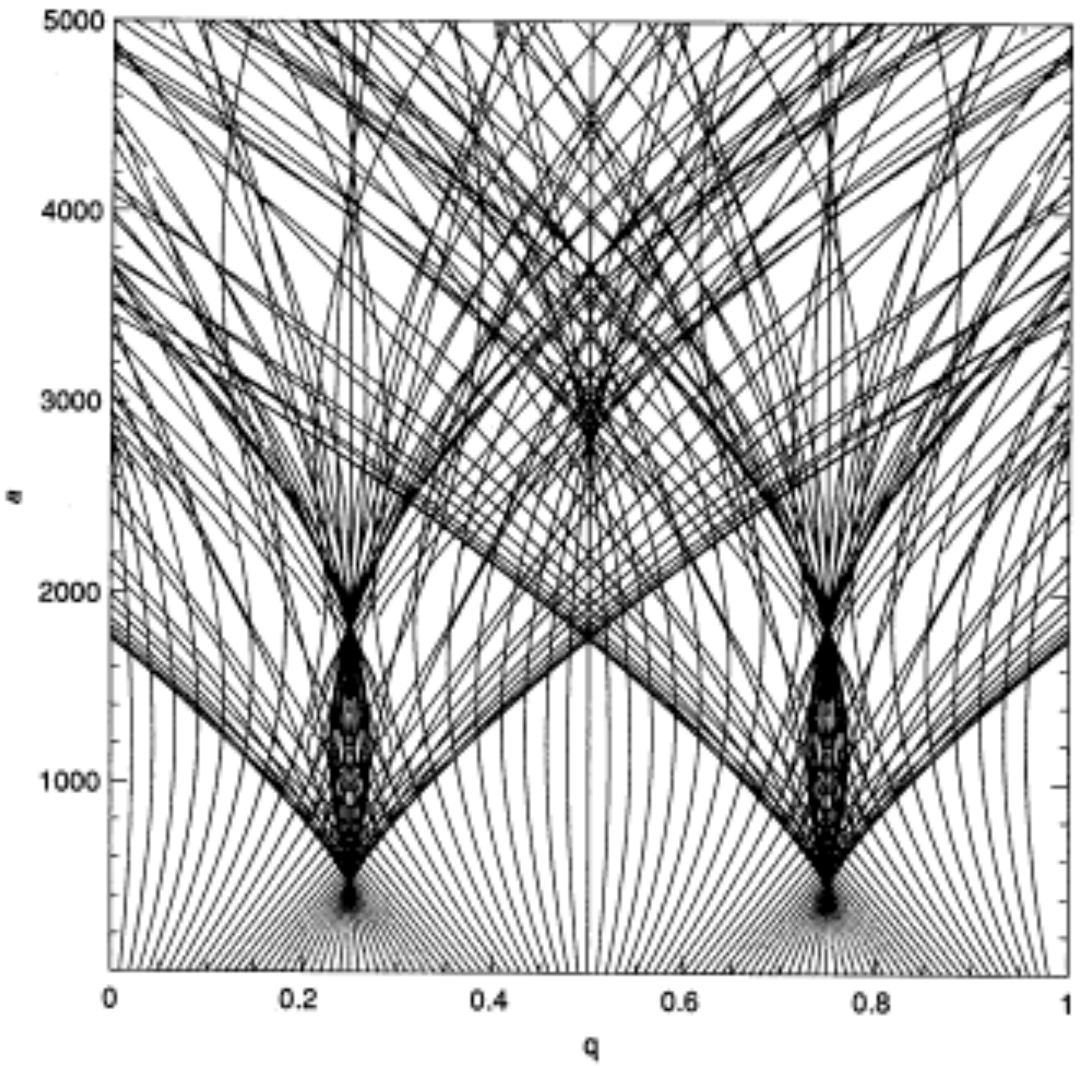
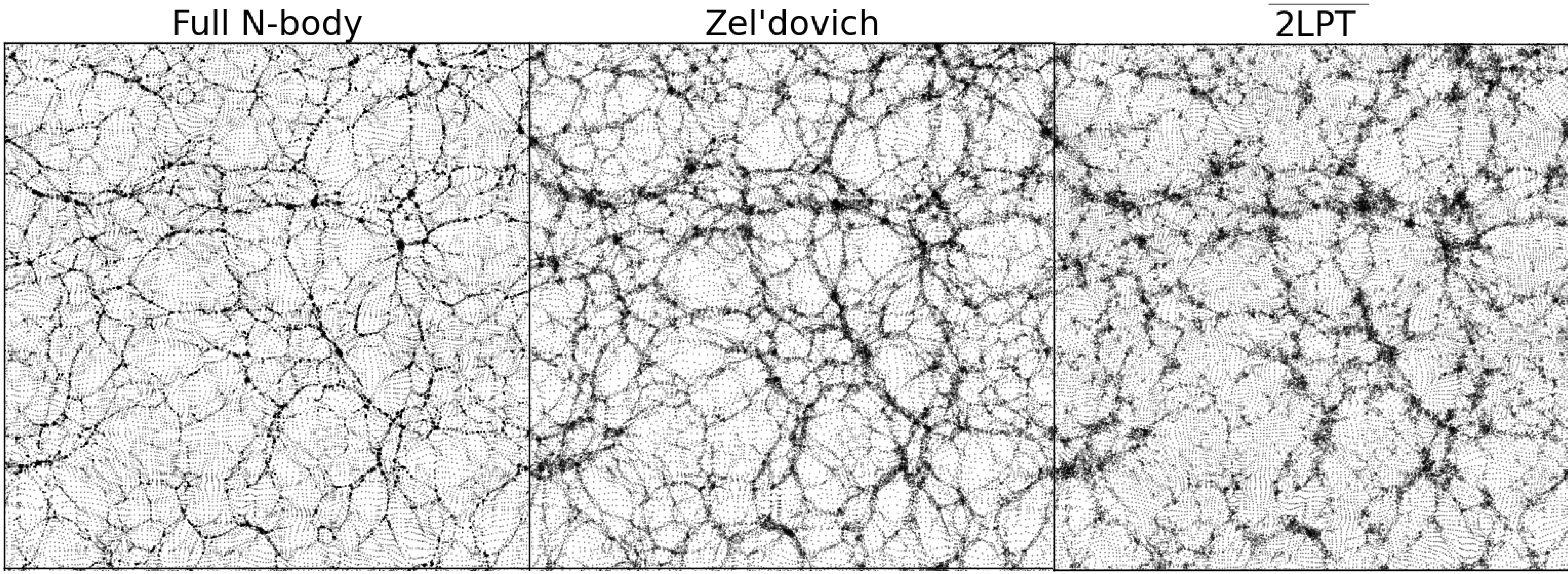


Figure 3. A family of trajectories corresponding to the model presented in Fig. 1 is shown for the first-order (upper panel) and second-order (lower panel) approximations. The trajectories end in the Eulerian space–time section ($y = 0.5$, t) centred at a cluster. These plots illustrate that the three-stream system that develops after the first shell-crossing performs a self-oscillation due to the action of self-gravity.

Buchert & Ehlers ('93)



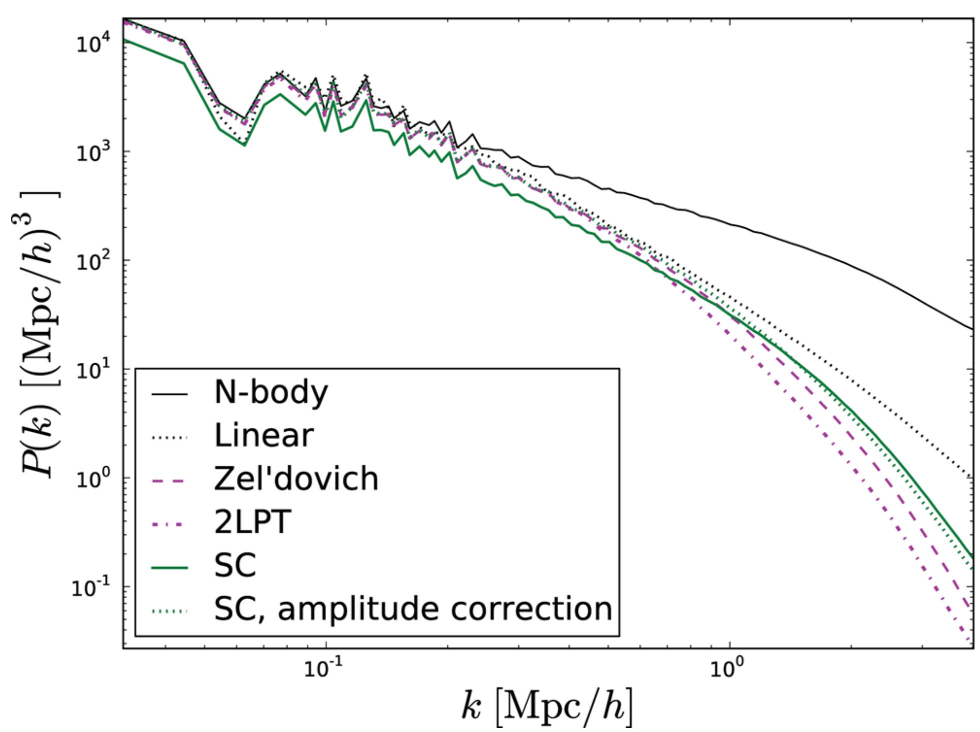
N_particle = 256^3

L=200Mpc/h

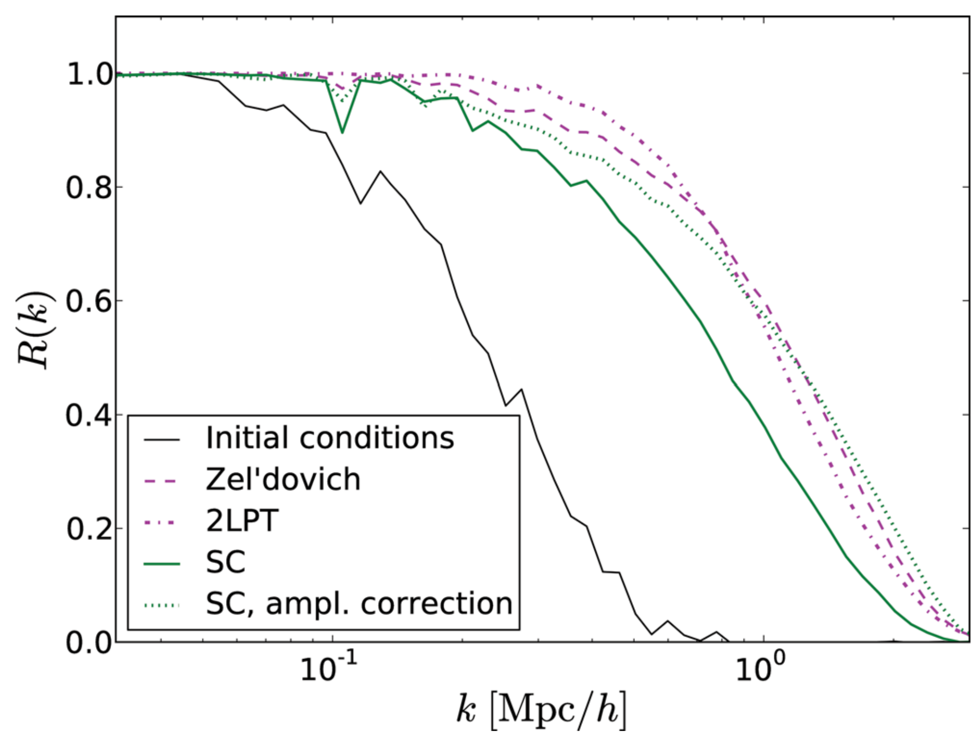
ΛCDM

Neyrink ('13)

power spectrum



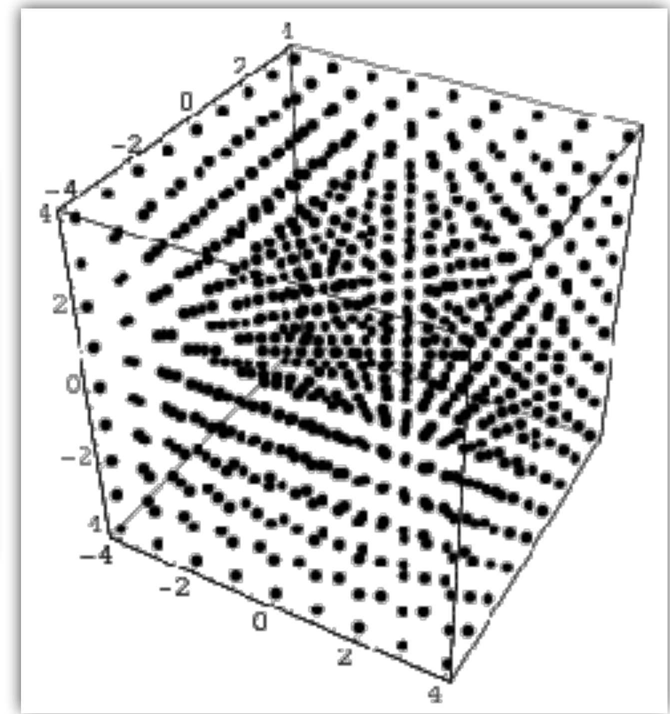
cross correlation coeff.



Zel’dovich近似の応用

N体シミュレーションの初期条件生成に有用

粒子を格子状に並べてZel’dovich近似でずらす：



$$\boldsymbol{x} = \boldsymbol{q} + \boldsymbol{\Psi}(\boldsymbol{q}) \qquad \boldsymbol{v} = a\dot{\boldsymbol{x}} = a\dot{\boldsymbol{\Psi}}(\boldsymbol{q})$$

格子点の位置　変移場ベクトル

Zel’dovich近似：

$$\Psi(\boldsymbol{k}) \simeq \frac{i\,\boldsymbol{k}}{k^2} D_+(z)\,\delta_0(\boldsymbol{k})$$

初期密度ゆらぎ

計算の流れ

1. フーリエ空間上にランダムな初期密度場を生成 $\delta_0(\boldsymbol{k})$
2. 変移場ベクトルを計算： $\boldsymbol{\Psi}(\boldsymbol{k}) \xrightarrow{\text{FFT}} \boldsymbol{\Psi}(\boldsymbol{q})$
3. 変移場ベクトルを使って粒子を移動： $\boldsymbol{\Psi}(\boldsymbol{q}) \qquad \dot{\boldsymbol{\Psi}}(\boldsymbol{q}) = \frac{\dot{D}_+(z)}{D_+(z)}\boldsymbol{\Psi}(\boldsymbol{q})$

Higher-order Lagrangian PT

e.g,. Matsubara ('15)

PT expansion: $\boldsymbol{\Psi}(\boldsymbol{q},\,t)=\boldsymbol{\Psi}^{(1)}(\boldsymbol{q},\,t)+\boldsymbol{\Psi}^{(2)}(\boldsymbol{q},\,t)+\boldsymbol{\Psi}^{(3)}(\boldsymbol{q},\,t)+\cdots$

Under Einstein-de Sitter approximation: $\boldsymbol{\Psi}^{(n)}_{\rm EdS}(\boldsymbol{q};\,a(t))\longrightarrow\boldsymbol{\Psi}^{(n)}(\boldsymbol{q};\,D_1(t))$

$\eta\equiv\ln D_1(t)$

Longitudinal:
$$\left(\frac{\partial^2}{\partial\eta^2}+\frac{1}{2}\frac{\partial}{\partial\eta}-\frac{3}{2}\right)\Psi^{(n)}_{k,k}$$
$$=-\sum_{m_1+m_2=n}\underline{\epsilon_{ijk}\epsilon_{ipq}\,\Psi^{(m_1)}_{j,p}\left(\frac{\partial^2}{\partial\eta^2}+\frac{1}{2}\frac{\partial}{\partial\eta}-\frac{3}{4}\right)\psi^{(m_2)}_{k,q}}$$
vanished in 1D
$$-\frac{1}{2}\sum_{m_1+m_2+m_3=n}\underline{\epsilon_{ijk}\epsilon_{pqr}\,\Psi^{(m_1)}_{i,p}\Psi^{(m_2)}_{j,q}\left(\frac{\partial^2}{\partial\eta^2}+\frac{1}{2}\frac{\partial}{\partial\eta}-\frac{1}{2}\right)\Psi^{(m_3)}_{k,r}},$$
vanished in 2D

Transverse:
$$\epsilon_{ijk}\left(\frac{\partial^2}{\partial\eta^2}+\frac{1}{2}\frac{\partial}{\partial\eta}\right)\Psi^{(n)}_{j,k}=-\sum_{m_1+m_2=n}\underline{\epsilon_{ijk}\,\Psi^{(m_1)}_{p,j}\left(\frac{\partial^2}{\partial\eta^2}+\frac{1}{2}\frac{\partial}{\partial\eta}\right)\Psi^{(m_2)}_{p,k}}.$$
vanished in 1D

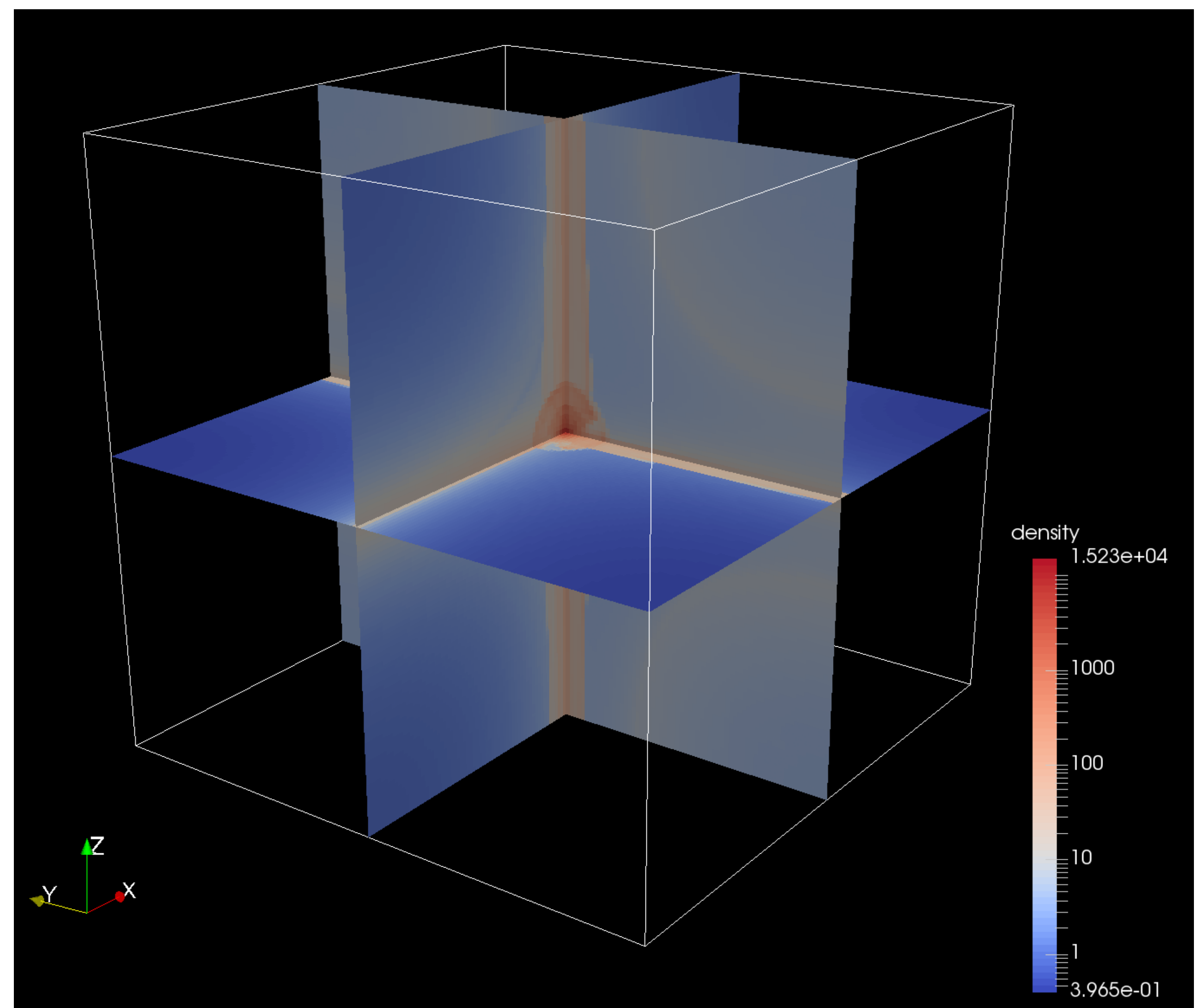
Performance of Lagrangian PT

Saga, AT & Colombi, arXiv:1805.08787

初期条件
（Zel'dovich近似解）

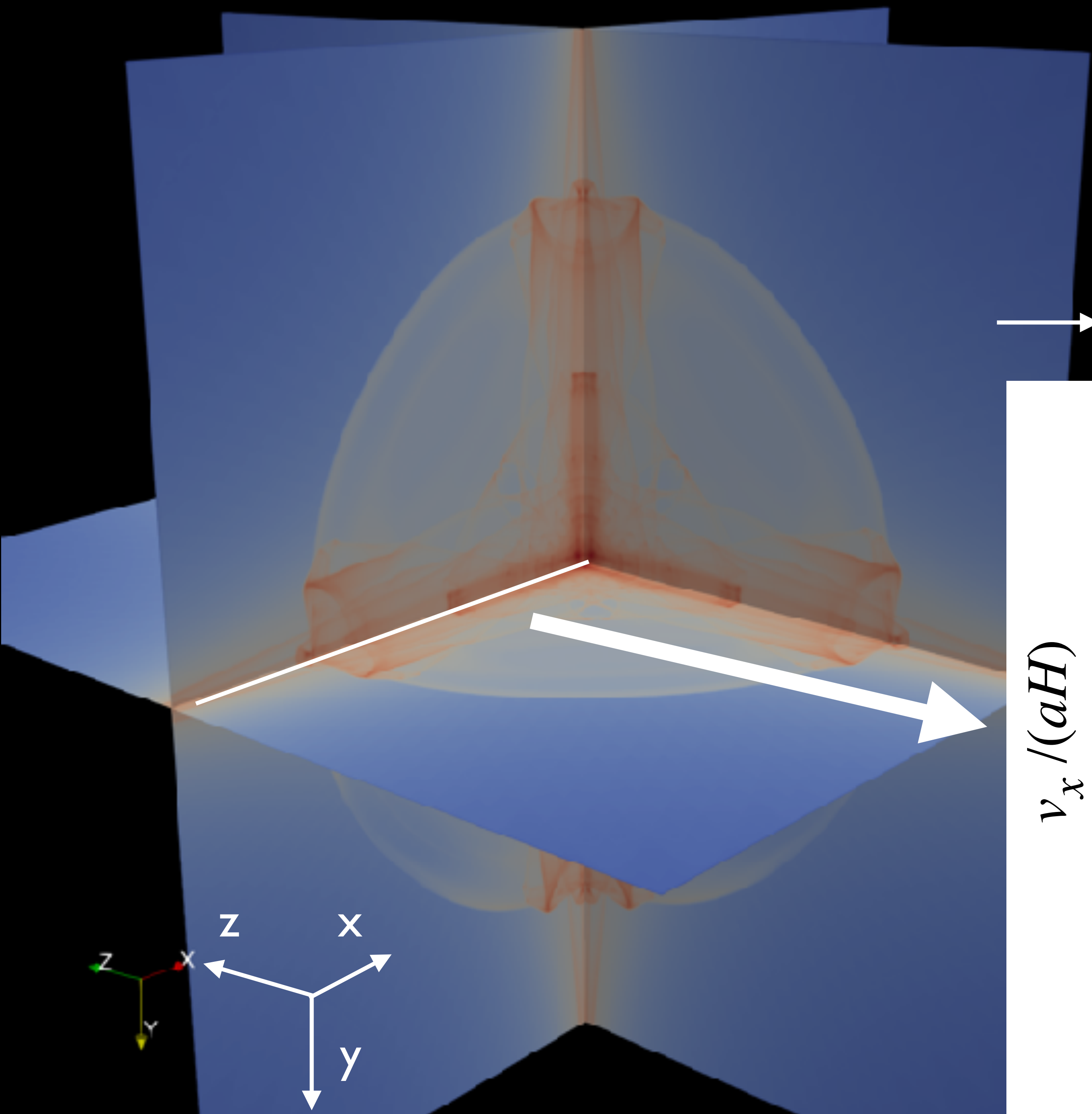
$$\boldsymbol{\Psi}(\boldsymbol{q}) = a_{\rm init} \begin{pmatrix} \epsilon_{\rm x} \sin q_x \\ \epsilon_{\rm y} \sin q_y \\ \epsilon_{\rm z} \sin q_z \end{pmatrix}$$

10次まで計算



Results

Saga, AT & Colombi, arXiv:1805.08787

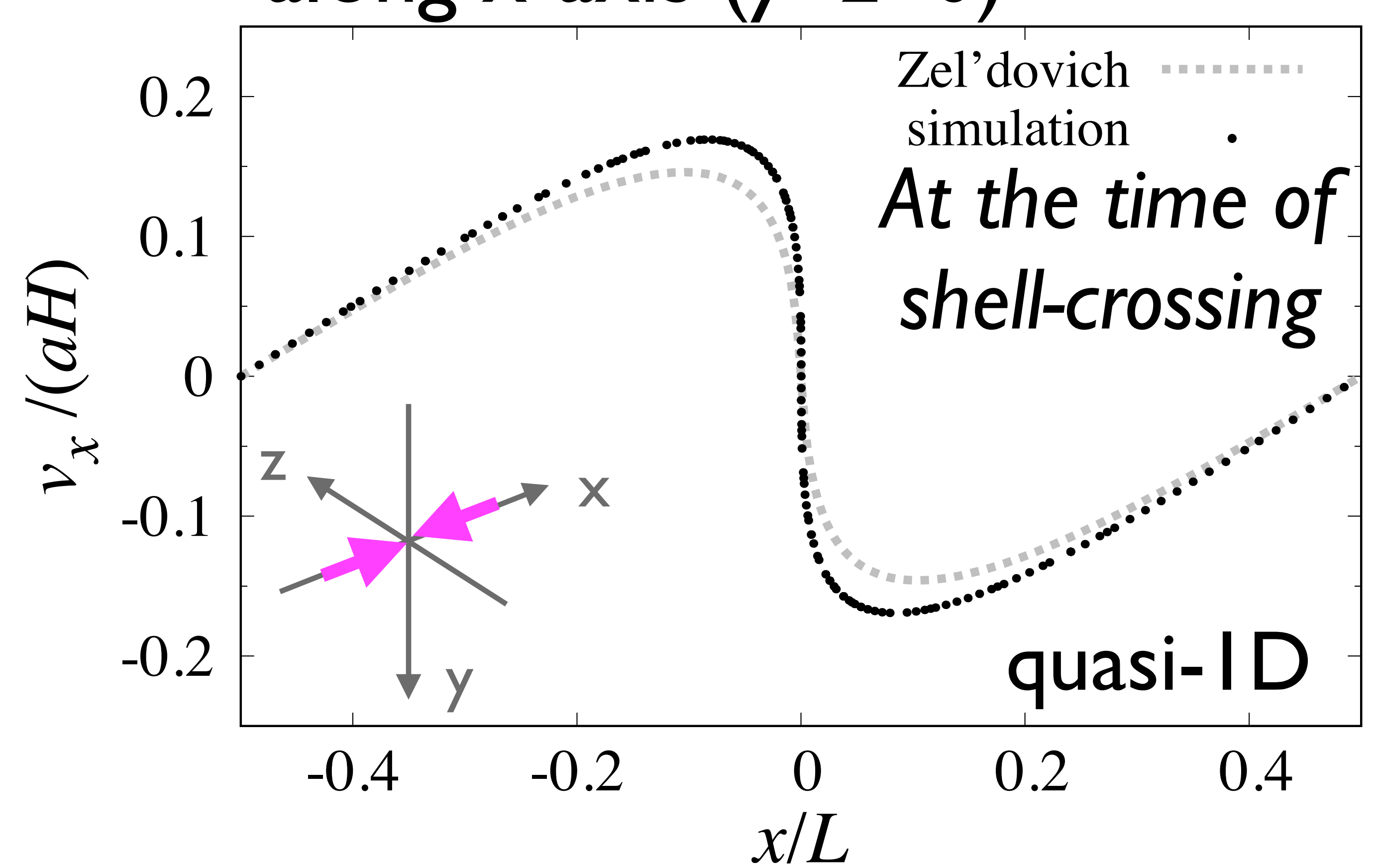


Single density peak in a periodic boundary box

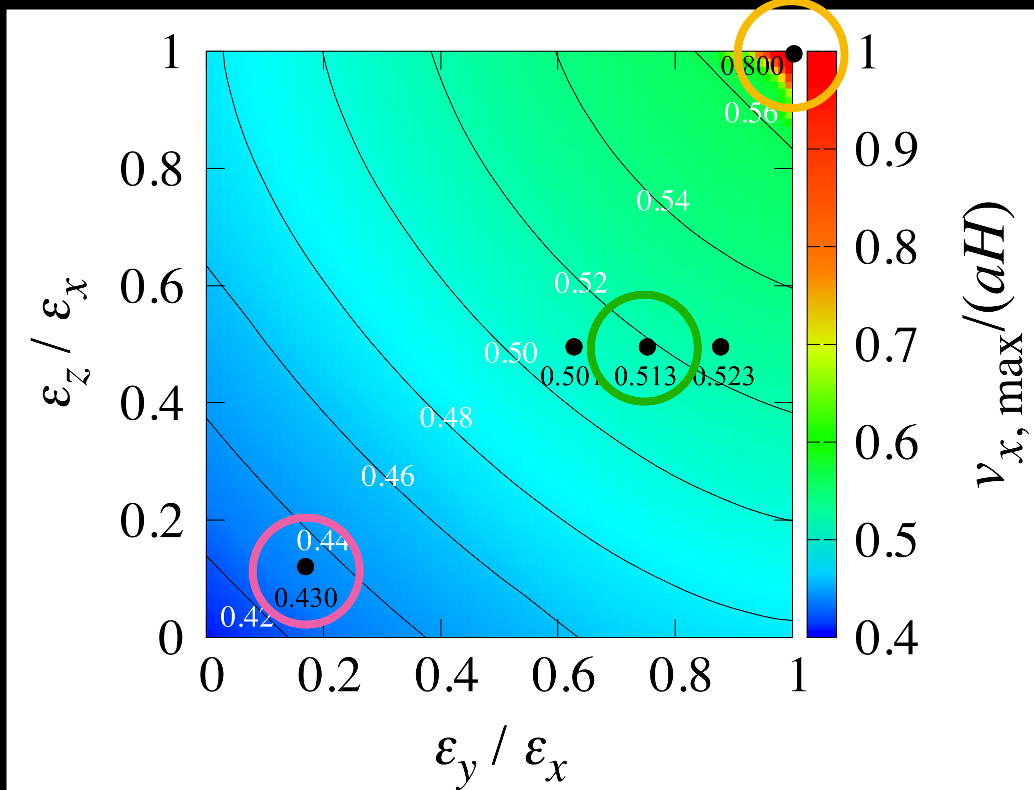
(sinusoidal function)

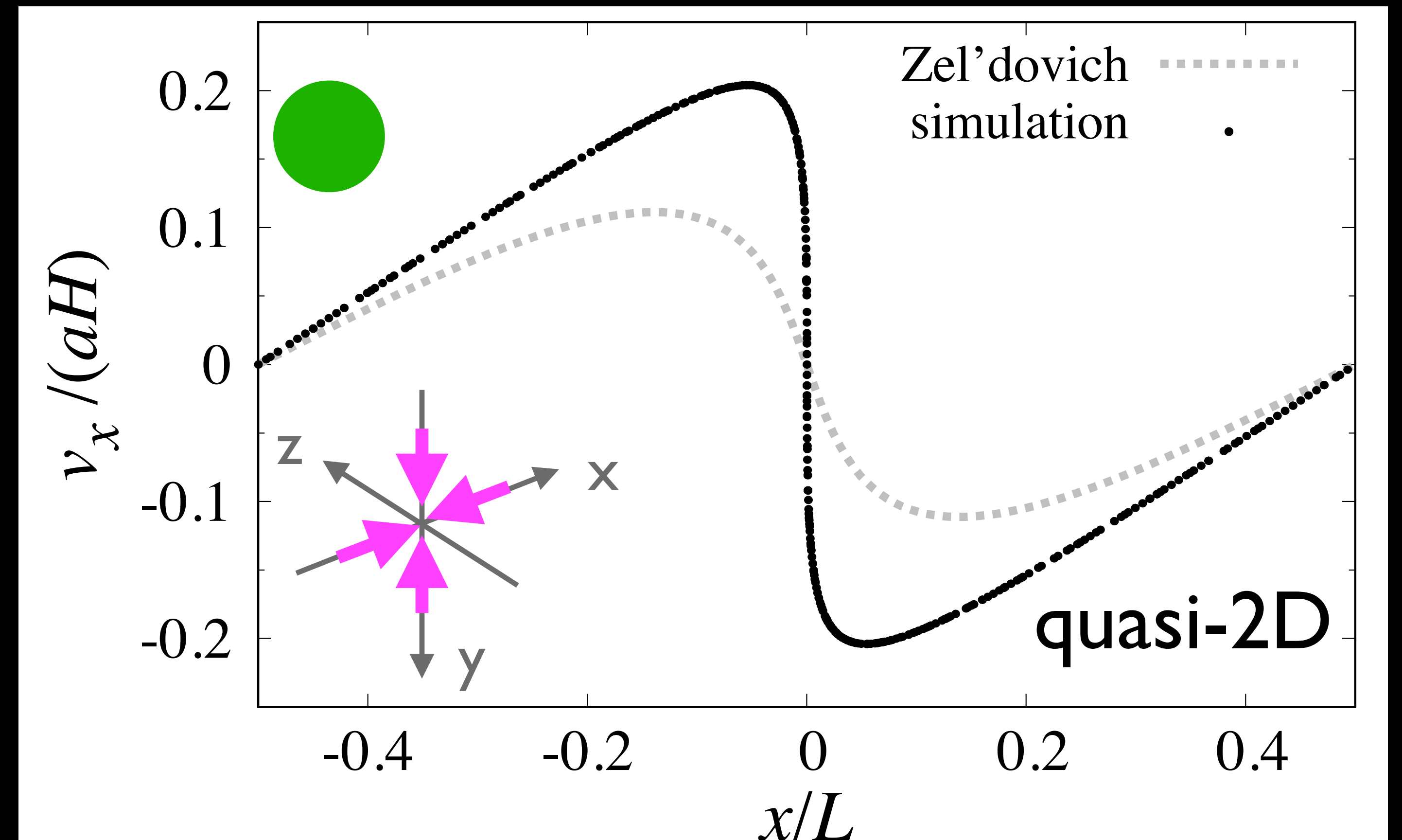
→ Shell-crossing happens at origin

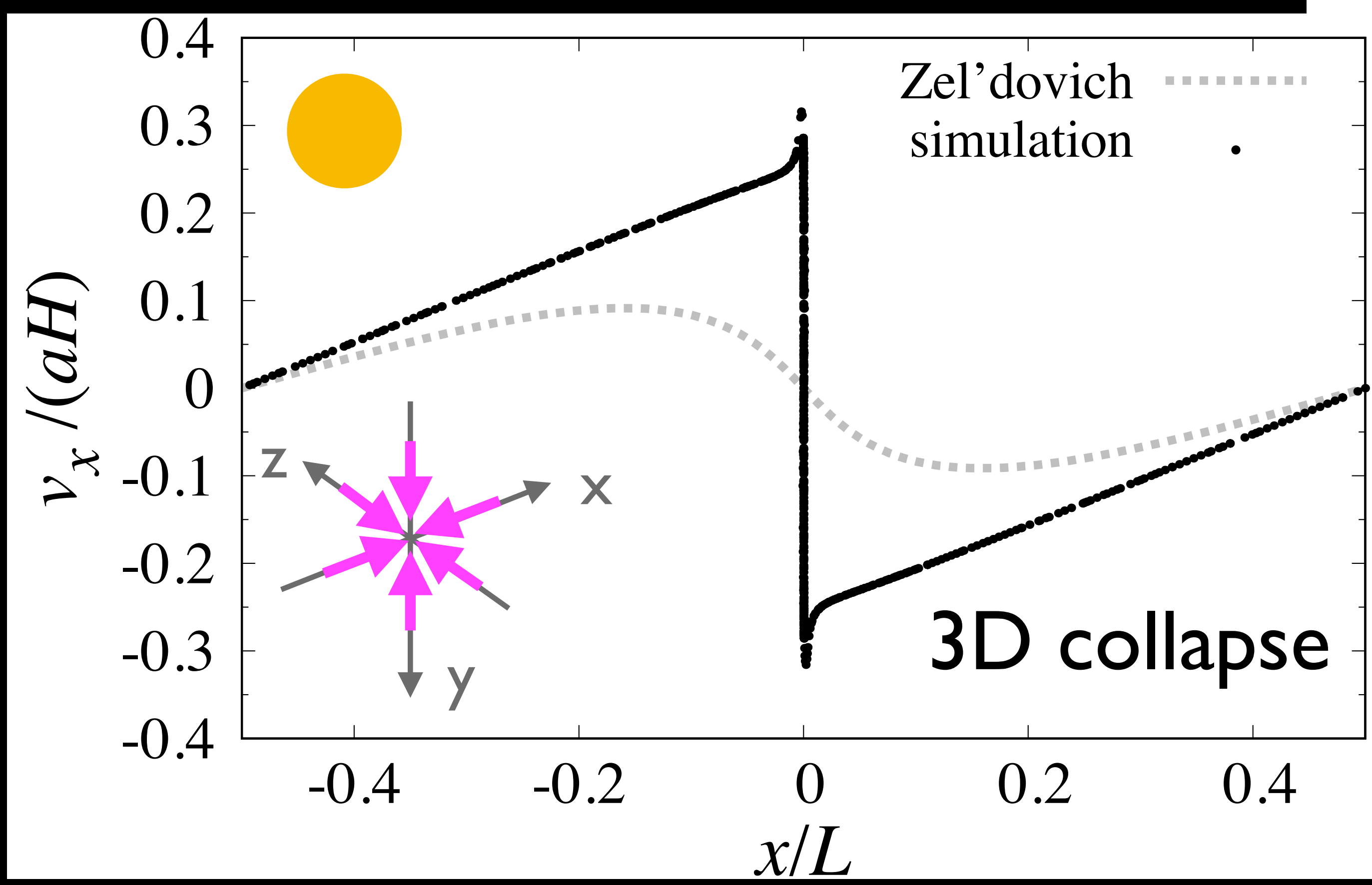
along x-axis (y=z=0)

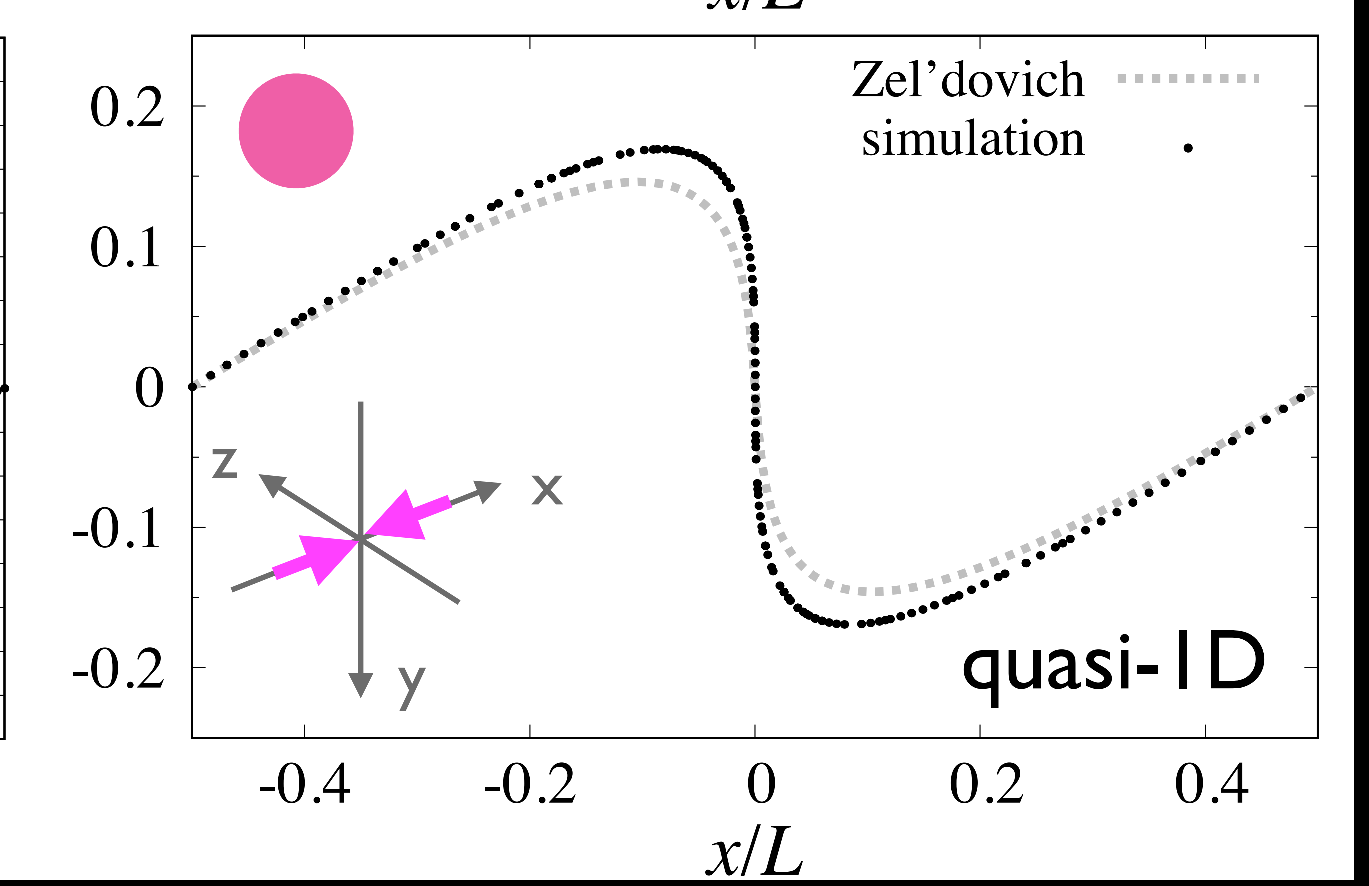


Vlasov simulation vs LPT





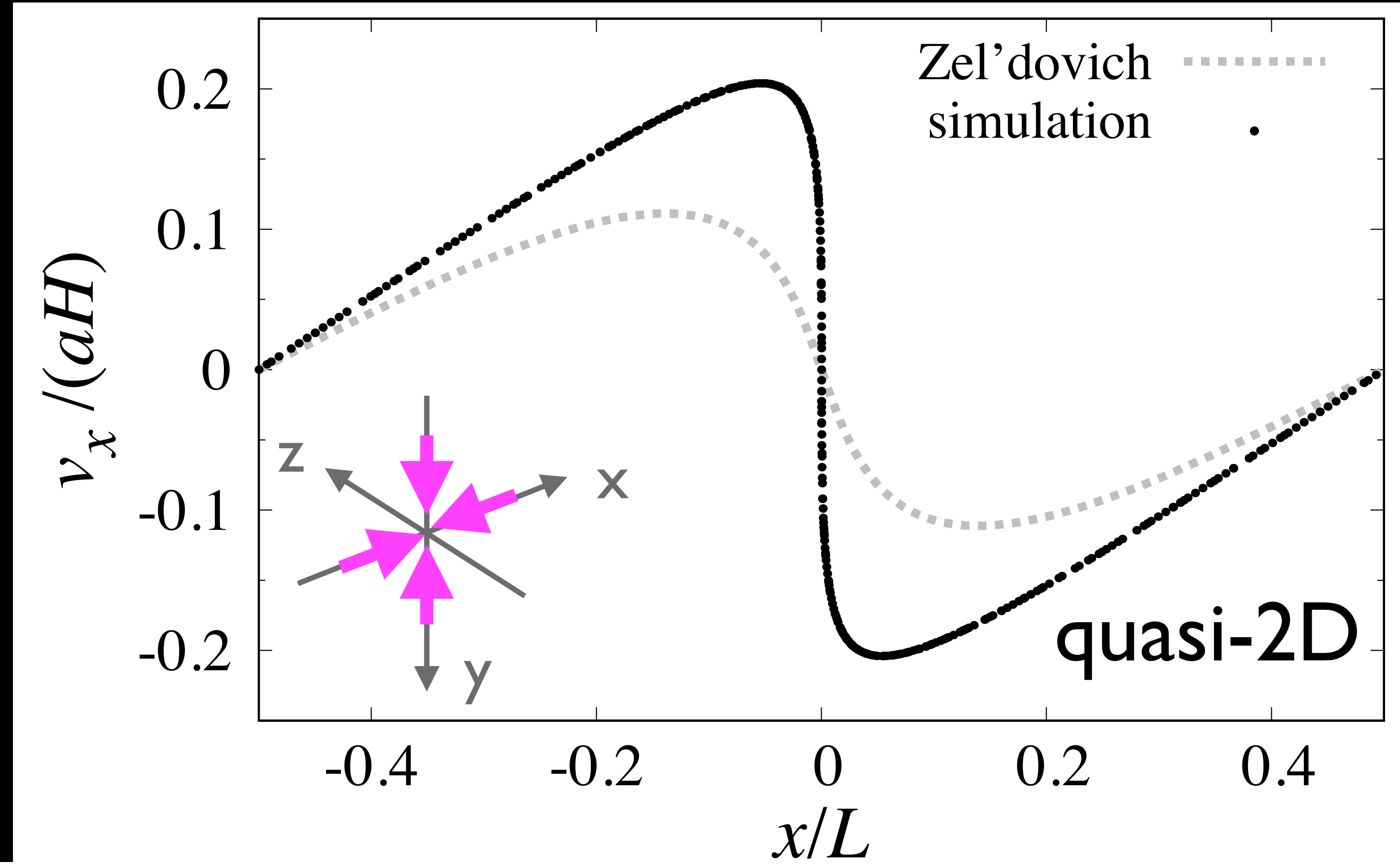


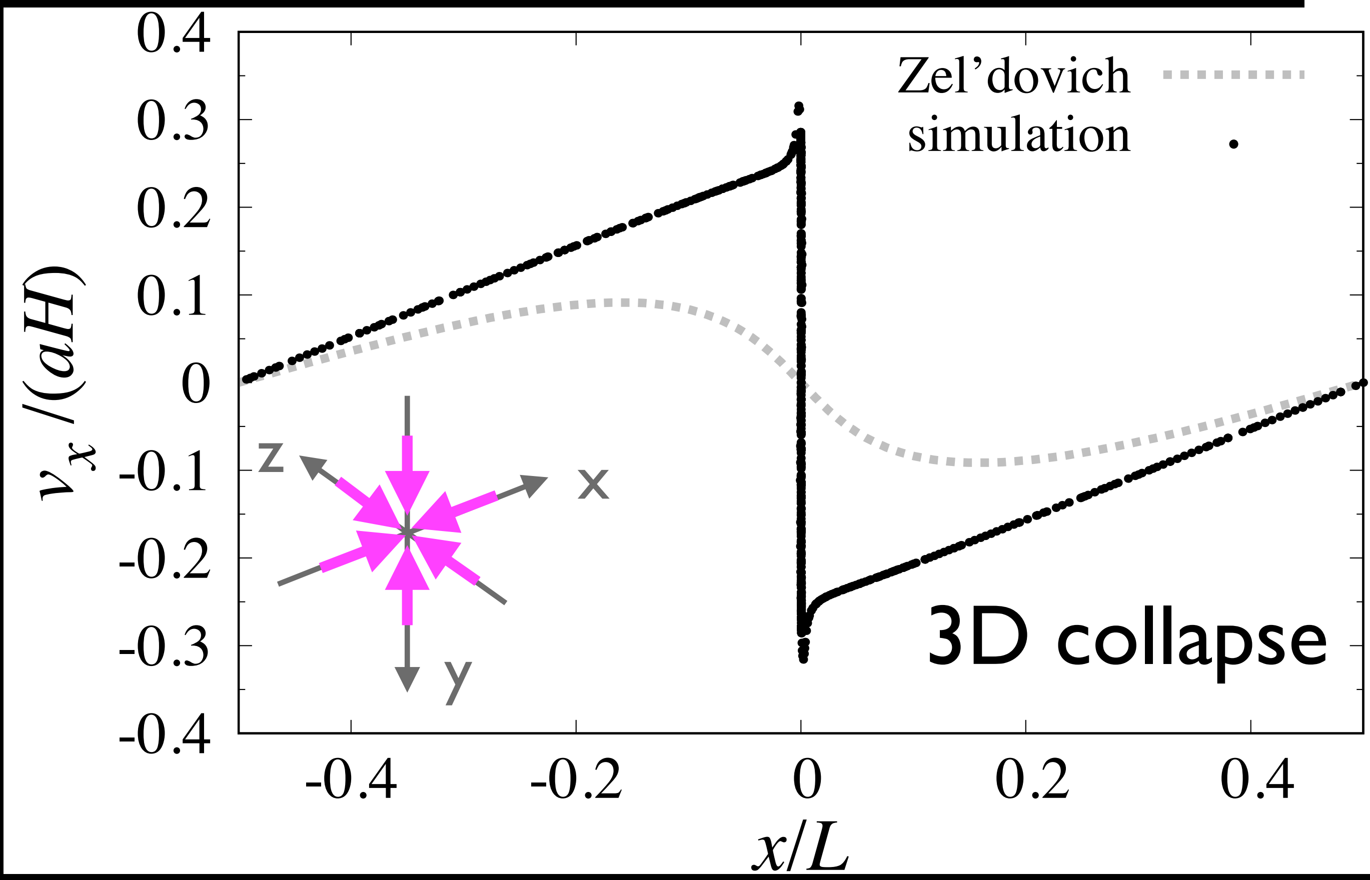


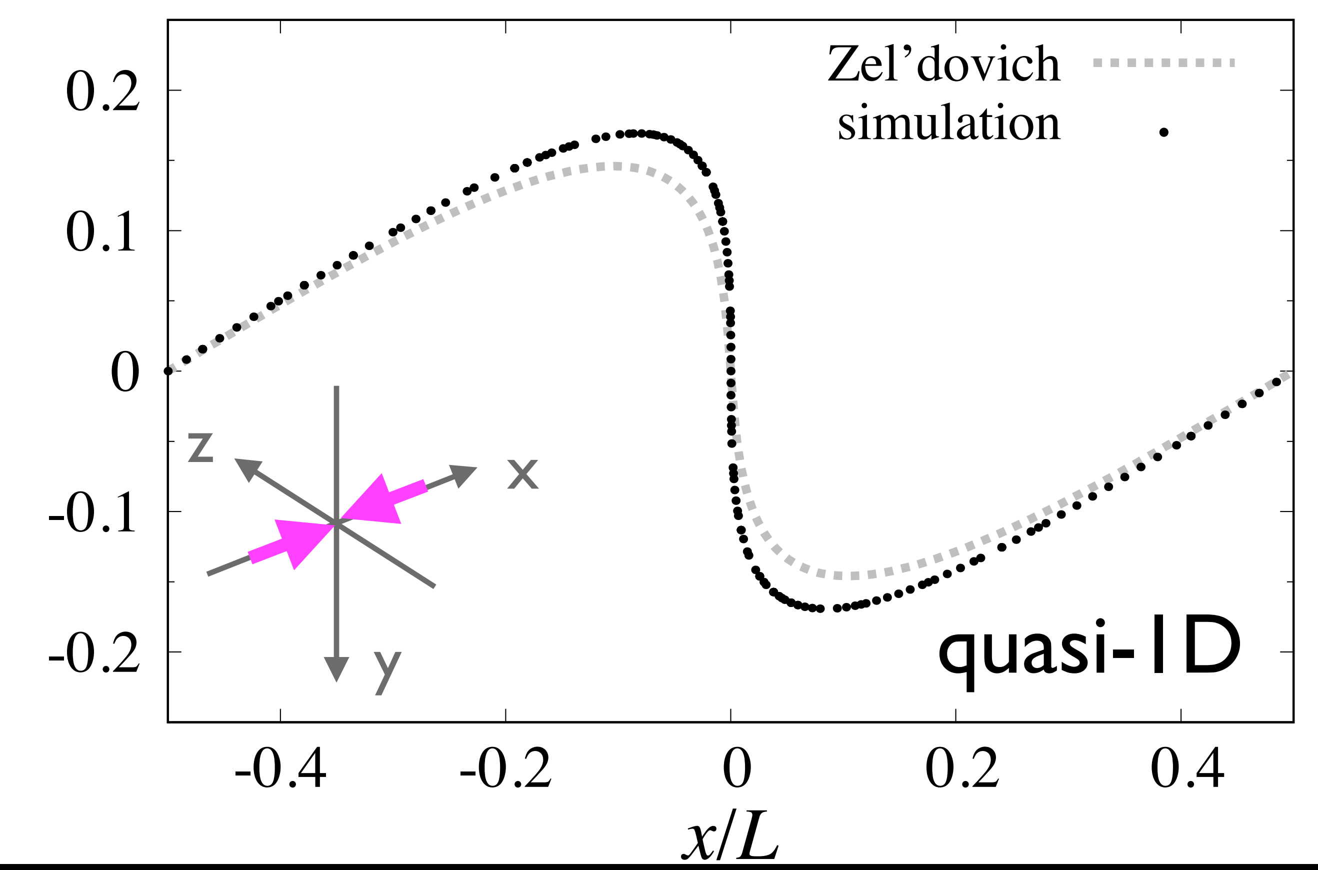
Vlasov simulation vs LPT

シェクルロッシング
時における比較

（原点で密度場発散）

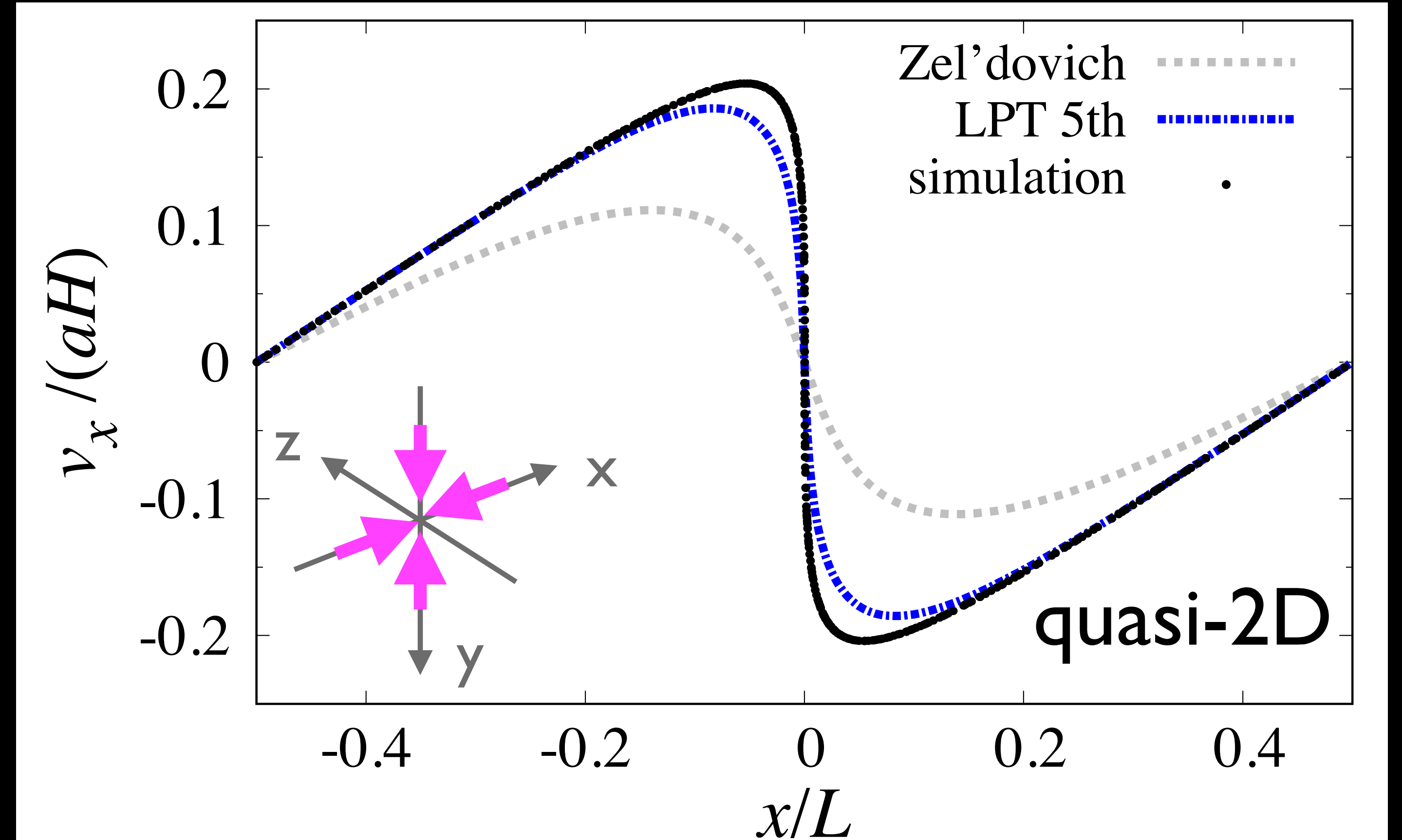


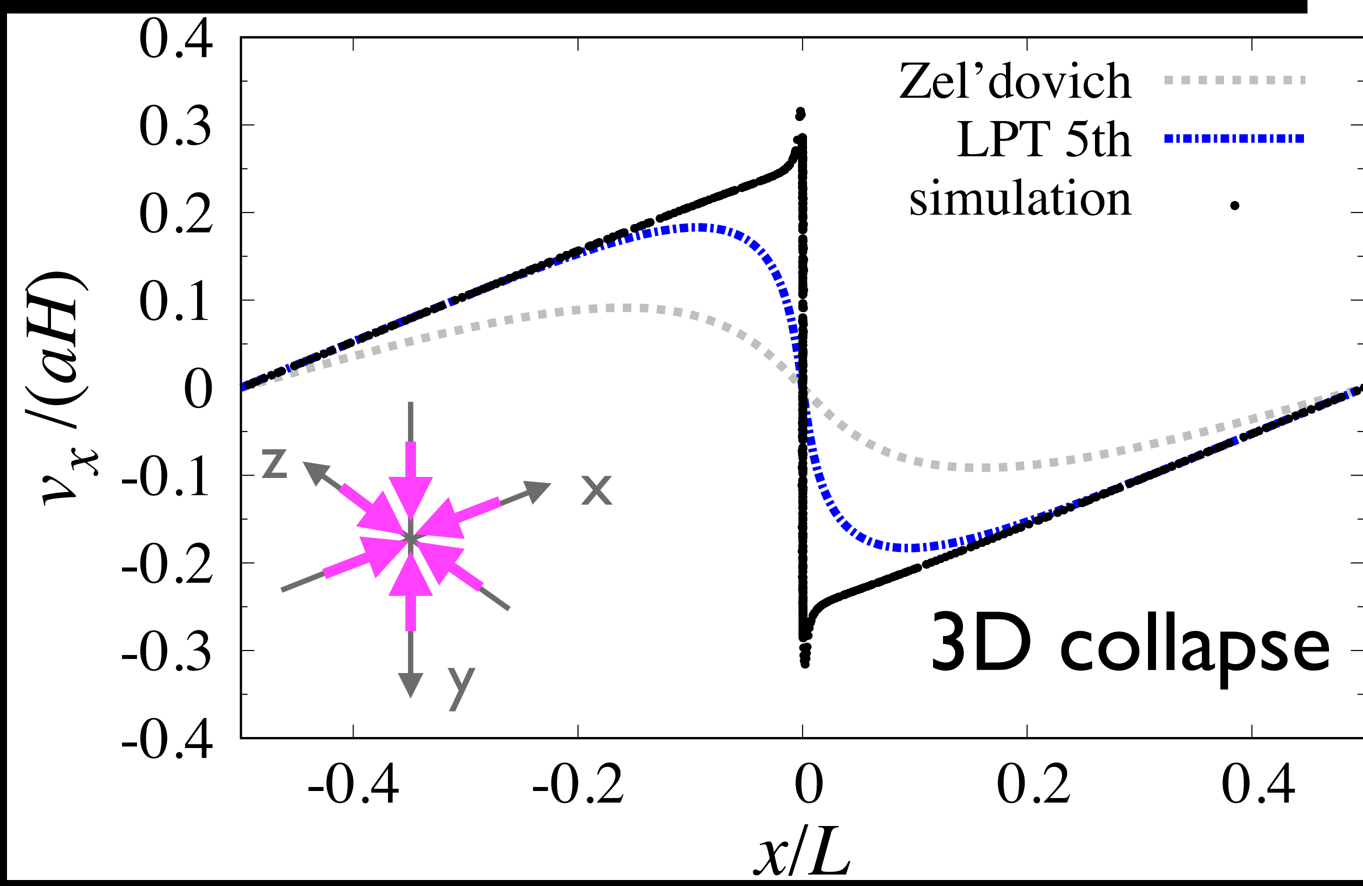


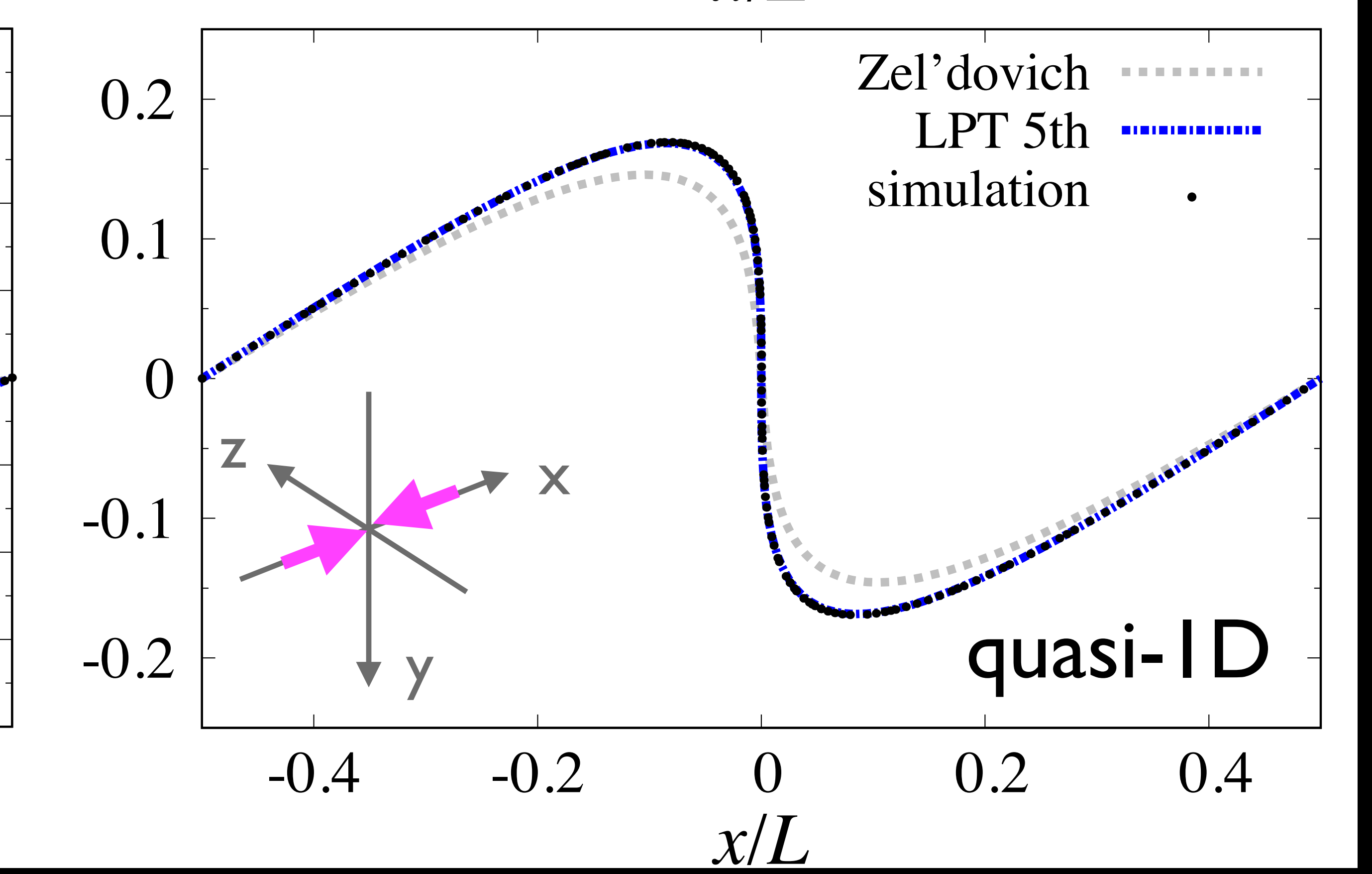


Vlasov simulation vs LPT

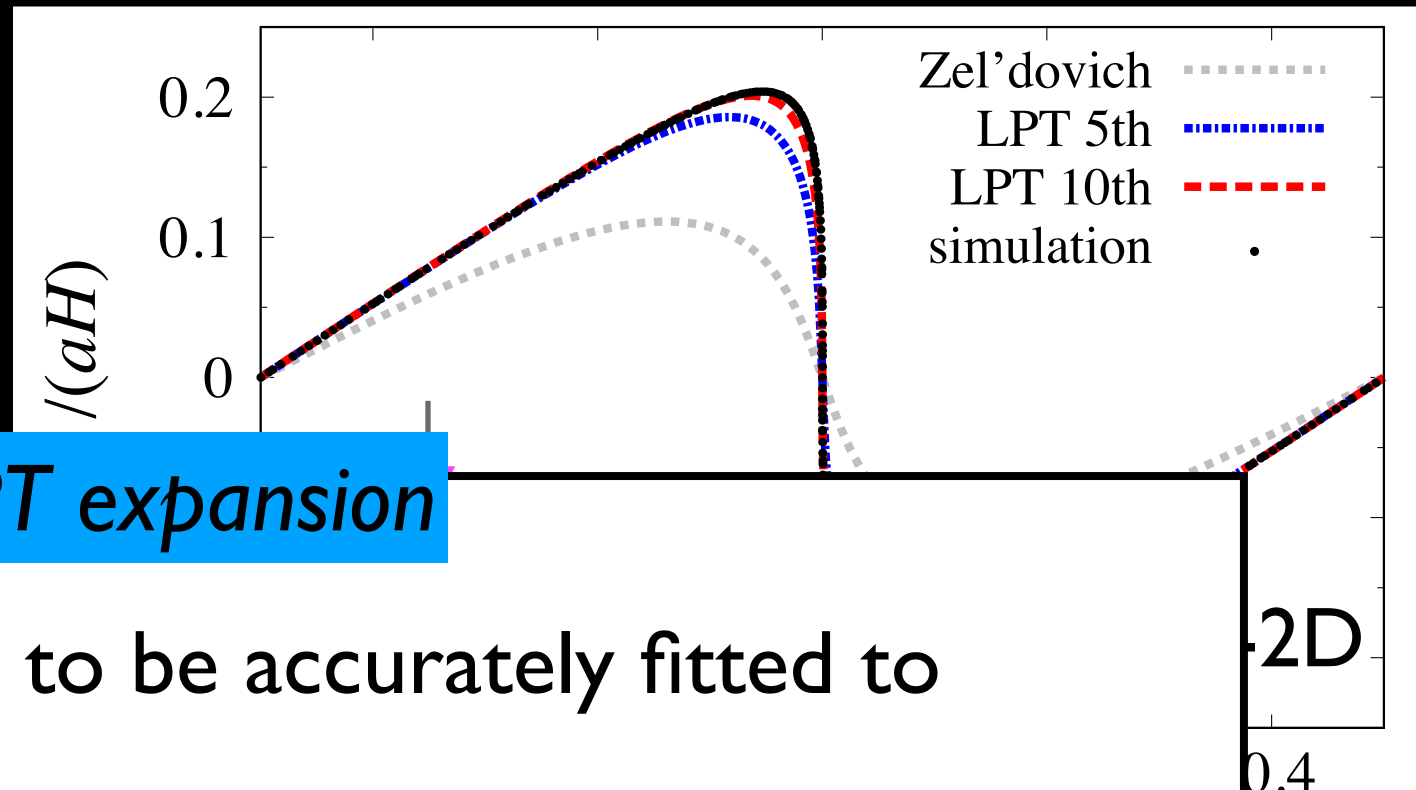
5th-order Lagrangian PT







Vlasov simulation vs LPT



Making use of convergence of LPT expansion

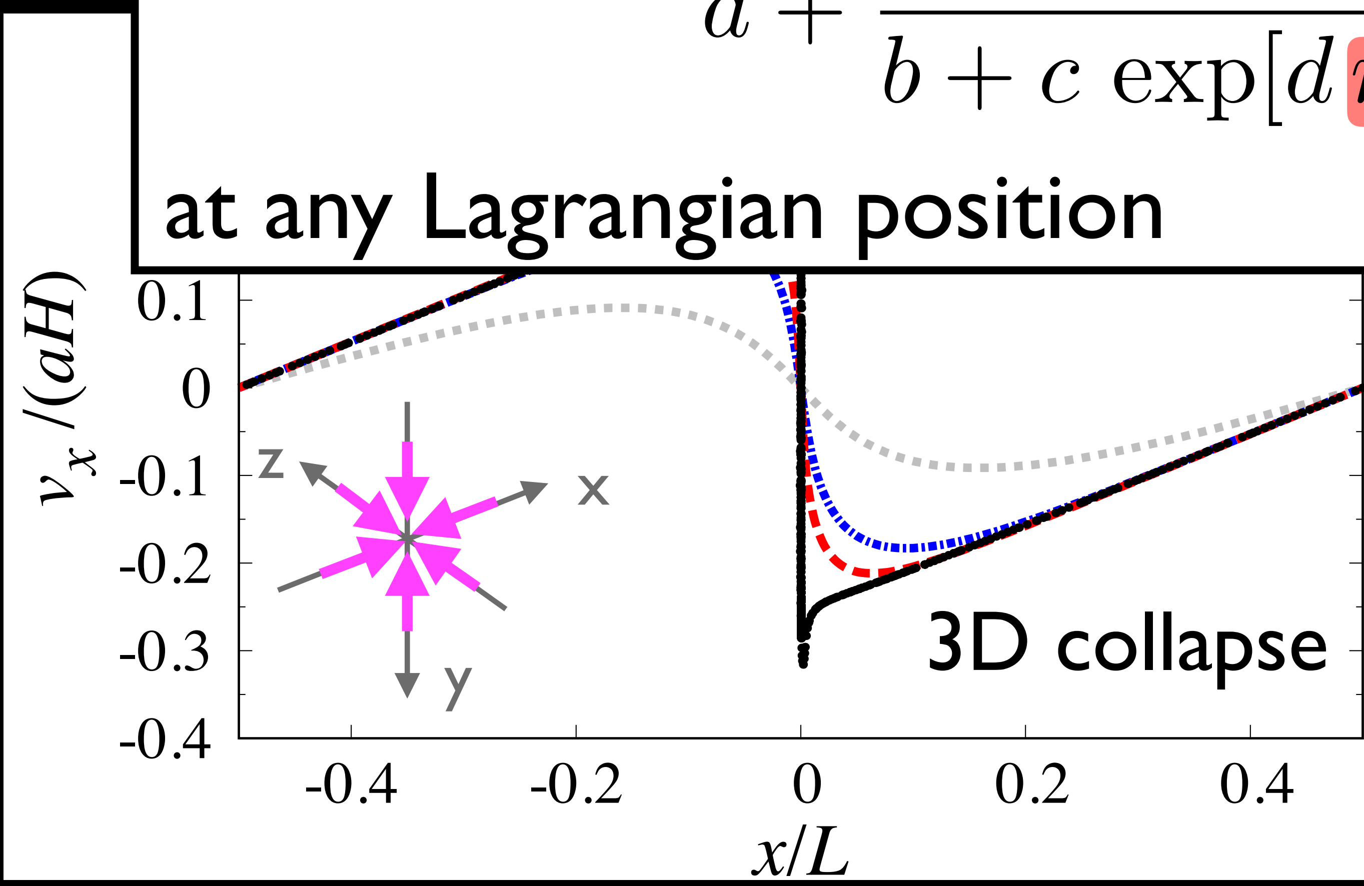
$x_{\rm LPT}(\boldsymbol{q})$ at *n*-th order is found to be accurately fitted to

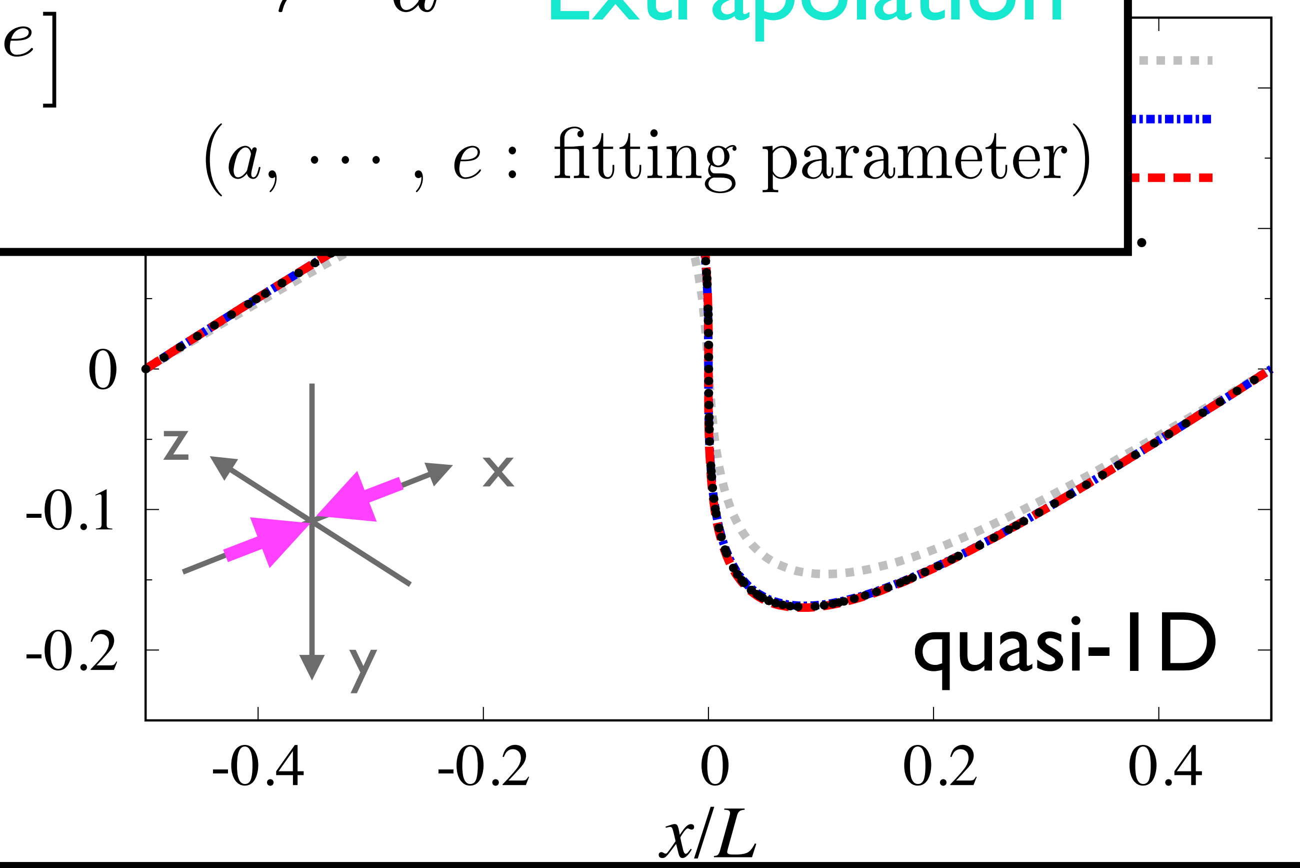
$$a+\frac{1}{b+c\,\exp[d\,n^{e}]}\;\xrightarrow{n\to\infty}\;a$$

Extrapolation

$(a,\cdots,e:\ \text{fitting parameter})$

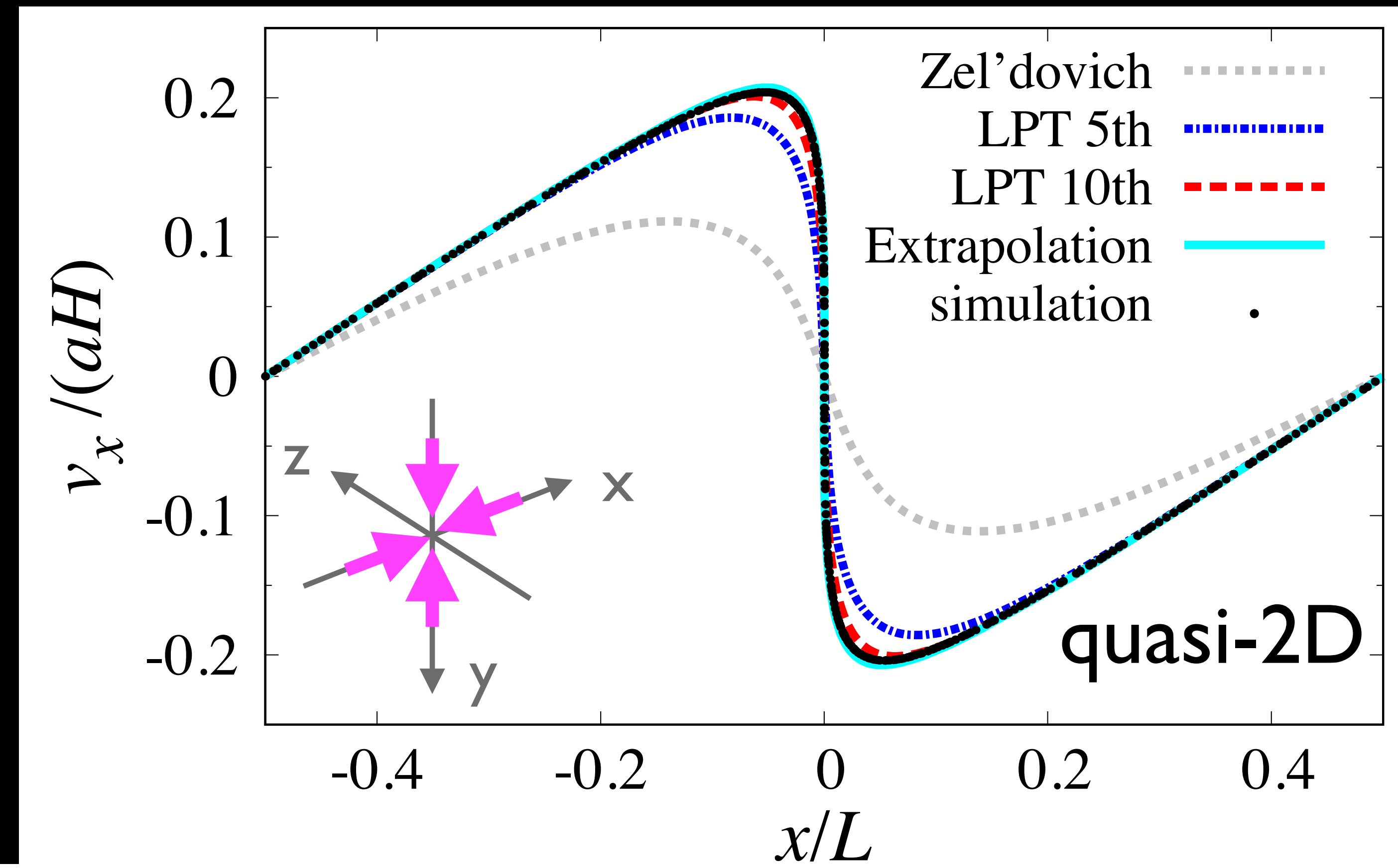
at any Lagrangian position

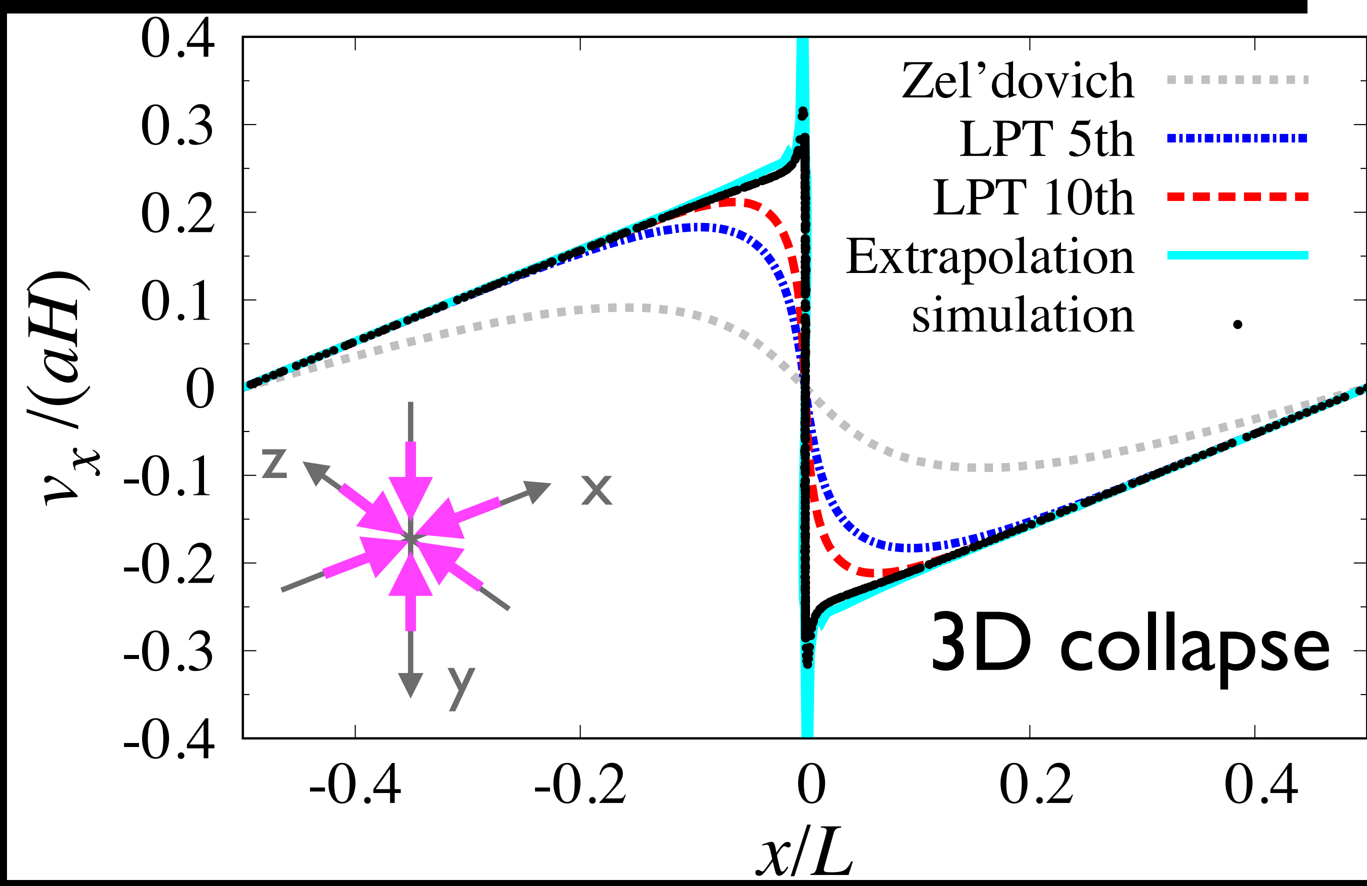


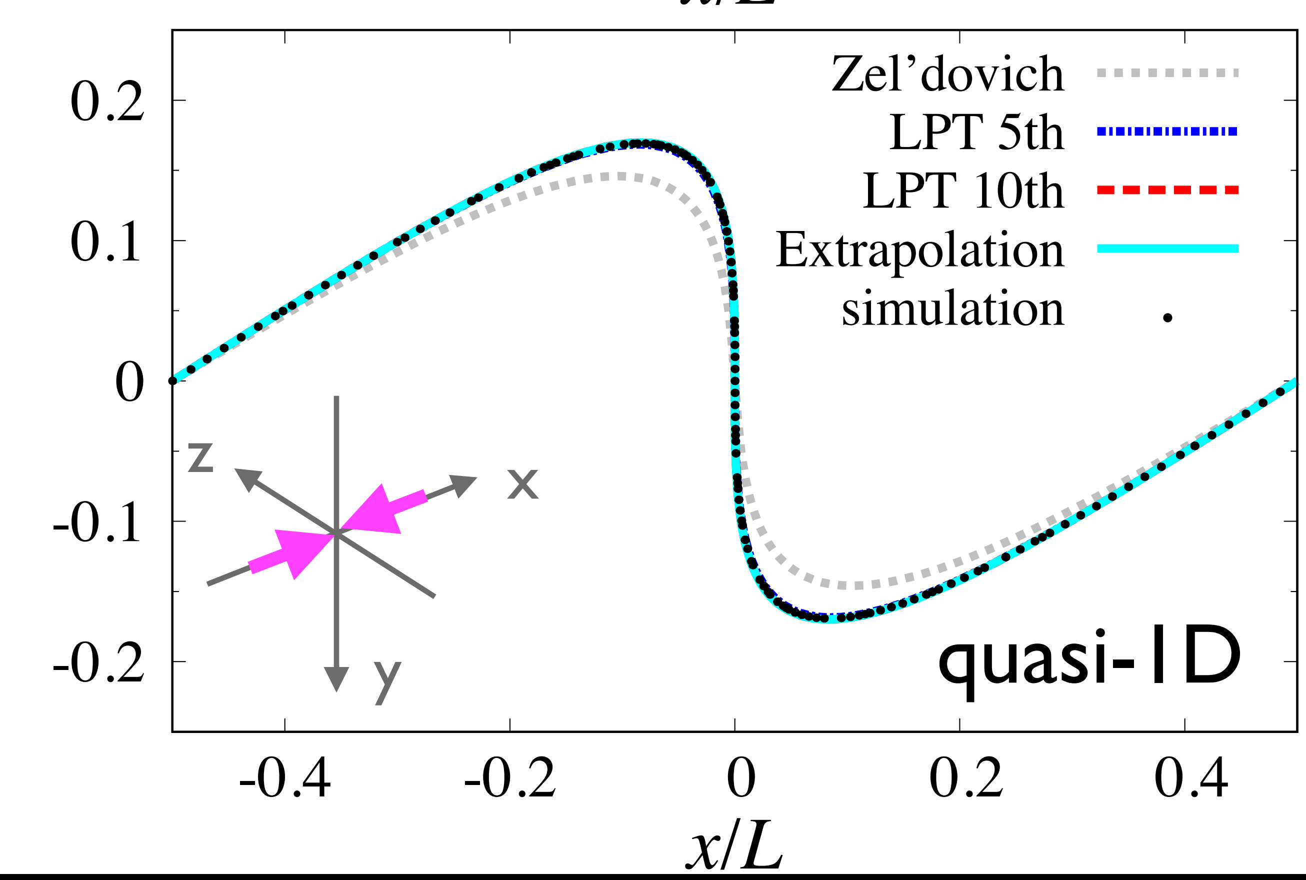


Vlasov simulation vs LPT

Extrapolation based on LPT

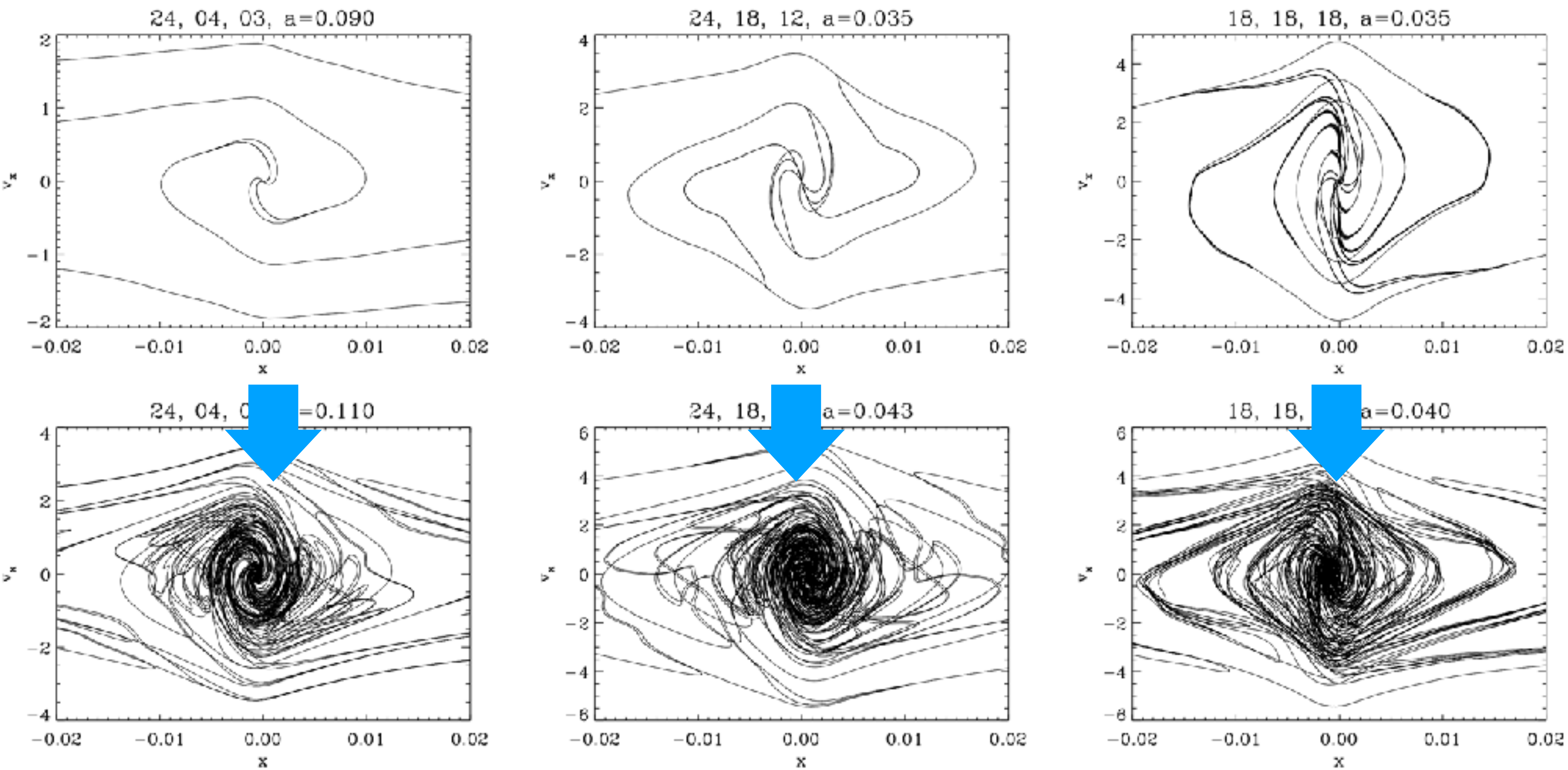






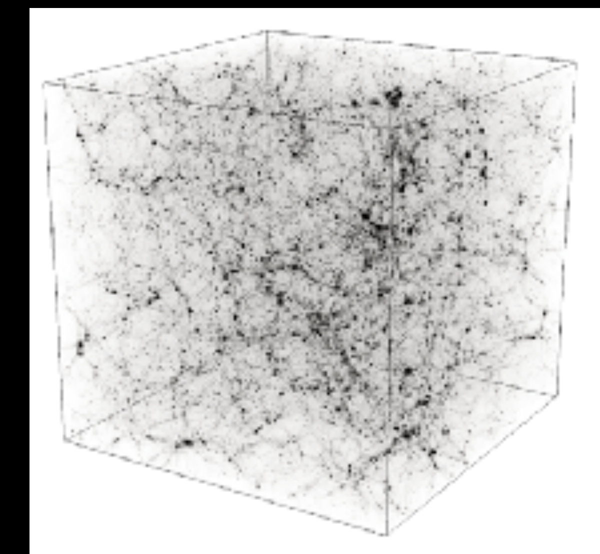
After shell-crossing,

quasi-1D collapse　　quasi-2D collapse　　3D collapse



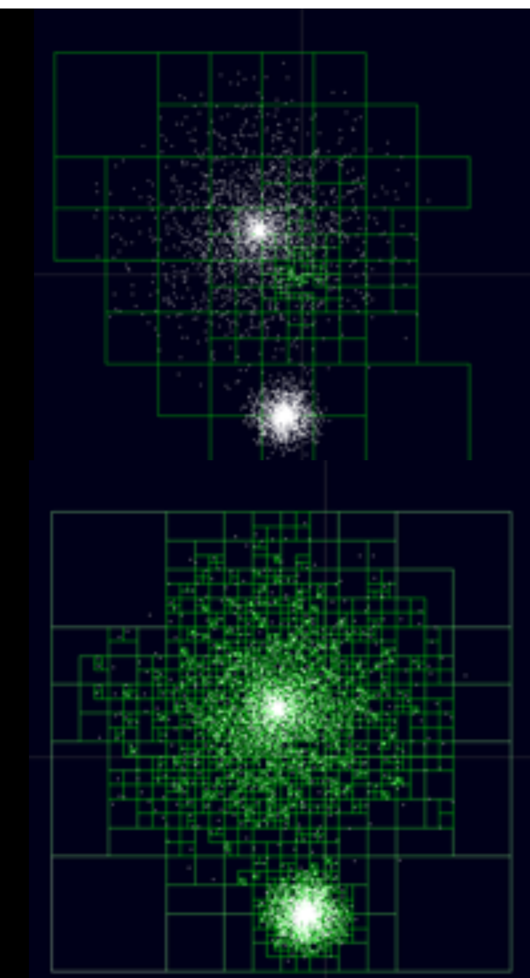
マルチストリーム構造が発達（ハローが形成）

Cosmological *N*-body simulations



Directly solve equation of motion for *N* particles

⟶ Run *N*-body simulation many times with a large number of particles in a huge box



To reduce O(N^2) operation for force calculation,

O(N log N)

- Tree algorithm
- Particle-Mesh algorithm (using FFT)

For cosmological study

N~1,000^3, L~1,000 Mpc,   >50 runs

Still extensive but very useful for practical purposes :

mock data analysis, locating 'galaxies' in dark matter halo, …

Tree-PM method for force calculation

In Fourier space,

$$\phi_{\boldsymbol{k}} = \phi_{\boldsymbol{k}}^{\text{long}} + \phi_{\boldsymbol{k}}^{\text{short}}$$

$$\phi_{\boldsymbol{k}}^{\text{long}} = \phi_{\boldsymbol{k}} \exp\left(-\boldsymbol{k}^2 r_s^2\right)$$

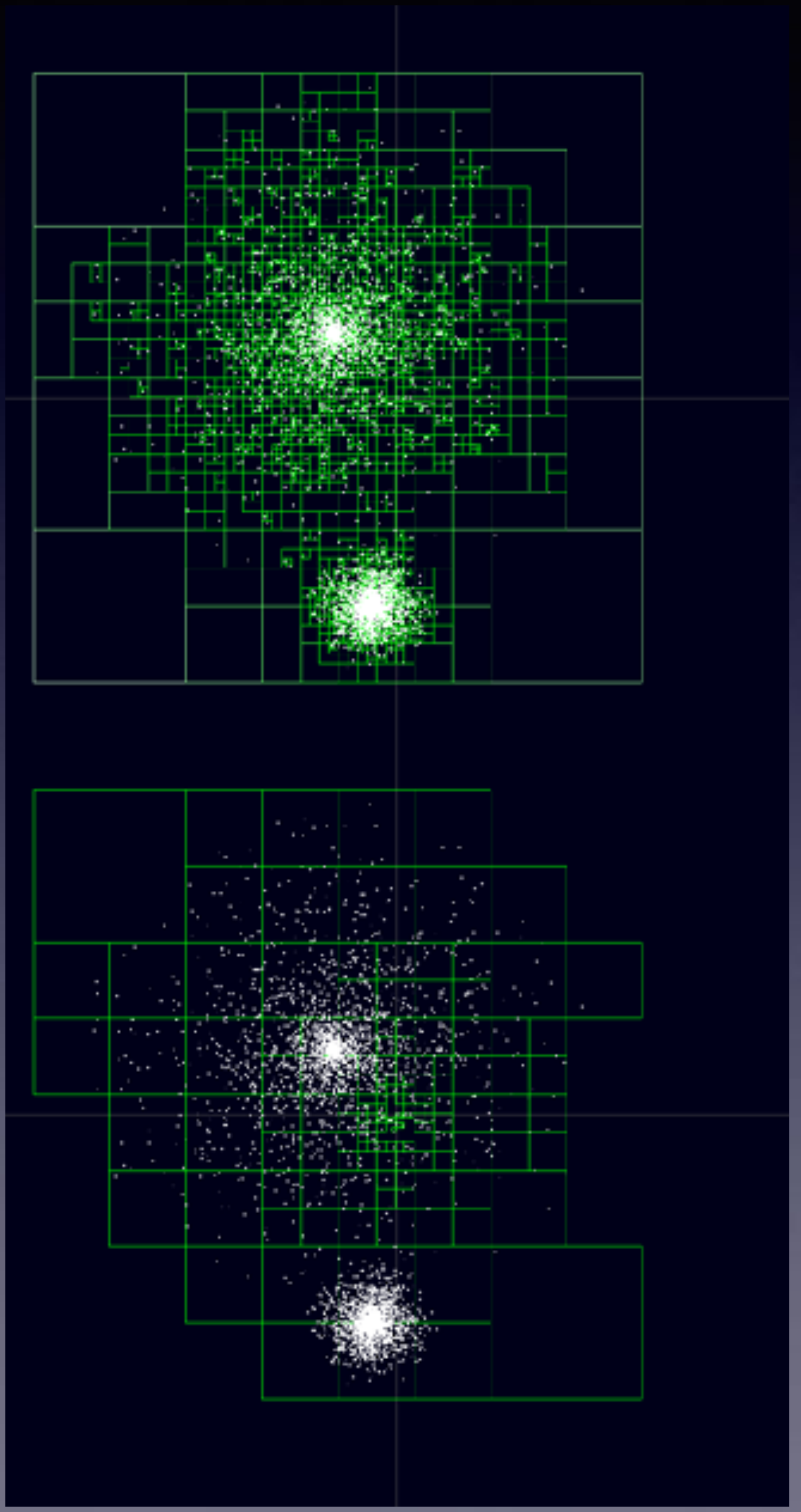
$$\phi_k^{\text{short}} = \phi_k \left[1 - \exp(-\boldsymbol{k}^2 r_s^2)\right] \longrightarrow \phi^{\text{short}}(\boldsymbol{x}) = -G \sum_i \frac{m_i}{r_i} \operatorname{erfc}\left(\frac{r_i}{2r_s}\right)$$

$$r_i = \min(|\boldsymbol{x} - \boldsymbol{r}_i - \boldsymbol{n}L|)$$

- long-range: PM method with FFT
- short-range: Tree algorithm

(Barnes-Hut oct-tree)

http://arborjs.org/docs/barnes-hut



Performance of each method is O(N log N)

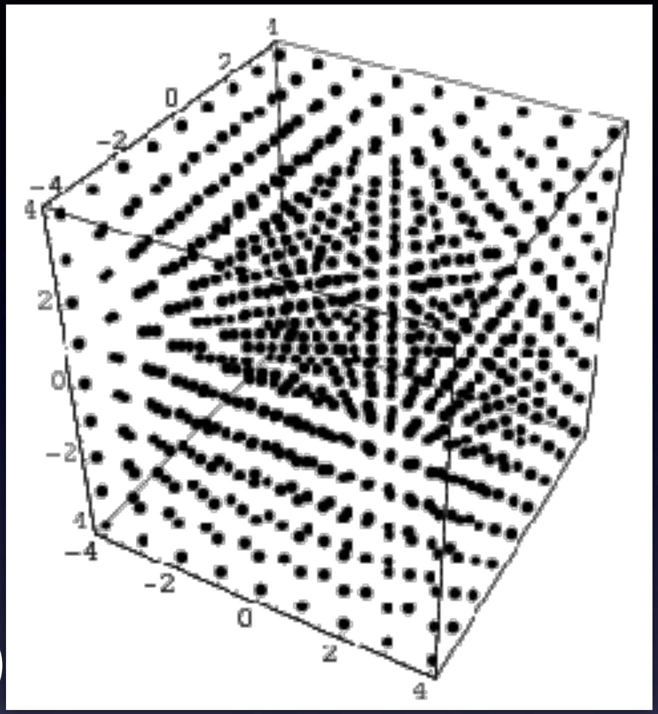
Cosmological initial condition

For particle assigned on each grid:

$$\boldsymbol{x} = \boldsymbol{q} + \boldsymbol{\Psi}(\boldsymbol{q})$$

initial position   Lagrangian displacement

Note $\boldsymbol{v} = a\dot{\boldsymbol{x}} = a\dot{\boldsymbol{\Psi}}(\boldsymbol{q})$



'q' is called Lagrangian coordinate (homogeneous mass dist)

leading order
(Zel'dovich approx.)

$$\Psi(\boldsymbol{k}) \simeq \frac{i\,\boldsymbol{k}}{k^2} D_+(z)\,\delta_0(\boldsymbol{k})$$

initial density field (random)

General procedure

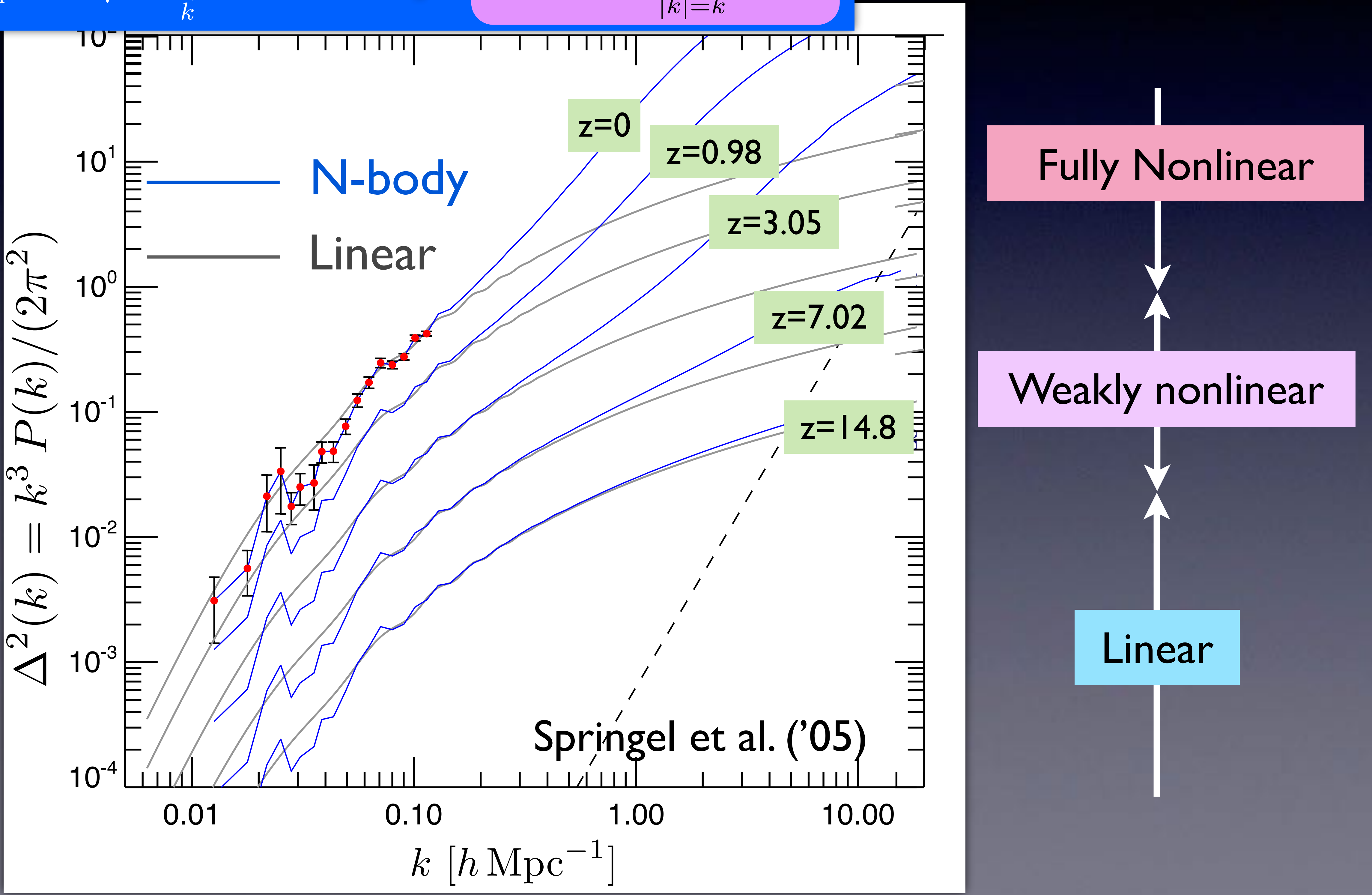
Improved initial condition generator with Lagrangian PT (2LPT code)

1. generate random field $\delta_0(\boldsymbol{k})$
2. calculate displacement field $\boldsymbol{\Psi}(\boldsymbol{k}) \xrightarrow{\text{FFT}} \boldsymbol{\Psi}(\boldsymbol{q})$
3. move particles according to displacement field $\boldsymbol{\Psi}(\boldsymbol{q})$ $\dot{\boldsymbol{\Psi}}(\boldsymbol{q}) = \frac{\dot{D}_+(z)}{D_+(z)}\boldsymbol{\Psi}(\boldsymbol{q})$

Perturbation theory of large-scale structure

Nonlinear gravitational evolution

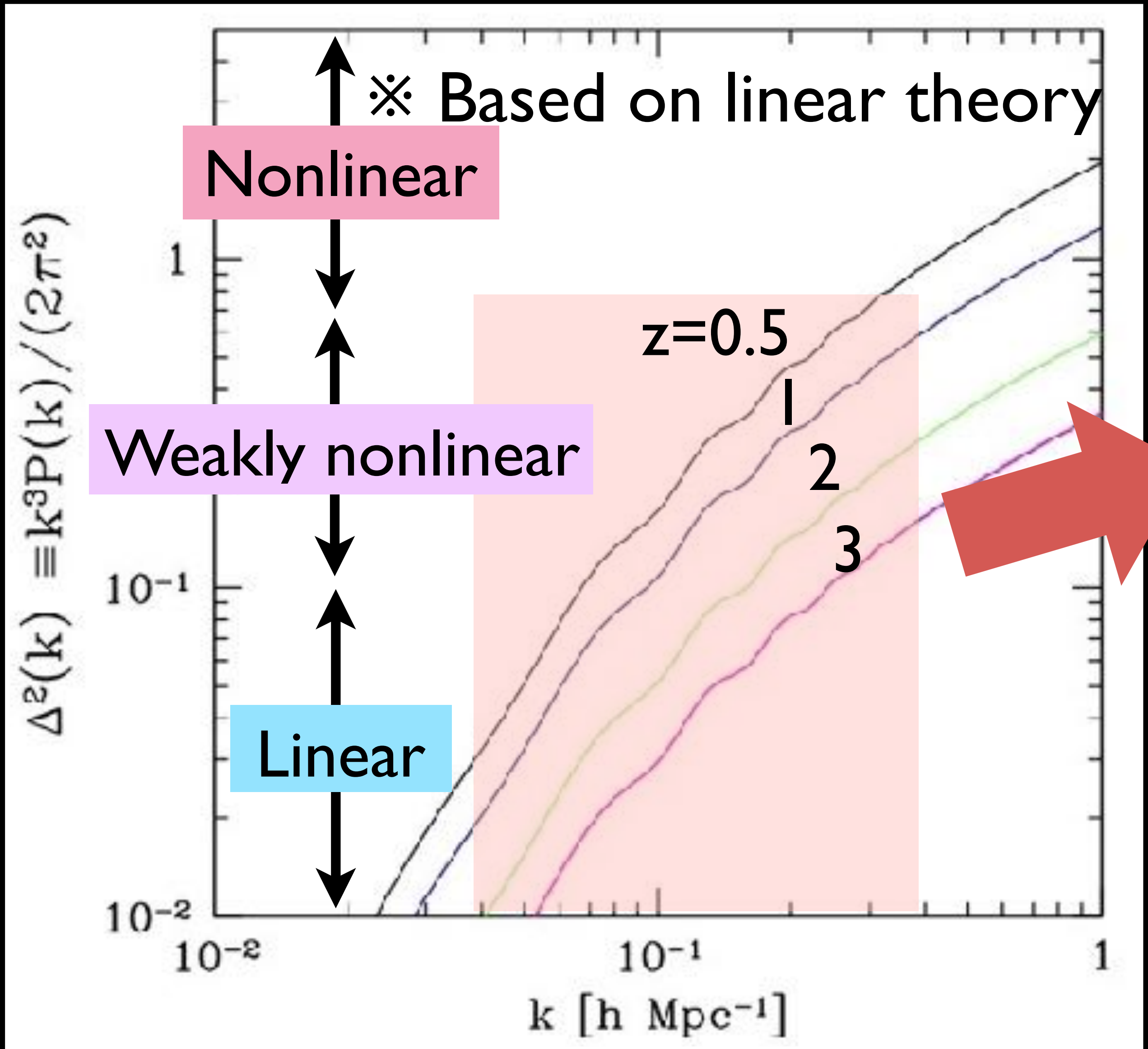
$$\delta(\vec{x}) \equiv \frac{\delta\rho_{\rm m}(\vec{x})}{\overline{\rho}_{\rm m}} = \frac{1}{\sqrt{V}}\sum_{\vec{k}} \delta(\vec{k})\, e^{i\vec{k}\cdot\vec{x}} \Rightarrow P(k) = \frac{1}{N_k}\sum_{|\vec{k}|=k} |\delta(\vec{k})|^2$$

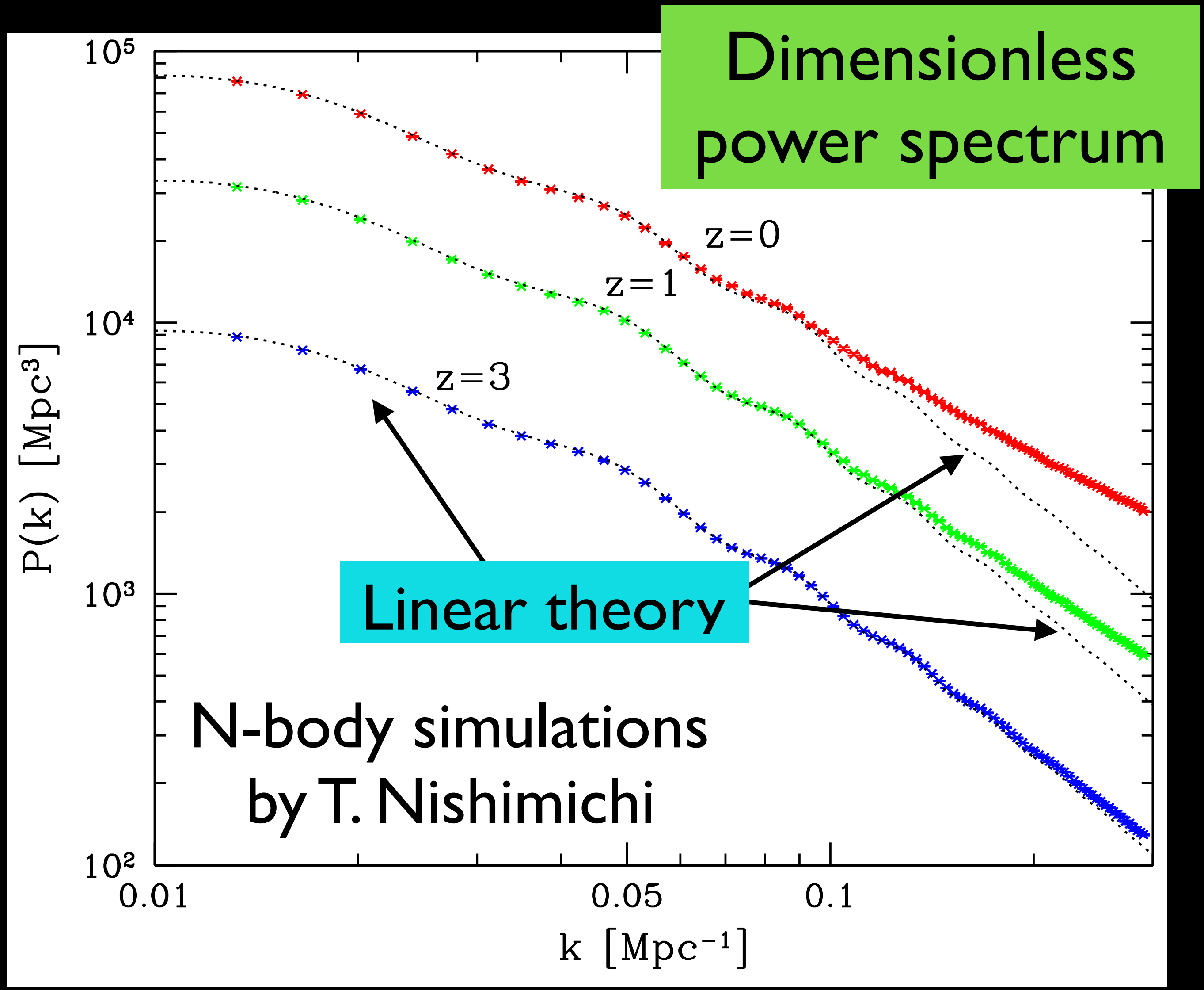


Regime of our interest

Most of interesting cosmological information (BAO, RSD, signature of massive neutrinos, …) lies at k<0.2-0.3 h/Mpc

→ Weakly nonlinear regime





Range of applicability

	Methods (Gravitational evolution)	Other systematics
		very difficult
Fully nonlinear ($\Delta^2 > 1$)	**N-body simulation** most powerful, but extensive & time-consuming **(c.f. fitting formula)**	**Baryon physics** (weak lensing)
weakly nonlinear ($\Delta^2 \lesssim 1$)	**Perturbation theory** limited range of application, but analytical & very fast	• **Galaxy bias** • **Redshift-space distortion** (galaxy surveys)
linear ($\Delta^2 \ll 1$)	**Linear theory** **(CMB Boltzmann code)**	*relatively easy*

Perturbation theory (PT)

Theory of large-scale structure based on gravitational instability

Juszkiewicz ('81), Vishniac ('83), Goroff et al. ('86), Suto & Sasaki ('91), Jain & Bertschinger ('94), ...

Cold dark matter + baryons = pressureless & irrotational fluid

Basic eqs.

$$\frac{\partial \delta}{\partial t} + \frac{1}{a}\vec{\nabla}\cdot\left[(1+\delta)\vec{v}\right] = 0$$

$$\frac{\partial \vec{v}}{\partial t} + \frac{\dot{a}}{a}\vec{v} + \frac{1}{a}(\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{1}{a}\vec{\nabla}\Phi$$

$$\frac{1}{a^2}\nabla^2\Phi = 4\pi G\,\bar{\rho}_{\rm m}\,\delta$$

Single-stream approx. of collisionless Boltzmann eq.

standard PT
$|\delta| \ll 1$

$$\delta = \delta^{(1)} + \delta^{(2)} + \delta^{(3)} + \cdots \qquad \langle \delta(\boldsymbol{k};t)\delta(\boldsymbol{k}';t)\rangle = (2\pi)^3\,\delta_{\rm D}(\boldsymbol{k}+\boldsymbol{k}')\,P(|\boldsymbol{k}|;t)$$

Equations of motion

$$\partial_\tau \delta + \partial_i \left[(1+\delta) v^i \right] = 0 \, ,$$

$$\partial_\tau v^i + \mathcal{H} \, v_l^i + \partial^i \phi + v_l^j \, \partial_j v^i = 0$$

$$\triangle \, \phi = \frac{3}{2} \mathcal{H}^2 \Omega_m \, \delta \, .$$

τ: conformal time

(adτ = dt)

$$\int_{\boldsymbol{q}} \equiv \int \frac{d^3\boldsymbol{q}}{(2\pi)^3}$$

Fourier expansion

$$\theta \equiv \nabla \cdot \boldsymbol{v}$$

$$\partial_\tau \delta(\boldsymbol{k},\tau) + \theta(\boldsymbol{k},\tau) = - \int_{\boldsymbol{q}} \alpha(\boldsymbol{q}, \boldsymbol{k}-\boldsymbol{q}) \, \theta(\boldsymbol{q},\tau) \delta(\boldsymbol{k}-\boldsymbol{q},\tau) \, ,$$

$$\partial_\tau \theta(\boldsymbol{k},\tau) + \mathcal{H} \, \theta(\boldsymbol{k},\tau) + \frac{3}{2} \Omega_m \mathcal{H}^2 \delta(\boldsymbol{k},\tau) = - \int_{\boldsymbol{q}} \beta(\boldsymbol{q}, \boldsymbol{k}-\boldsymbol{q}) \, \theta(\boldsymbol{q},\tau) \theta(\boldsymbol{k}-\boldsymbol{q},\tau)$$

$$\alpha(\boldsymbol{q}_1, \boldsymbol{q}_2) \equiv \frac{\boldsymbol{q}_1 \cdot (\boldsymbol{q}_1 + \boldsymbol{q}_2)}{q_1^2} \, , \qquad \beta(\boldsymbol{q}_1, \boldsymbol{q}_2) \equiv \frac{1}{2} (\boldsymbol{q}_1 + \boldsymbol{q}_2)^2 \frac{\boldsymbol{q}_1 \cdot \boldsymbol{q}_2}{q_1^2 \, q_2^2} \, .$$

Standard perturbation theory

$$\delta(\boldsymbol{k},a) = \sum_{i=1}^{\infty}\delta_{(i)}(\boldsymbol{k},a)\,, \qquad \theta(\boldsymbol{k},a) = -\mathcal{H}\,f(a)\sum_{i=1}^{\infty}\theta_{(i)}(\boldsymbol{k},a)$$

$$f(a) \equiv d\ln D_1/d\ln a$$

$D_1(a)$: Linear growth factor

Adopting the E-dS approximation,

$$\delta_{(n)}(\boldsymbol{k},a) = \underline{D_1^n(a)}\,\delta_n(\boldsymbol{k})\,, \qquad \theta_{(n)}(\boldsymbol{k},a) = \underline{D_1^n(a)}\,\theta_n(\boldsymbol{k})\,.$$

Linear density field (Gaussian)

$$\delta_n(\boldsymbol{k}) = \int_{\boldsymbol{q}_1}\cdots\int_{\boldsymbol{q}_n}(2\pi)^3\delta_D^{(3)}(\boldsymbol{k}-\boldsymbol{q}_1\cdots-\boldsymbol{q}_n)\,F_n(\boldsymbol{q}_1,\ldots,\boldsymbol{q}_n)\,\delta_0(\boldsymbol{q}_1)\ldots\delta_0(\boldsymbol{q}_n)\,,$$

$$\theta_n(\boldsymbol{k}) = \int_{\boldsymbol{q}_1}\cdots\int_{\boldsymbol{q}_n}(2\pi)^3\delta_D^{(3)}(\boldsymbol{k}-\boldsymbol{q}_1\cdots-\boldsymbol{q}_n)\,G_n(\boldsymbol{q}_1,\ldots,\boldsymbol{q}_n)\,\delta_0(\boldsymbol{q}_1)\ldots\delta_0(\boldsymbol{q}_n)\,,$$

standard PT kernel $(F_1 = G_1 = 1)$

Recursion relation for PT kernels

$$\mathcal{F}_a^{(n)}(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_n) \equiv \begin{bmatrix} F_n(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_n) \\ G_n(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_n) \end{bmatrix}$$

$$\mathcal{F}_a^{(n)}(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_n) = \sum_{m=1}^{n-1} \sigma_{ab}^{(n)}\,\gamma_{bcd}(\boldsymbol{q}_1,\boldsymbol{q}_2)\,\mathcal{F}_c^{(m)}(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_m)\,\mathcal{F}_d^{(n-m)}(\boldsymbol{k}_{m+1},\cdots,\boldsymbol{k}_n)$$

$$\boldsymbol{q}_1 = \boldsymbol{k}_1 + \cdots + \boldsymbol{k}_m$$
$$\boldsymbol{q}_2 = \boldsymbol{k}_{m+1} + \cdots + \boldsymbol{k}_n$$

$$\sigma_{ab}^{(n)} = \frac{1}{(2n+3)(n-1)} \begin{pmatrix} 2n+1 & 2 \\ 3 & 2n \end{pmatrix}$$

$$\gamma_{abc}(\boldsymbol{k}_1,\boldsymbol{k}_2) = \begin{cases} \frac{1}{2}\left\{1+\frac{\boldsymbol{k}_2\cdot\boldsymbol{k}_1}{|\boldsymbol{k}_2|^2}\right\}; & (a,b,c)=(1,1,2) \\ \frac{1}{2}\left\{1+\frac{\boldsymbol{k}_1\cdot\boldsymbol{k}_2}{|\boldsymbol{k}_1|^2}\right\}; & (a,b,c)=(1,2,1) \\ \frac{(\boldsymbol{k}_1\cdot\boldsymbol{k}_2)|\boldsymbol{k}_1+\boldsymbol{k}_2|^2}{2|\boldsymbol{k}_1|^2|\boldsymbol{k}_2|^2}; & (a,b,c)=(2,2,2) \\ 0; & \text{otherwise.} \end{cases}$$

Note—. repetition of the same subscripts (a,b,c) indicates the sum over all multiplet components

PT kernels constructed from recursion relation should be symmetrized

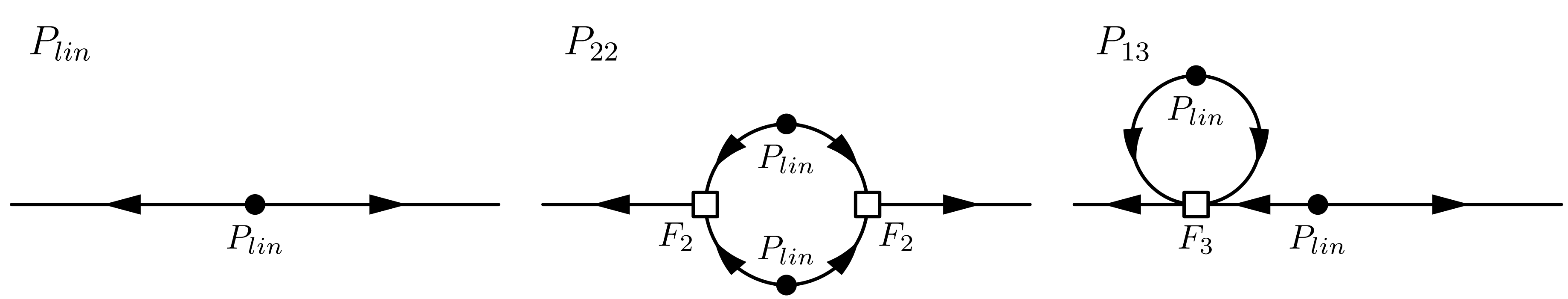
Power spectrum

$$\langle \delta(\boldsymbol{k}_1, a)\delta(\boldsymbol{k}_2, a)\rangle \equiv (2\pi)^3 \delta_D^{(3)}(\boldsymbol{k}_1 + \boldsymbol{k}_2)\, P(k_1, a)$$

$$P_{SPT}(k) \;=\; \underbrace{P_{lin}(k)}_{\text{linear}} + \underbrace{P_{22}(k) + P_{13}(k)}_{\text{1-loop}} + \text{ higher order loops .}$$

$$P_{22}(k) \;=\; 2\int_{\boldsymbol{q}} P_{lin}(q) P_{lin}(|\boldsymbol{k}-\boldsymbol{q}|)\, F_2^2(\boldsymbol{q}, \boldsymbol{k}-\boldsymbol{q}) \;,$$

$$P_{13}(k) \;=\; 6P_{lin}(k)\int_{\boldsymbol{q}} P_{lin}(q)\, F_3(\boldsymbol{k}, \boldsymbol{q}, -\boldsymbol{q}) \;,$$

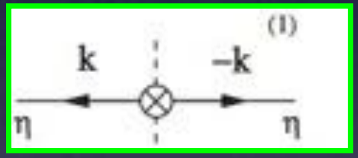


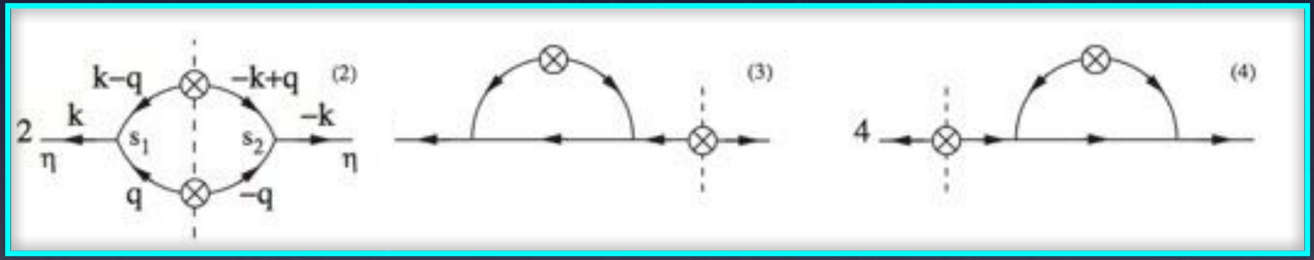
Next-to-next-to leading order

up to 2-loop order

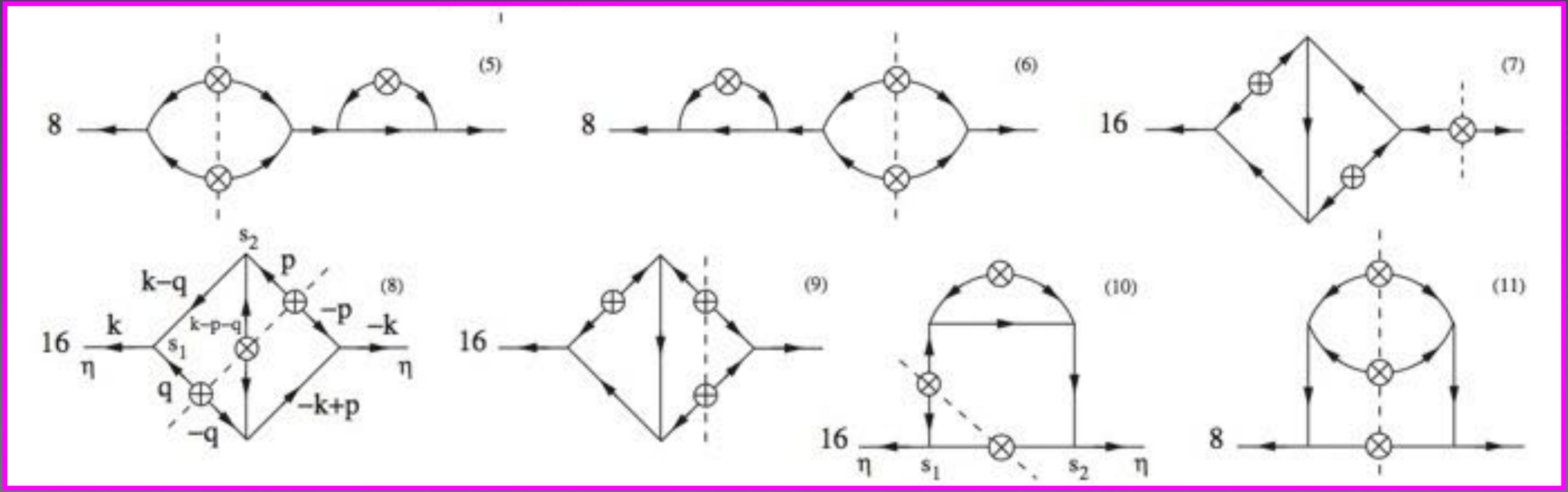
$$P^{(mn)} \simeq \langle \delta^{(m)} \delta^{(n)} \rangle$$

$$P(k) = \underbrace{P^{(11)}(k)}_{\text{Linear (tree)}} + \underbrace{\left(P^{(22)}(k) + P^{(13)}(k)\right)}_{\text{1-loop}} + \underbrace{\left(P^{(33)}(k) + P^{(24)}(k) + P^{(15)}(k)\right)}_{\text{2-loop}} + \cdots$$





Crocce & Scoccimarro ('06)



Calculation involves multi-dimensional numerical integration

(time-consuming)

Comparison with simulations

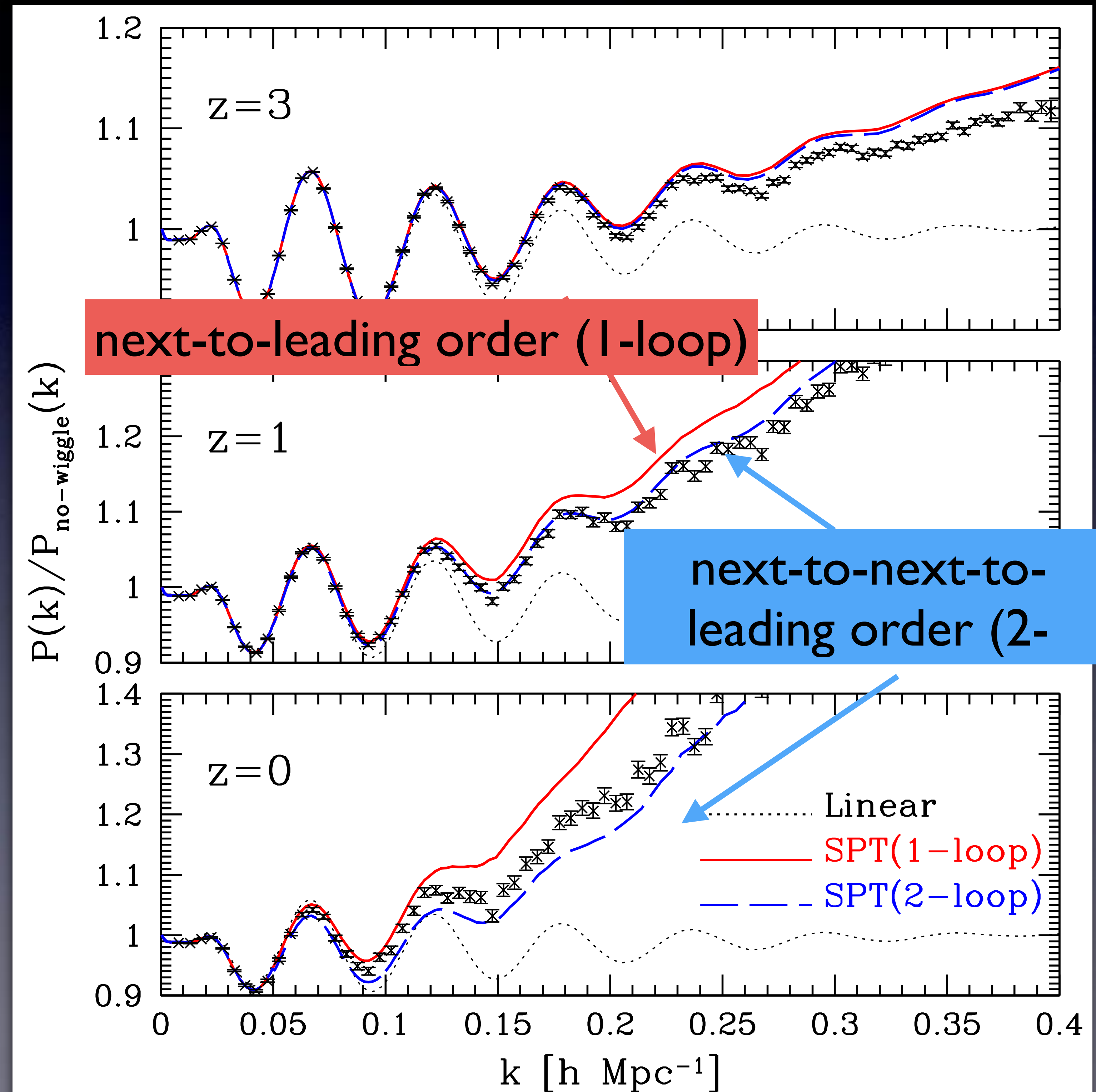
Standard PT qualitatively explains scale-dependent nonlinear growth, however,

1-loop :
overestimates simulations

2-loop :
overestimates at high-z, while it turn to underestimate at low-z

Standard PT produces ill-behaved PT expansion !!

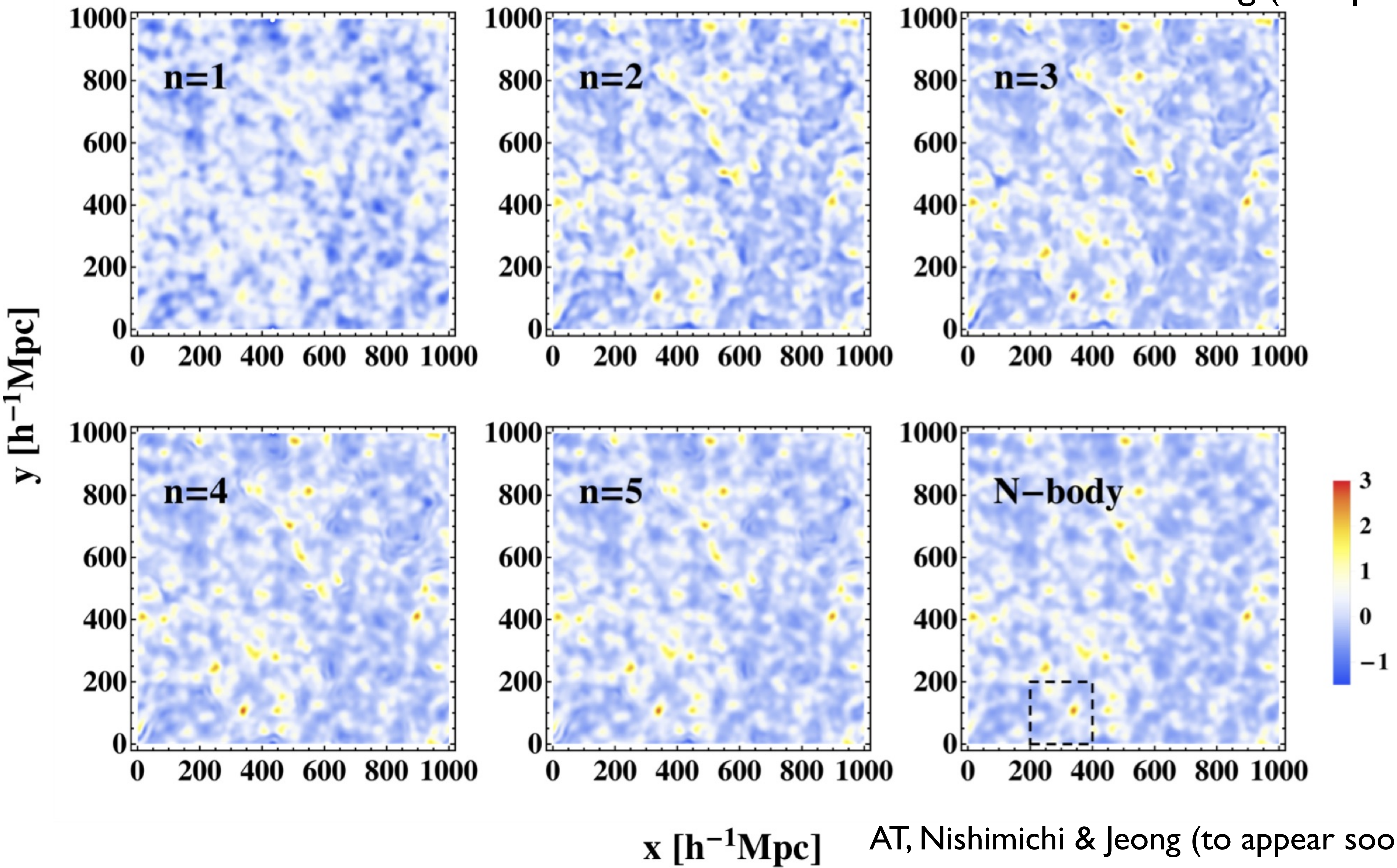
… need to be improved



AT et al. ('09)

Density field in standard PT

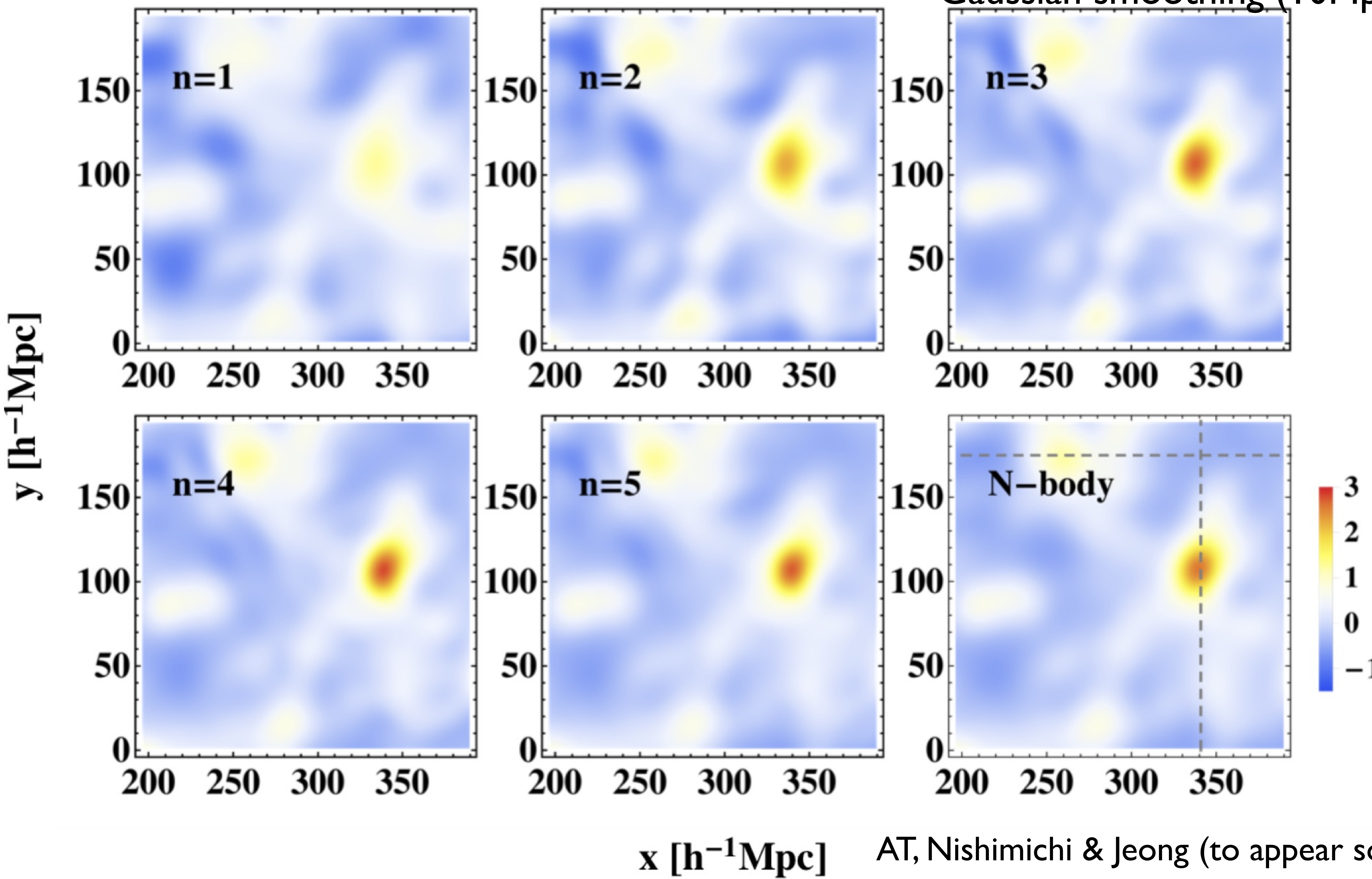
Gaussian smoothing (10Mpc/h)



AT, Nishimichi & Jeong (to appear soon)

Density field in standard PT

Gaussian smoothing (10Mpc/h)

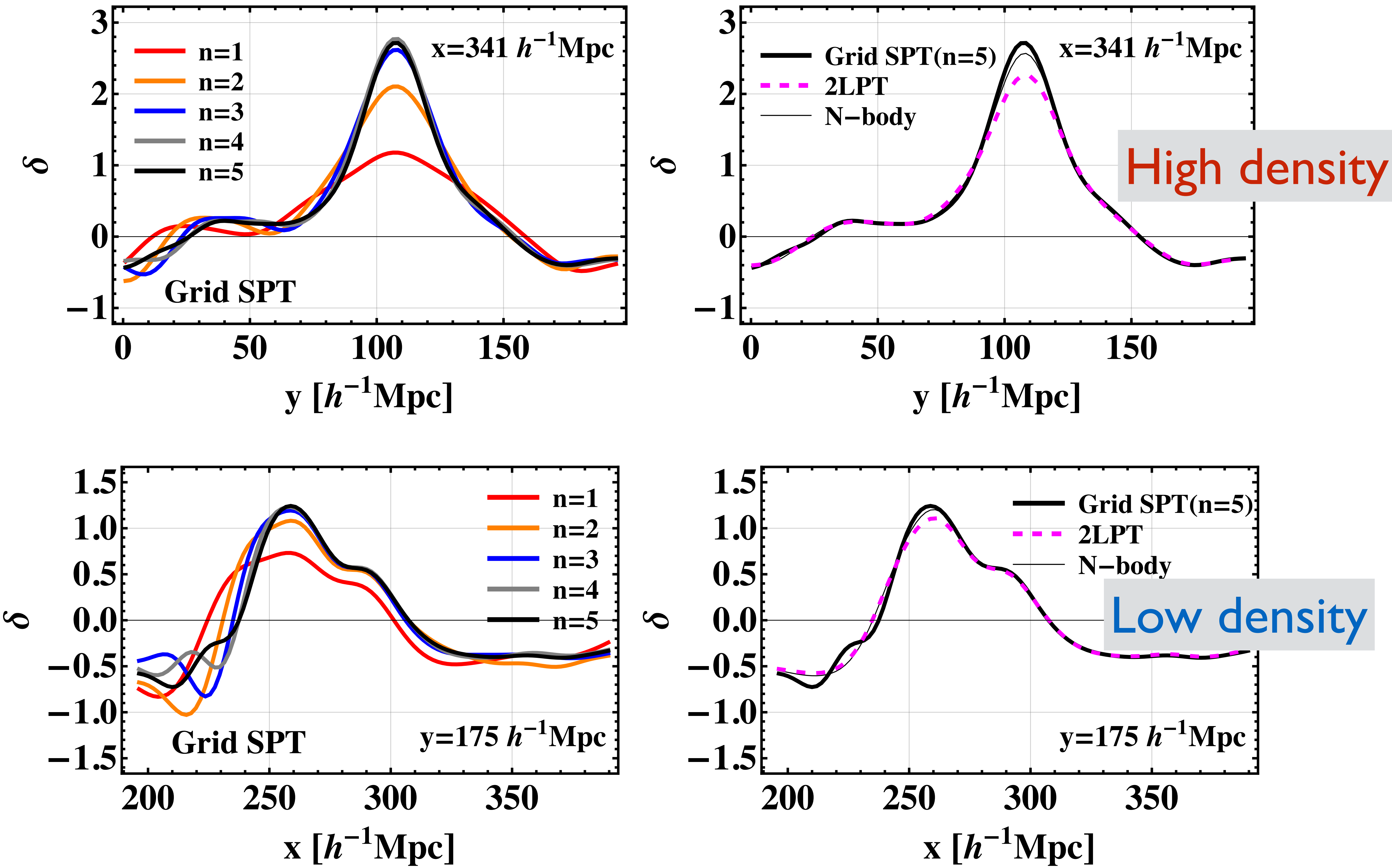


AT, Nishimichi & Jeong (to appear soon)

1D density field in standard PT

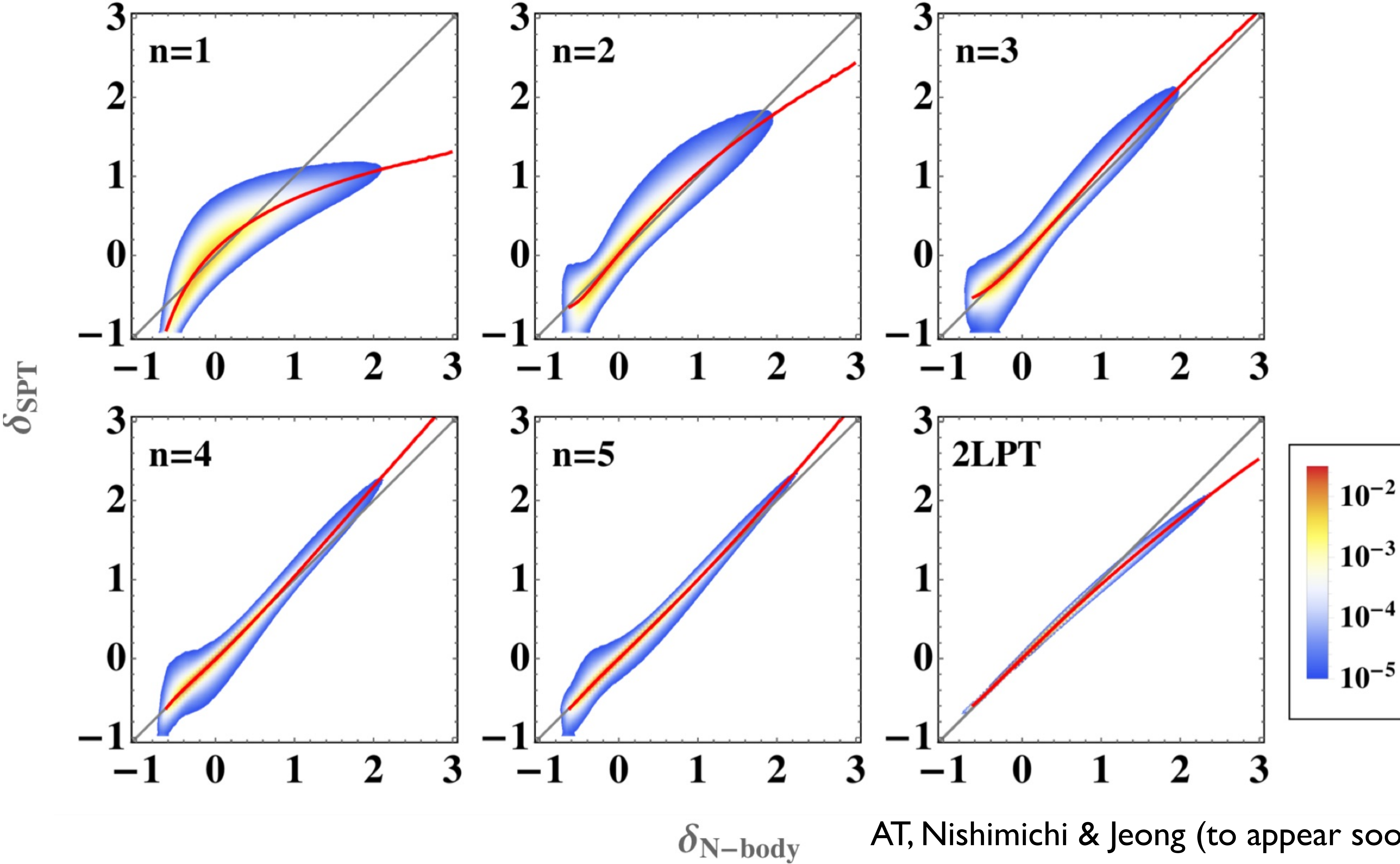
Gaussian smoothing (10Mpc/h)

AT, Nishimichi & Jeong (to appear soon)



Correlation between N-body and SPT

Gaussian smoothing (10Mpc/h)



AT, Nishimichi & Jeong (to appear soon)

Improving PT predictions

Basic idea Reorganizing standard PT expansion by introducing non-perturbative statistical quantities

$\delta_0(\boldsymbol{k})$

initial density field (Gaussian)

$\delta(\boldsymbol{k};z)$

Evolved density field (non-Gaussian)

Initial power spectrum

$P_0(k)$

from linear theory

(CMB Boltzmann code)

Nonlinear mapping

Observables

$P(k;z)$

$B(k_1,k_2,k_3;z)$

$T(k_1,k_2,k_3,k_4;z)$

$\vdots$

of dark matter/galaxies/halos

Concept of '*propagator*' in physics/mathematics may be useful

Propagator in physics

- ✦ Green's function in linear differential equations
- ✦ Probability amplitude in quantum mechanics

Schrödinger Eq.

$$\left(-i\hbar\frac{\partial}{\partial t}+H_x\right)\psi(x,t)=0$$

$$G(x,t;x',t')\equiv\frac{\delta\psi(x,t)}{\delta\psi(x',t')}$$

$$\left(-i\hbar\frac{\partial}{\partial t}+H_x\right)G(x,t;x',t')=-i\hbar\,\delta_D(x-x')\delta_D(t-t')$$

$$\Rightarrow\quad \psi(x,t)=\int_{-\infty}^{+\infty}dx'\,G(x,t;x',t')\,\psi(x',t')\ ;\quad t>t'$$

Cosmic propagators

Propagator should carry information on
non-linear evolution & statistical properties

Crocce & Scoccimarro ('06)

Evolved (non-linear) density field

$$\left\langle \frac{\delta\,\delta_{\rm m}(\boldsymbol{k};t)}{\delta\,\delta_0(\boldsymbol{k'})} \right\rangle \equiv \delta_{\rm D}(\boldsymbol{k}-\boldsymbol{k'})\,\boxed{\Gamma^{(1)}(k;t)}$$

Propagator

Initial density field

Ensemble w.r.t randomness of initial condition

Contain statistical information on *full-nonlinear* evolution

(Non-linear extension of Green's function)

Multi-point propagators

Bernardeau, Crocce & Scoccimarro ('08)
Matsubara ('11) $\longrightarrow$ *integrated PT*

As a natural generalization,

Multi-point propagator

$$\left\langle \frac{\delta^n\,\delta_{\rm m}(\boldsymbol{k};t)}{\delta\,\delta_0(\boldsymbol{k}_1)\cdots\delta\,\delta_0(\boldsymbol{k}_n)} \right\rangle = (2\pi)^{3(1-n)}\,\delta_{\rm D}(\boldsymbol{k}-\boldsymbol{k'})\,\Gamma^{(n)}(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_n;t)$$

With this multi-point prop.

- Building blocks of a new perturbative theory (PT) expansion

  ·········· Γ-expansion or Wiener-Hermite expansion

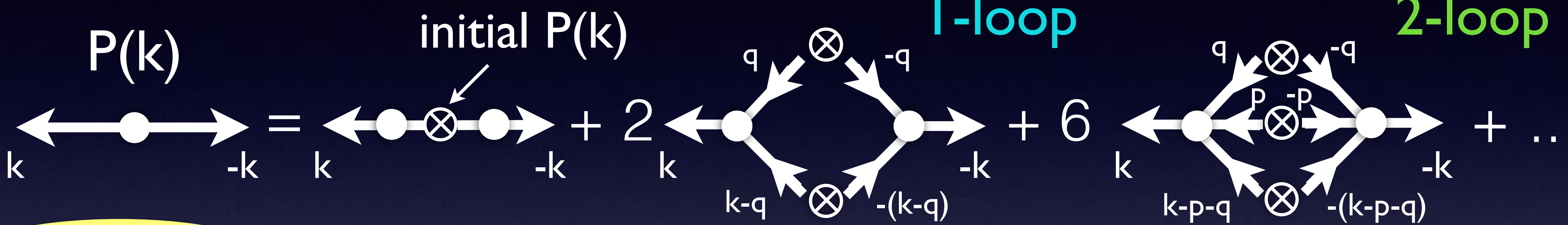
- A good convergence of PT expansion is expected

  (c.f. standard PT)

Power spectrum

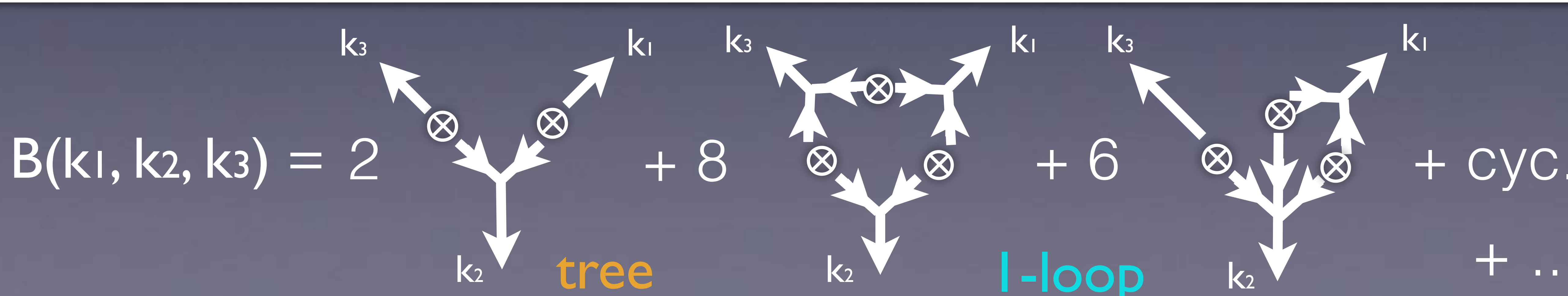
Initial power spectrum

$$P(k;t) = \left[\Gamma^{(1)}(k;t)\right]^2 P_0(k) + 2\int \frac{d^3\boldsymbol{q}}{(2\pi)^3}\left[\Gamma^{(2)}(\boldsymbol{q},\boldsymbol{k}-\boldsymbol{q};t)\right]^2 P_0(q)P_0(|\boldsymbol{k}-\boldsymbol{q}|)$$
$$+\,6\int \frac{d^6\boldsymbol{p}d^3\boldsymbol{q}}{(2\pi)^6}\left[\Gamma^{(3)}(\boldsymbol{p},\boldsymbol{q},\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q};t)\right]^2 P_0(p)P_0(q)P_0(|\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q}|) + \cdots$$



Bispectrum

$$B(k_1,k_2,k_3) = 2\,\Gamma^{(2)}(\mathbf{k}_1,\mathbf{k}_2)\,\Gamma^{(1)}(k_1)\,\Gamma^{(1)}(k_2)\,P_0(k_1)P_0(k_2) + \text{cyc.}$$
$$+\left[8\int d^3q\,\Gamma^{(2)}(\mathbf{k}_1-\mathbf{q},\mathbf{q})\Gamma^{(2)}(\mathbf{k}_2+\mathbf{q},-\mathbf{q})\Gamma^{(2)}(\mathbf{q}-\mathbf{k}_1,-\mathbf{k}_2-\mathbf{q})P_0(|\mathbf{k}_1-\mathbf{q}|)P_0(|\mathbf{k}_2+\mathbf{q}|)P_0(q)\right.$$
$$\left.+\;6\int d^3q\,\Gamma^{(3)}(-\mathbf{k}_3,-\mathbf{k}_2+\mathbf{q},-\mathbf{q})\Gamma^{(2)}(\mathbf{k}_2-\mathbf{q},\mathbf{q})\Gamma^{(1)}(\mathbf{k}_3)P_0(|\mathbf{k}_2-\mathbf{q}|)P_0(q)P_0(k_3) + \text{cyc.}\right].$$

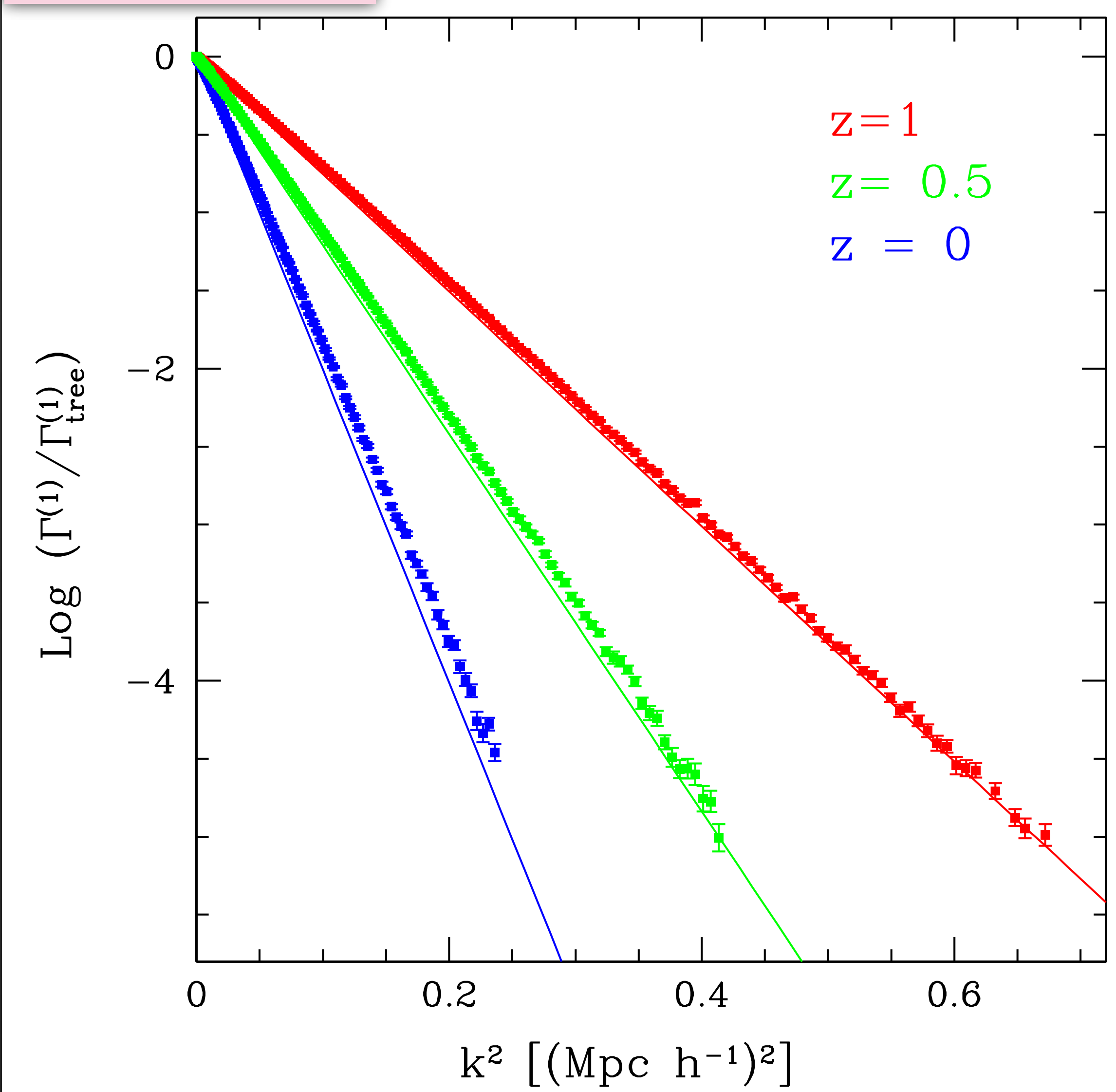


Generic property of propagators

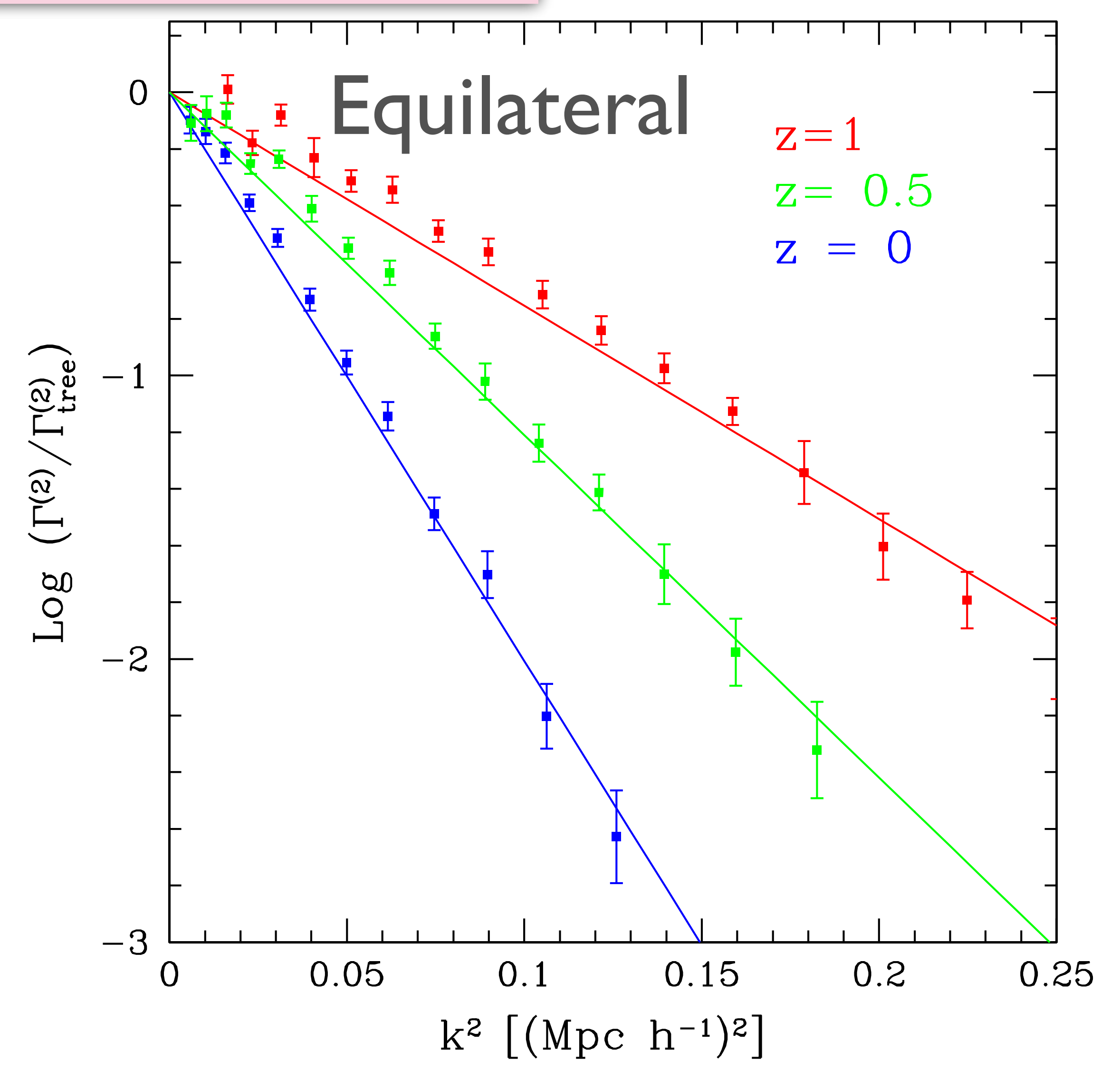
Crocce & Scoccimarro ’06, Bernardeau et al. ’08

$$\Gamma^{(n)} \xrightarrow{k\to+\infty} {}^{s}F_n(\boldsymbol{k}_1,\cdots\boldsymbol{k}_n)\, e^{-k^2\sigma_{\rm v}^2/2} \quad ; \quad \sigma_{\rm v}^2=\int \frac{dq}{6\pi^2}\,P_{\theta\theta}(q)$$

$\Gamma^{(1)}(k)$



$\Gamma^{(2)}(k_1,k_2,k_3)$



Constructing regularized propagators

- UV property (k >>1) :

$$\Gamma^{(n)} \xrightarrow{k\to+\infty} \Gamma^{(n)}_{\rm tree}\, e^{-k^2\sigma_{\rm v}^2/2} \quad ; \quad \sigma_{\rm v}^2=\int\frac{dq}{6\pi^2}P_{\theta\theta}(q)$$

Bernardeau, Crocce & Scoccimarro ('08), Bernardeau, Van de Rijt, Vernizzi ('11)

- IR behavior (k<<1) can be described by standard PT calculations :

$$\Gamma^{(n)}=\Gamma^{(n)}_{\rm tree}+\Gamma^{(n)}_{\rm 1\text{-}loop}+\Gamma^{(n)}_{\rm 2\text{-}loop}+\cdots$$

Importantly, each term behaves like $\Gamma^{(n)}_{p\text{-loop}} \xrightarrow{k\to+\infty} \frac{1}{p!}\left(-\frac{k^2\,\sigma_{\rm v}^2}{2}\right)^p \Gamma^{(n)}_{\rm tree}$

A regularization scheme that reproduces both UV & IR behaviors

Bernardeau, Crocce & Scoccimarro ('12)

Regularized propagator

Bernardeau, Crocce & Scoccimarro ('12)

A global solution that satisfies both UV (k>>1) & IR (k<<1) properties:

$$\Gamma^{(n)}_{\rm reg} = \left[\Gamma^{(n)}_{\rm tree}\left\{1+\underbrace{\frac{k^2\sigma_{\rm v}^2}{2}}_{\text{counter term}}\right\}+\Gamma^{(n)}_{\rm 1\text{-}loop}\right]\exp\left\{-\frac{k^2\sigma_{\rm v}^2}{2}\right\}; \qquad \sigma_{\rm v}^2=\int\frac{dq}{6\pi^2}P_{\theta\theta}(q)$$

.... *IR behavior is valid at 1-loop level*

Precision of IR behavior can be systematically improved by including higher-loop corrections and adding counter terms

e.g., For IR behavior valid at 2-loop level,

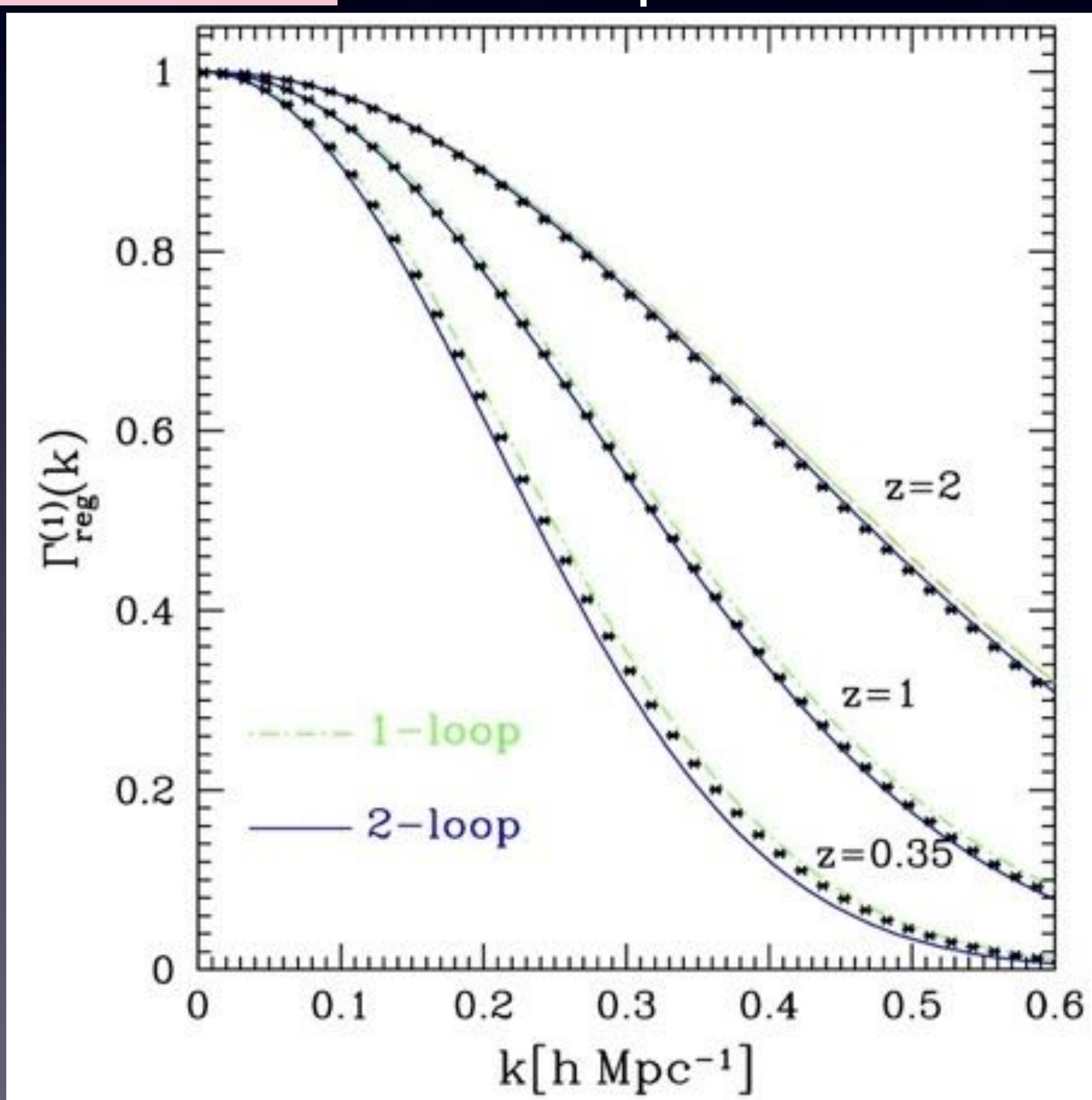
$$\Gamma^{(n)}_{\rm reg} = \left[\Gamma^{(n)}_{\rm tree}\left\{1+\underbrace{\frac{k^2\sigma_{\rm v}^2}{2}+\frac{1}{2}\left(\frac{k^2\sigma_{\rm v}^2}{2}\right)^2}_{\text{counter term}}\right\}+\Gamma^{(n)}_{\rm 1\text{-}loop}\left\{1+\underbrace{\frac{k^2\sigma_{\rm v}^2}{2}}_{\text{counter term}}\right\}+\Gamma^{(n)}_{\rm 2\text{-}loop}\right]\exp\left\{-\frac{k^2\sigma_{\rm v}^2}{2}\right\}$$

Propagators in N-body simulations

compared with '*Regularized*' propagators constructed analytically

$\Gamma^{(1)}(k)$

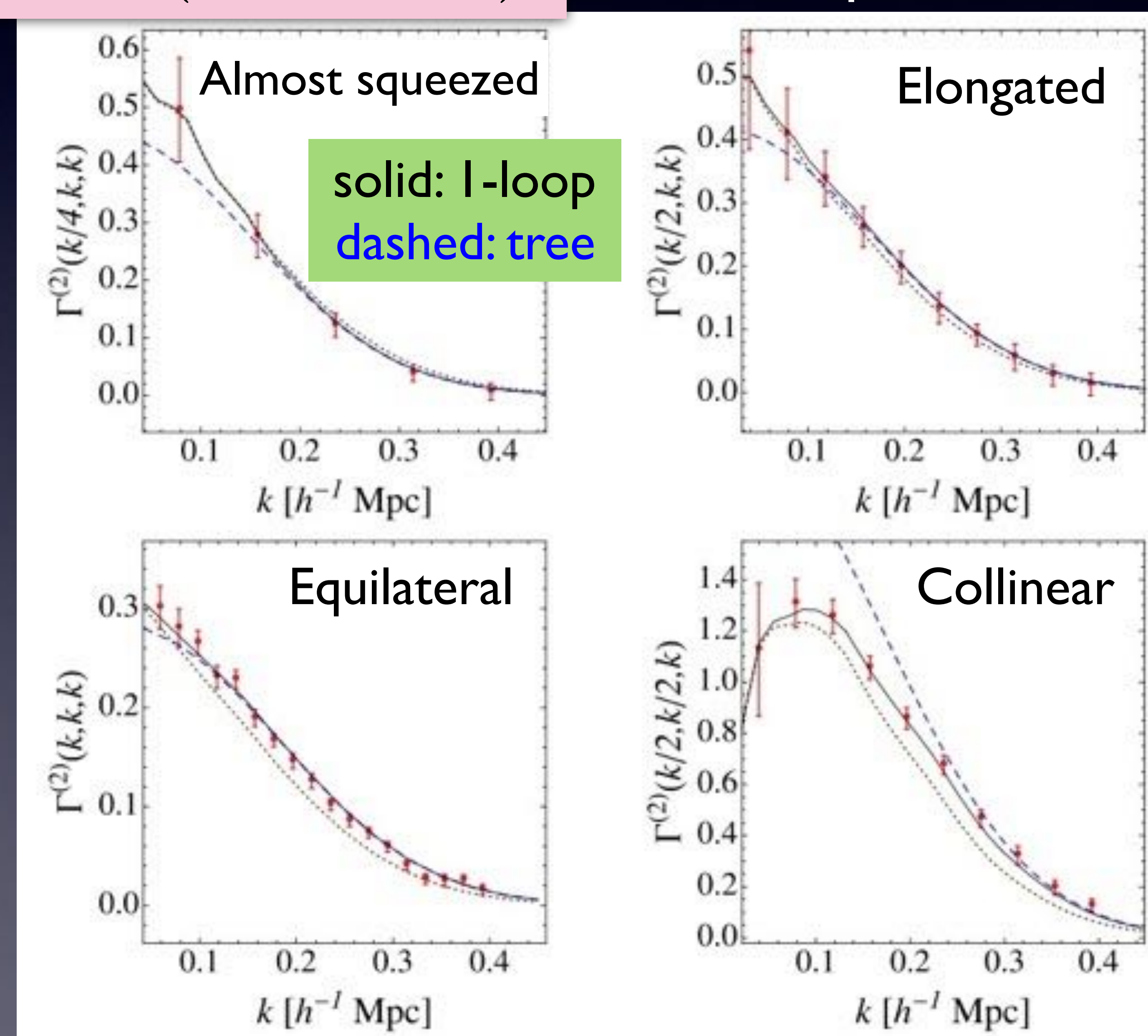
predictions up to 2-loop order



Bernardeau, AT & Nishimichi ('12)

$\Gamma^{(2)}(k_1, k_2, k_3)$

predictions up to 1-loop order



solid: 1-loop

dashed: tree

Bernardeau et al. ('12)

RegPT : fast PT code for P(k) & ξ(r)

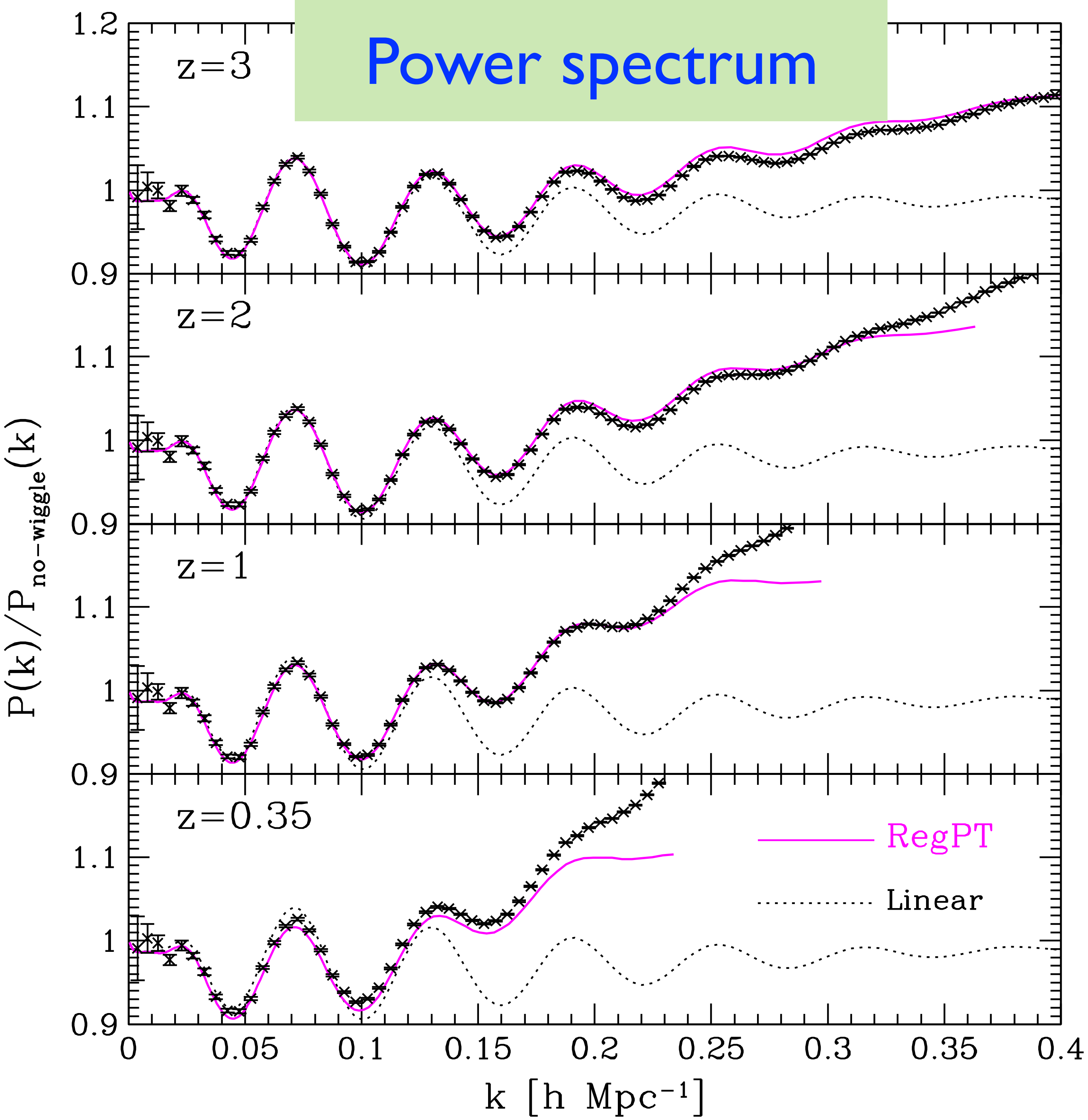
few sec.

(regularized)

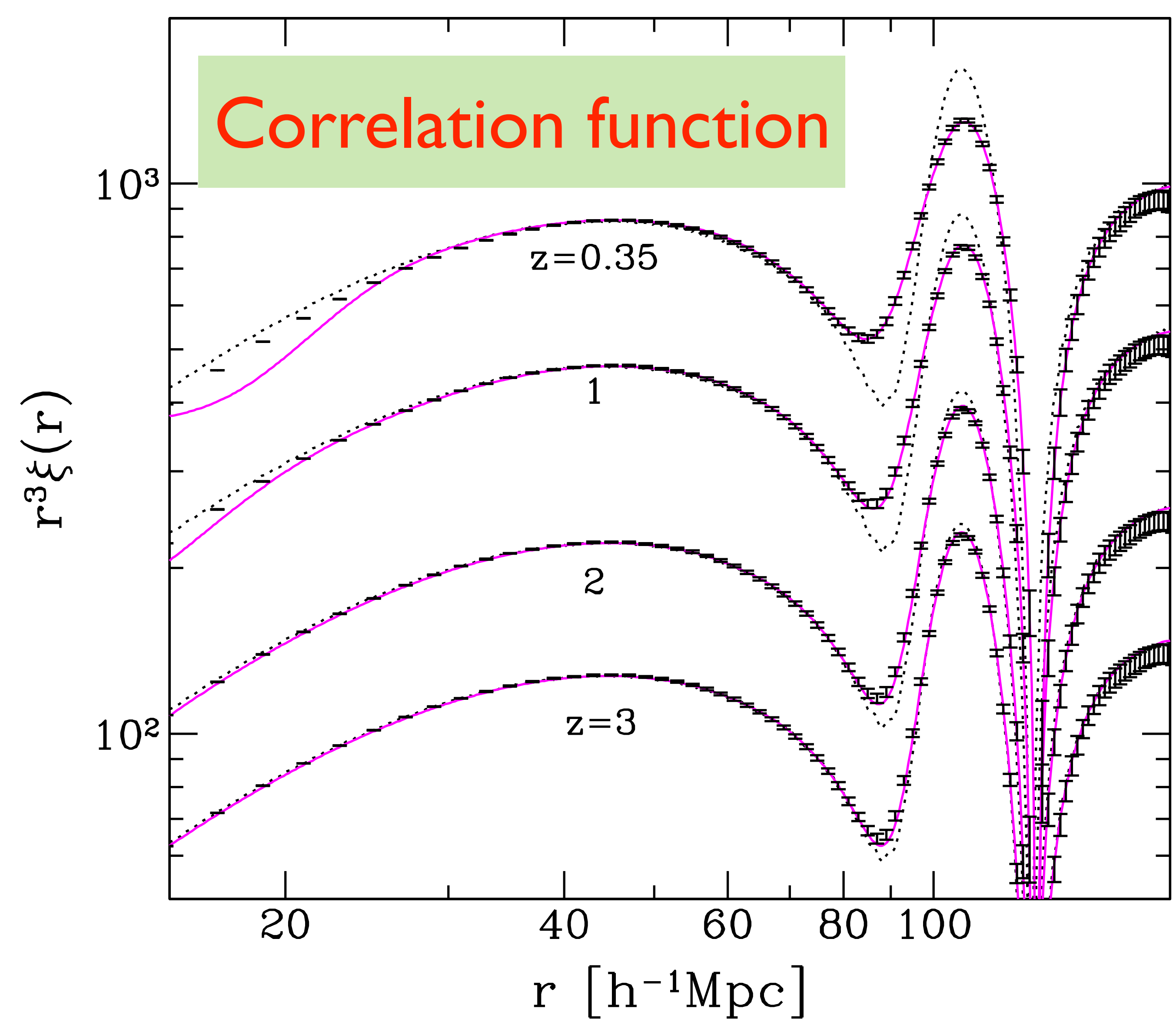
A public code based on multi-point propagators at *2-loop order*

http://www2.yukawa.kyoto-u.ac.jp/~atsushi.taruya/regpt_code.html

Power spectrum



Correlation function



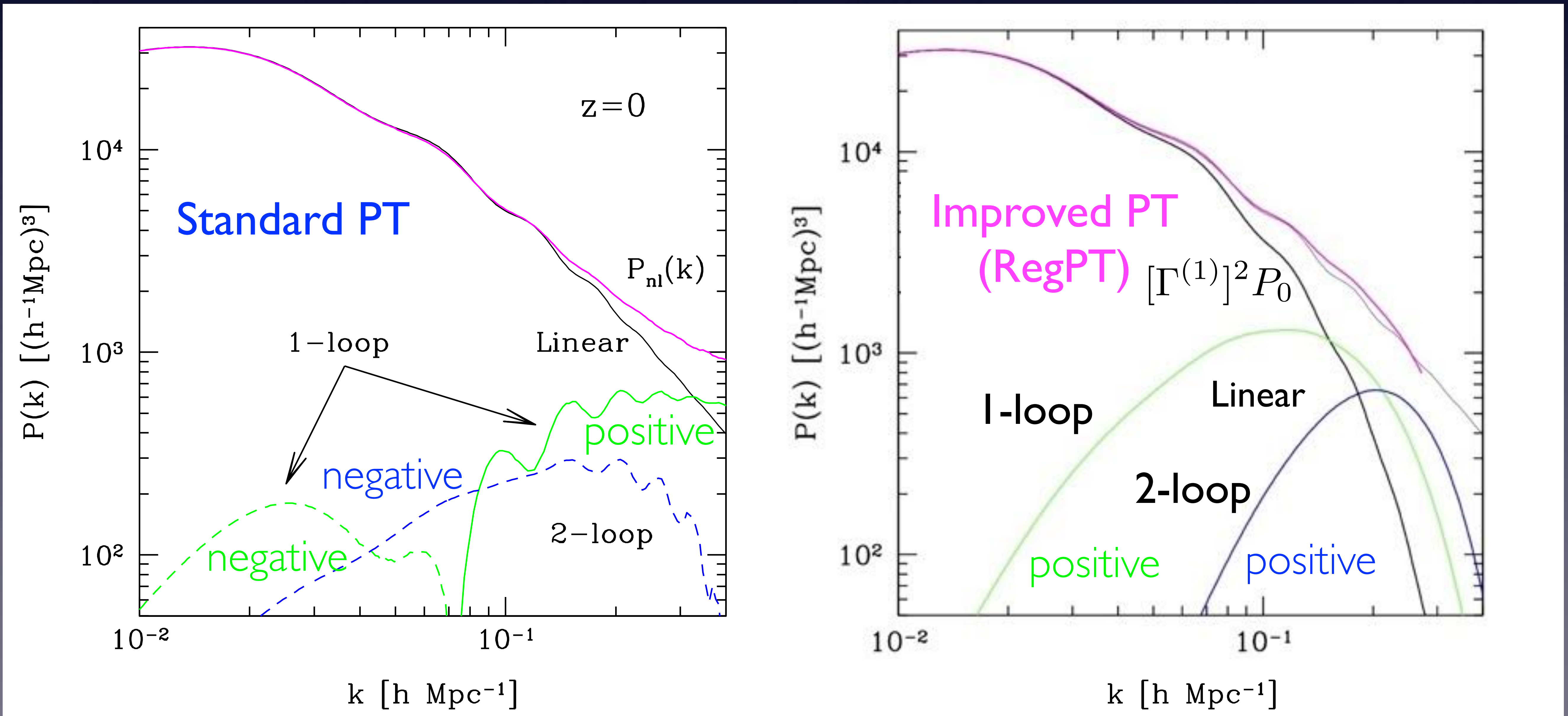
AT, Bernardeau, Nishimichi & Codis ('12)

Why improved PT works well?

AT, Bernardeau, Nishimichi, Codis ('12)
AT et al. ('09)

- All corrections become comparable at low-z.
- Positivity is not guaranteed.

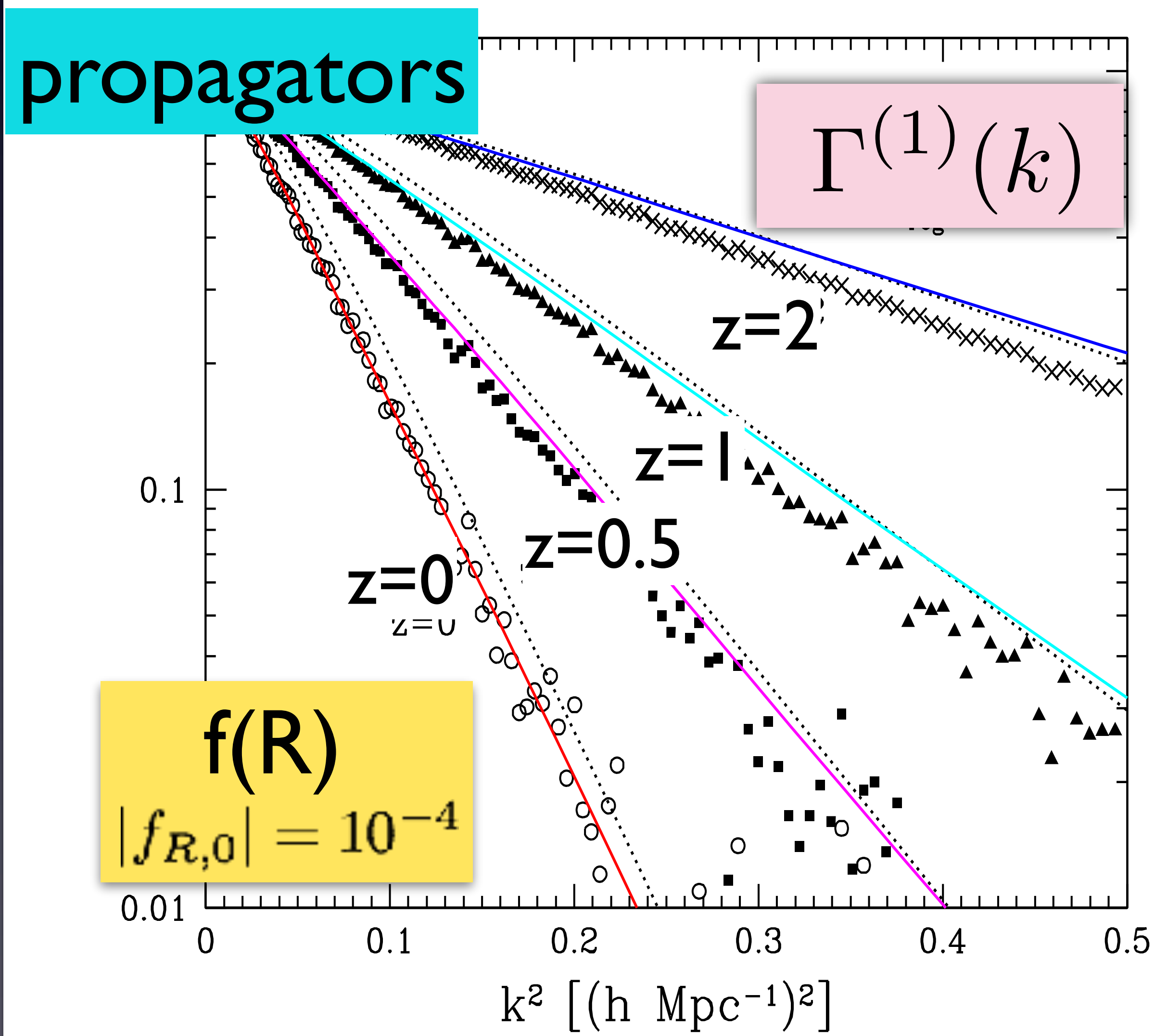
Corrections are positive & localized, shifted to higher-k for higher-loop

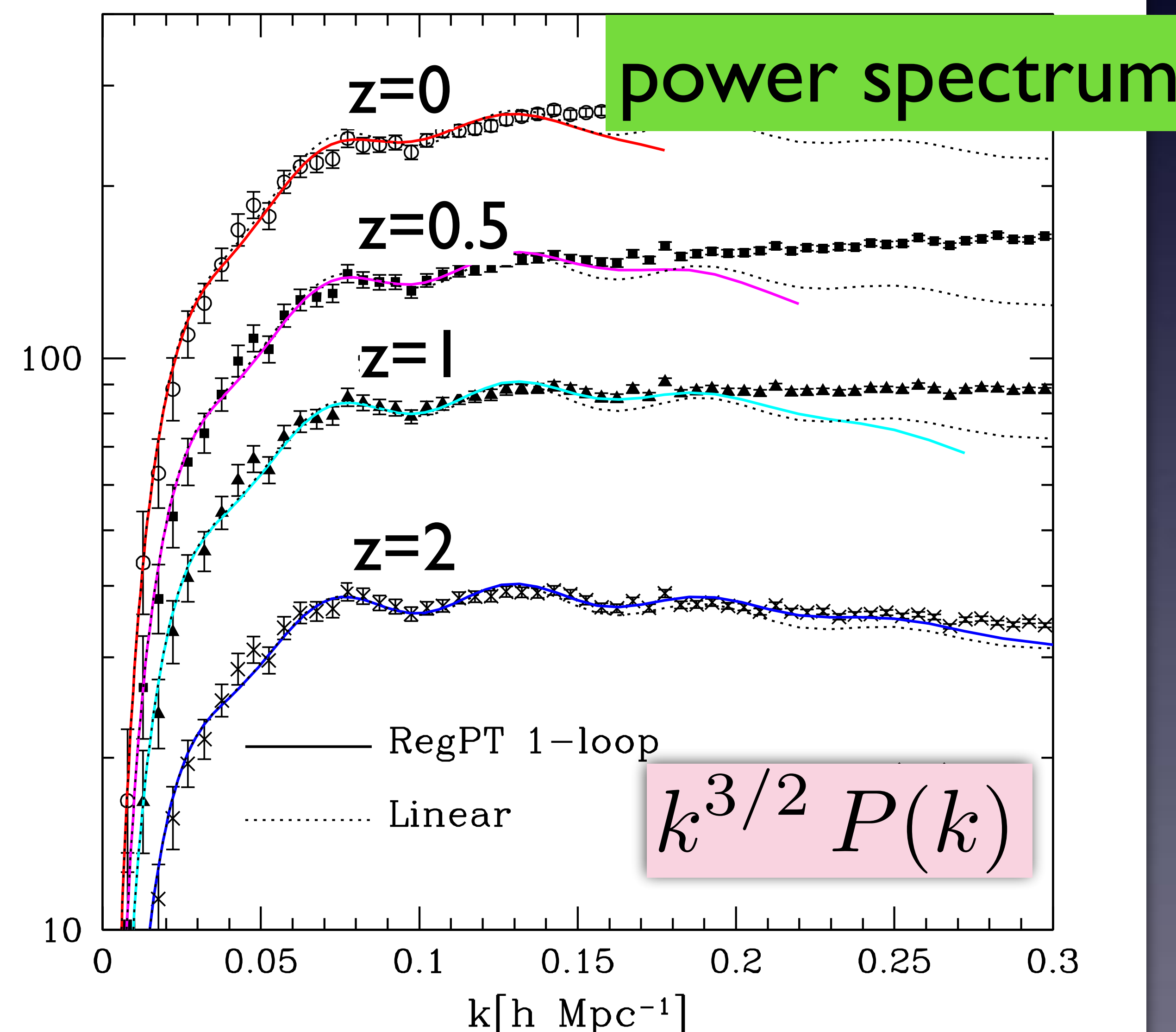


RegPT in modified gravity

Good convergence is ensured by
a *generic* damping behavior in propagators $\Gamma^{(n)} \xrightarrow{k\to\infty} \Gamma^{(n)}_{\rm tree}\, e^{-k^2\sigma_{\rm d}^2/2}$

Even in modified gravity, well-controlled expansion with RegPT





N-body data: Baojiu Li

AT, Nishimichi, Bernardeau,et al.('14)