Gravity in Noncommutative and/or Discrete Geometry

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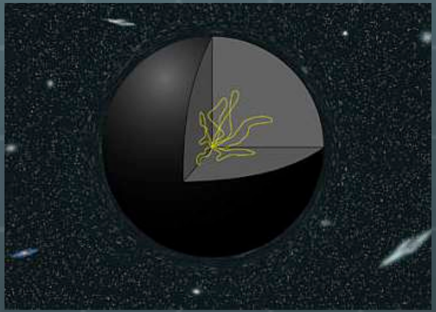
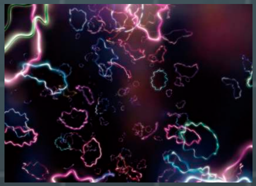
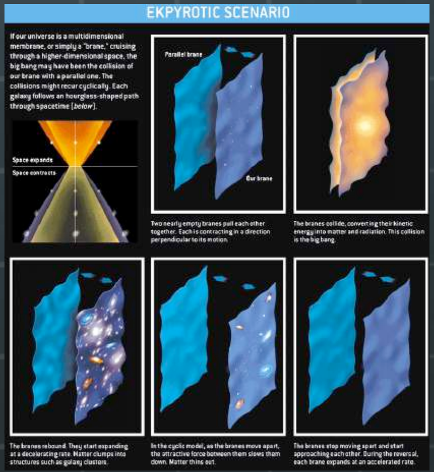
研究テーマ1：素粒子論・宇宙論
- ブラックホールの内部
- ビッグバン「以前」の宇宙
- 超弦理論に基づく宇宙論・ブラックホール物理



ブラックホールの特異点と弦



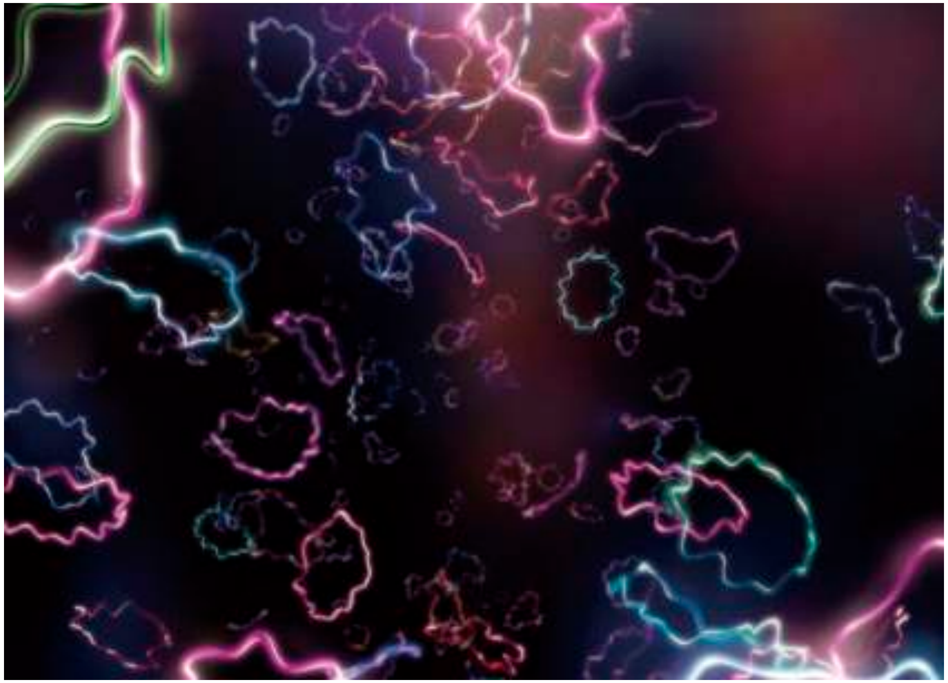
研究テーマ2：物理教育
- 身体感覚と理解のプロセス
  「難しいこと」を理解するには？
- 思春期における勉強への照れ・恥じをなくす教育
  アソビとマナビの接続，「勉強ってかっこいい？」
- 一般講座「世界を面白がるための物理」など



高次元宇宙とビッグバン

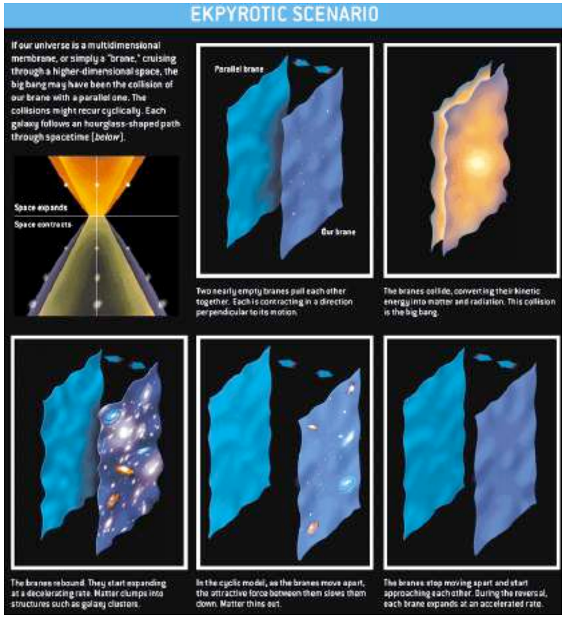
テーマ１： 超弦理論

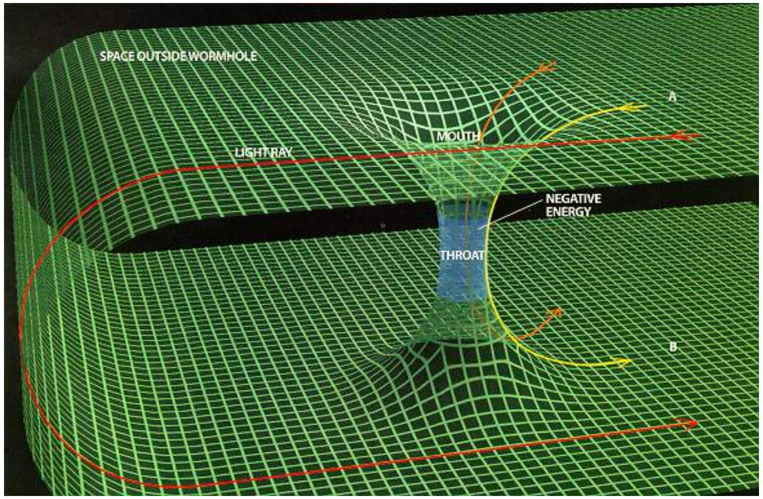
- 重力と物質を統一，量子重力の候補
- 「弦」という1次元物体の量子論



テーマ１： 超弦理論と宇宙論

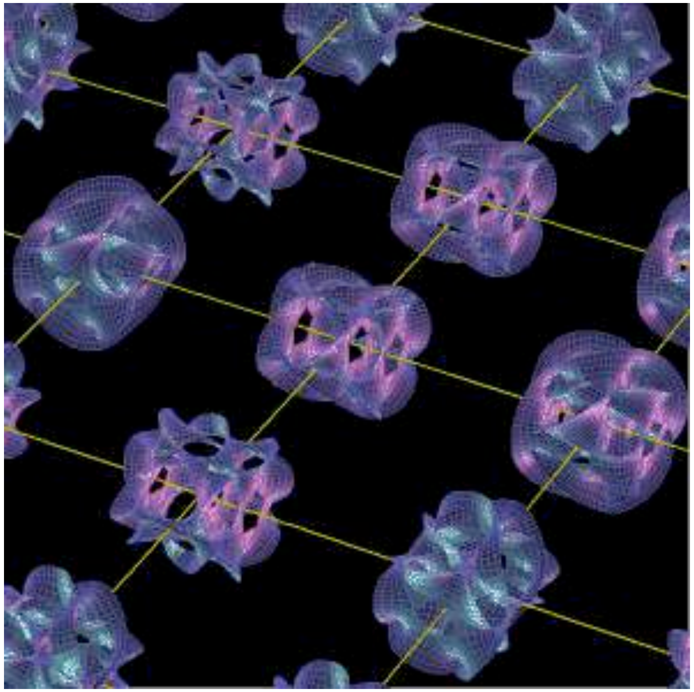
- 高次元時空の存在





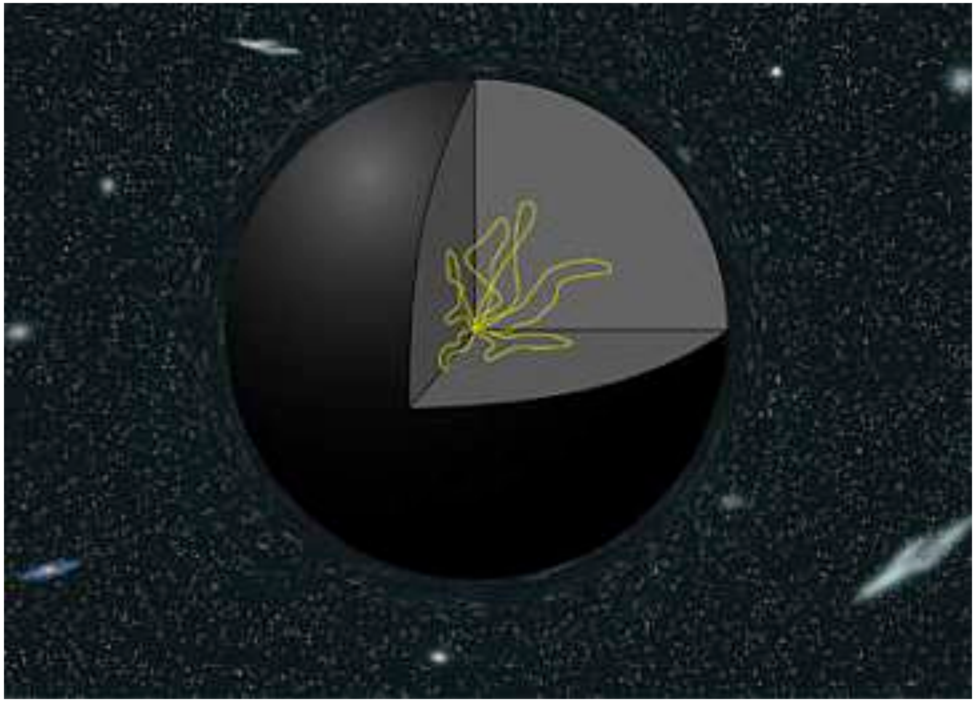
テーマ１： 超弦理論

- 時空間自体が弦からできている



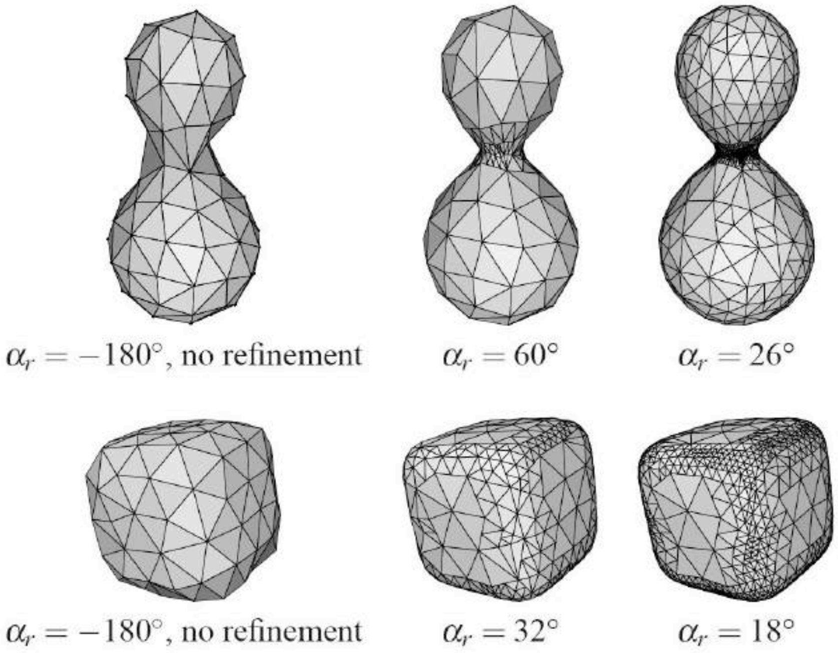
テーマ１： 超弦理論とBH

- ブラックホールの内部構造



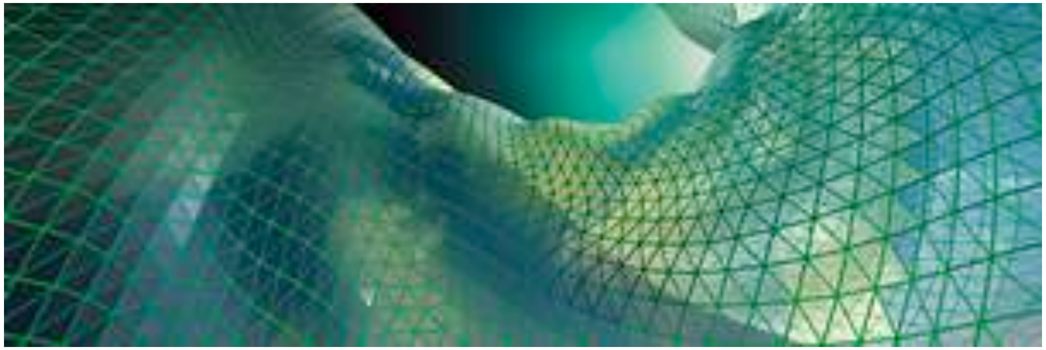
テーマ２：離散/非可換時空

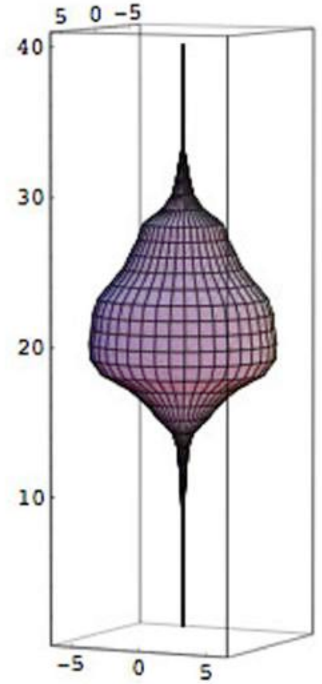
- Causal Dynamical Triangulation (CDT)
- 時空を単体（三角形や四面体）で分割



テーマ２：因果的動的単体分割

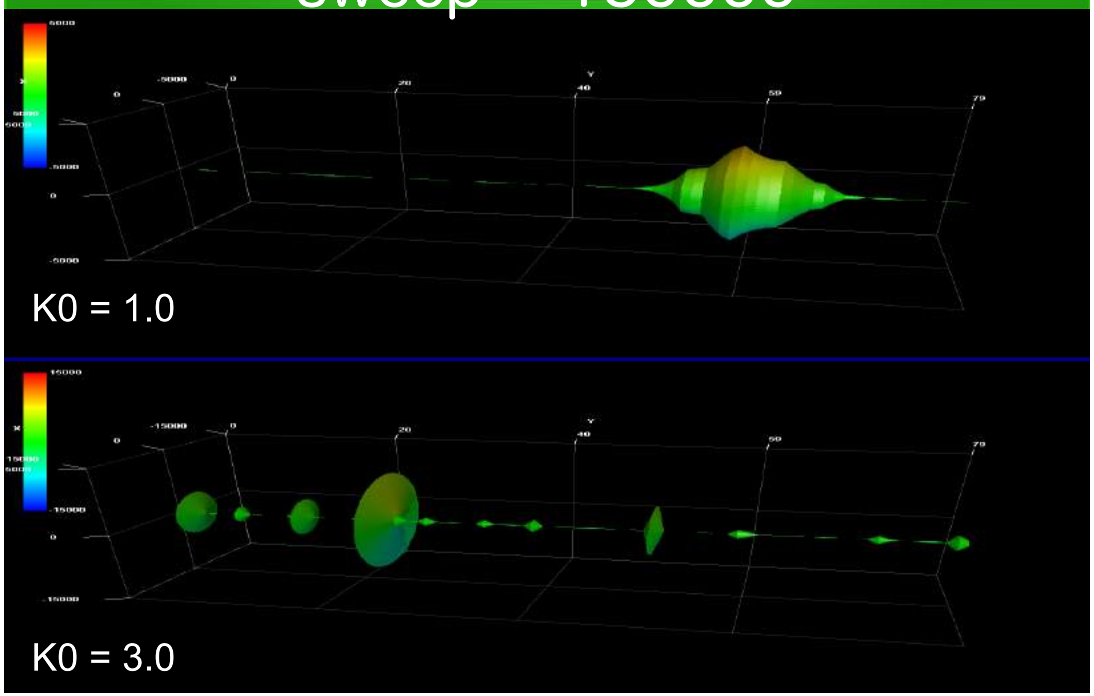
- どういった時空が創発しうるか数値的に計算
- 何故この世が4次元なのかを考える





テーマ２：因果的動的単体分割

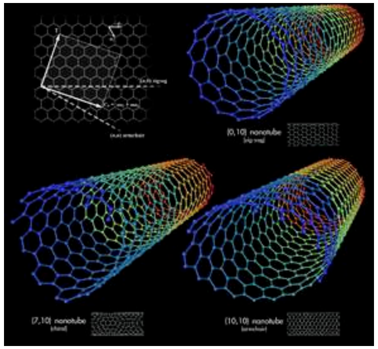
シミュレーションの結果
sweep = 150000

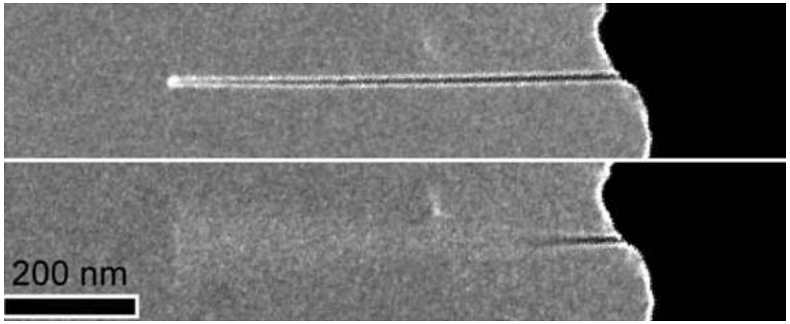


Misono-SK
（2012卒研）

テーマ３：より工学的なもの

- 1次元電子系と共形場の理論
  - カーボンナノチューブの電気特性
  - 応用：カーボンナノラジオ，トランジスタ

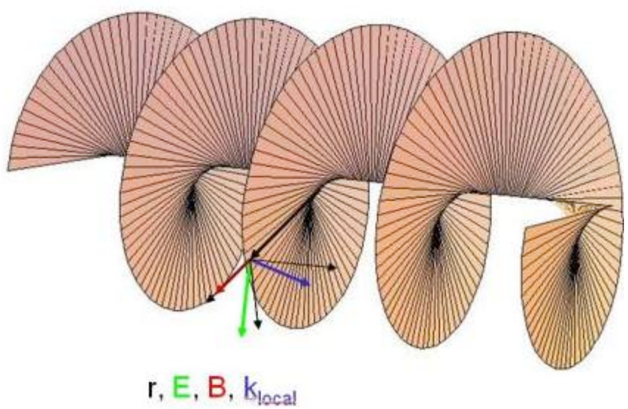




テーマ３：より工学的なもの

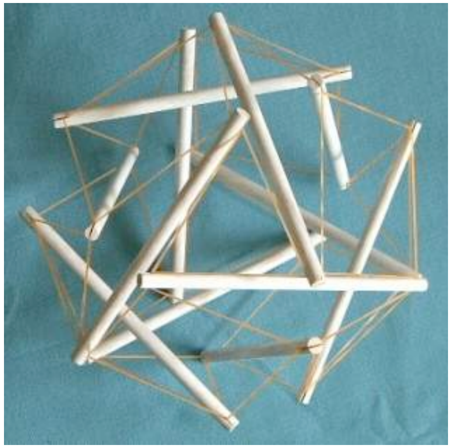
- Optical Vortex（光渦）とその応用
  - 軌道角運動量を持つ光
  - 系外惑星観測への応用
  - ドラッグデリバリーシステムへの応用

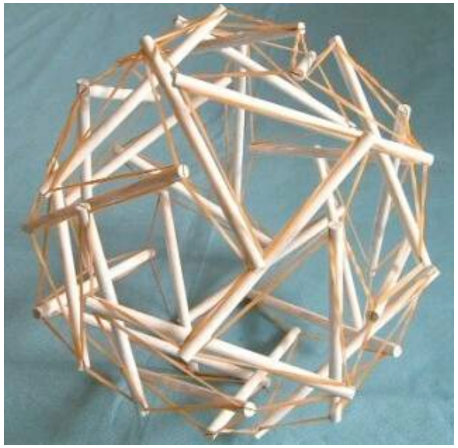
$$u(r, \varphi, z) = c(r)\ e^{ikz}\ e^{i\varphi}$$



テーマ３：より工学的なもの

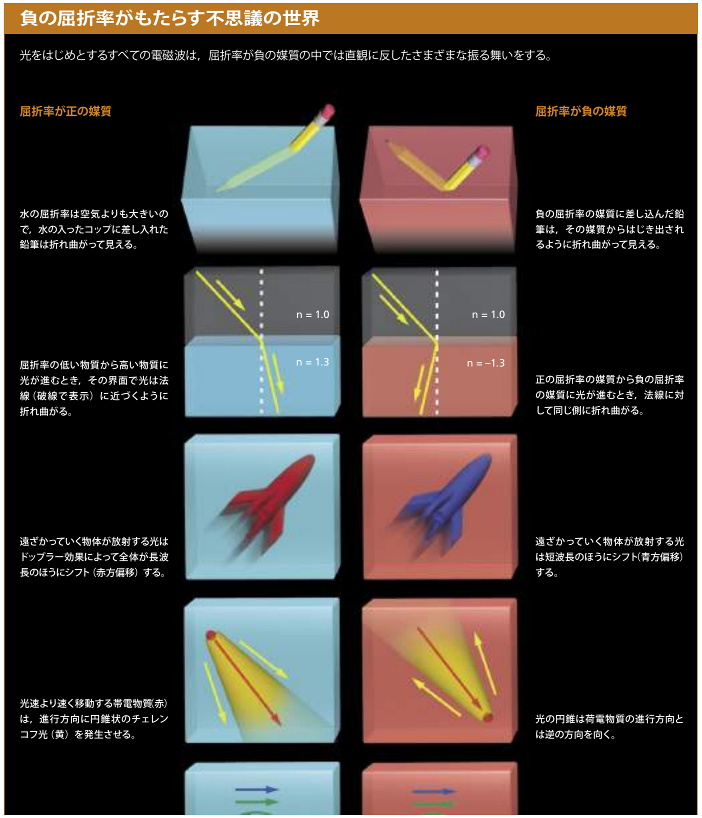
- テンセグリティの構造安定性
  - 群論を使う
  - 数値計算で安定な構造を探す





テーマ３：より工学的なもの

- 負の屈折率を持つメタマテリアル



$$n = \frac{1}{\sqrt{\varepsilon_r \mu_r}}$$

$$\varepsilon_r < 0, \mu_r < 0 \rightarrow n < 0$$

Repulsive gravity in BHT massive gravity

Geodesics and Repulsive Gravity in BHT Massive Gravity (tentative)

Keisuke Nakashi [1], Shinpei Kobayashi [2], Shu Ueda [3] and Hiromi Saida [4]

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4-1-1 Nukuikitamachi, Koganei, Tokyo 184-8501, JAPAN

[4] *Department of Physics, Daido University,*
Minami-ku, Nagoya, 457-8530, JAPAN

coming soon!

that our conclusion and the future discussion.

2 BHT massive gravity and static circularly symmetry black hole solution

The action of BHT massive gravity is given by [1]

$$S = \frac{1}{16\pi G}\int d^3x\sqrt{-g}\left(R - 2\lambda - \frac{1}{m^2}K\right), \tag{2.1}$$

where K is

$$K = R_{\mu\nu}R^{\mu\nu} - \frac{3}{8}R^2. \tag{2.2}$$

The source-free field equation can be read

$$G_{\mu\nu} + \lambda g_{\mu\nu} - \frac{1}{2m^2}K_{\mu\nu} = 0, \tag{2.3}$$

where

$$K_{\mu\nu} = 2\Box R_{\mu\nu} - \frac{1}{2}(\nabla_\mu\nabla_\nu R + g_{\mu\nu}\Box R) - 8R_{\mu\rho}R^{\rho}_{\nu} + \frac{9}{2}RR_{\mu\nu} + g_{\mu\nu}\left(3R^{\alpha\beta}R_{\alpha\beta} - \frac{13}{8}R^2\right). \tag{2.4}$$

When a spacetime has a constant curvatu

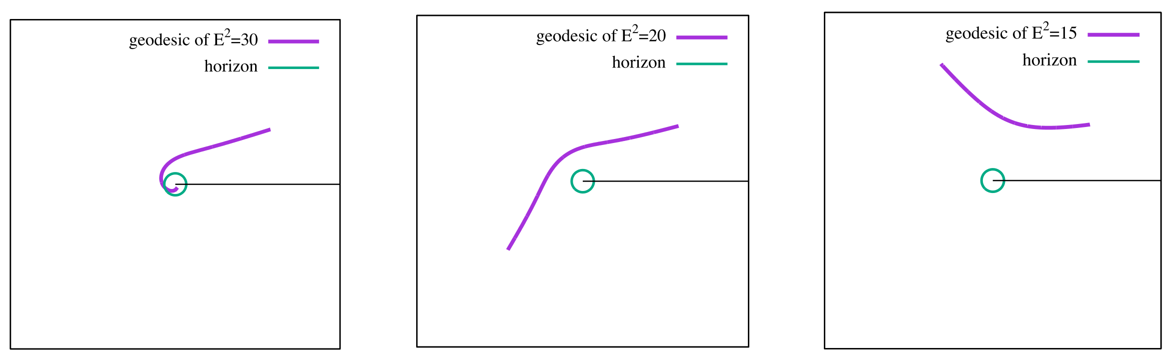
is also simplified as $K_{\mu\nu} = -\frac{1}{2}\Lambda g_{\mu\nu}$ [1, 5].

with two different radii, determined by

$$\Lambda_\pm = 2m($$

from Eq.(2.3). A special case defined as

$$m^2 =$$



region I region III_1 region III_2

Figure 4: Behaviors of the geodesics for the parameter region I and III.

Motivation

Quantum Gravity?

Quantum Geometry?

Natural? Criterion?

Candidates of quantum gravity

- string theory
  - string field theory
  - matrix models
  - ...
- loop quantum gravity
- (causal) dynamical triangulation
- noncommutative geometry
- …

Old history

PHYSICAL REVIEW VOLUME 71, NUMBER 1 JANUARY 1, 1947

Quantized Space-Time

HARTLAND S. SNYDER
Department of Physics, Northwestern University, Evanston, Illinois
(Received May 13, 1946)

It is usually assumed that space-time is a continuum. This assumption is not required by Lorentz invariance. In this paper we give an example of a Lorentz invariant discrete space-time.

THE problem of the interaction of matter and fields has not been satisfactorily solved to this date. The root of the trouble in present field theories seems to lie in the assumption of point interactions between matter and fields. On the other hand, no relativistically invariant Hamiltonian theory is known for any form of interaction other than point interactions.

Even for the case of point interactions the relativistic invariance is of a formal nature only, as the equations for quantized interacting fields have no solutions. The uses of source functions, or of a cut-off in momentum space to replace infinity by a finite number are distasteful arbitrary procedures, and neither process has yet been formulated in a relativistically invariant manner. It may not be possible to do this.

It is possible that the usual four-dimensional continuous space-time does not provide a suitable framework within which interacting matter and fields can be described. I have chosen the idea that a modification of the ordinary concept of space-time may be necessary because the "elementary" particles have fixed masses and associated Compton wave-lengths.

The special theory of relativity may be based on the invariance of the indefinite quadratic form

$$S^2 = c^2t^2 - x^2 - y^2 - z^2, \qquad (1)$$

Old history

that the usual assumptions concerning the continuous nature of space-time are not necessary for Lorentz invariance. This result is the minimum objective of this work.

The ten operators defined in (3) and (4) have a total of forty-five commutators. Only six of these commutators differ from the ordinary ones and these six are

$$
\begin{aligned}
[x, y] &= (ia^2/\hbar)L_z, \quad [t, x] = (ia^2/\hbar c)M_x, \\
[y, z] &= (ia^2/\hbar)L_x, \quad [t, y] = (ia^2/\hbar c)M_y, \\
[z, x] &= (ia^2/\hbar)L_y, \quad [t, z] = (ia^2/\hbar c)M_z.
\end{aligned}
\qquad (5)
$$

We see from these commutators that if we take the limit $a \to 0$ keeping $\hbar$ and c fixed, our quantized space-time changes to the ordinary continuous space-time.

commutators with L_x, L_y, L_z, M_x, M_y, M_z as do the usual expressions for the space or time displacement operators. In addition, each has a continuous spectrum running from minus infinity to plus infinity. Their commutators with the coordinates and time are not the same as usual and are given by

$$
\begin{aligned}
&[x, p_x] = i\hbar[1 + (a/\hbar)^2 p_x^2]; \\
&[t, p_t] = i\hbar[1 - (a/\hbar c)^2 p_t^2]; \\
&[x, p_y] = [y, p_x] = i\hbar(a/\hbar)^2 p_x p_y; \\
&[x, p_t] = c^2[p_x, t] = i\hbar(a/\hbar)^2 p_x p_t; \text{ etc.}
\end{aligned}
\qquad (8)
$$

Here we note that if all the components of the momentum are small compared to $\hbar/a$ and the energy is small compared to $\hbar c/a$ then these commutators approach those which are given in orinary quantum mechanics. Further, as we take

deformation of spacetime → deformation of algebraic structure of functions on it

A realization of noncommutativity

- Noncommutativity between space coordinates

$$[x, y] = i\theta, \qquad \theta \quad \text{: constant parameter}$$

$$\longrightarrow [z, \bar{z}] = 1 \qquad \left(z = \frac{x+iy}{\sqrt{2\theta}},\ \bar{z} = \frac{x-iy}{\sqrt{2\theta}} \right)$$

- a realization: Wick-Voros product

$$(f \star g)(z, \bar{z}) = \exp\left(\frac{\partial}{\partial \bar{z}'} \frac{\partial}{\partial z''} \right) f(z', \bar{z}') g(z'', \bar{z}'') \Big|_{z'=z''=z}$$

$$\Downarrow$$

$$[z, \bar{z}]_\star = z \star \bar{z} - \bar{z} \star z = 1 \qquad \text{cf.)} \quad [a, a^\dagger] = 1$$

A field theory on noncommutative space

on a commutative space:

$$S = \int dt d^2x \left(\partial_z \phi \partial_{\bar{z}} \phi + \frac{m}{2} \phi^2 + \cdots \right)$$

$$\Downarrow$$

on a noncommutative space:

$$S = \int dt d^2x \left(\partial_z \phi \star \partial_{\bar{z}} \phi + \frac{m}{2} \phi \star \phi + \cdots \right)$$

algebraic structure of functions is changed
by introducing noncommutativity

Applications to gravity

- Gauge theories of gravitation
  [Chamseddine, Chaichian-Setare-Tureanu-Zet, Banados-Chandia-Grandi-Schaposnik-Silva, ...]
  - (2+1)dim CS theory $\sim$ (2+1)dim gravity $\rightarrow$ NC version

$$d\mathcal{A} + \mathcal{A}\cdot\mathcal{A} = 0 \quad \rightarrow \quad d\mathcal{A} + \mathcal{A}\star\mathcal{A} = 0 \quad \rightarrow \quad \mathcal{L} = e \star R$$

- Twisted diffeomorphism [Aschieri-Dimitrijevic-Meyer-Wess]

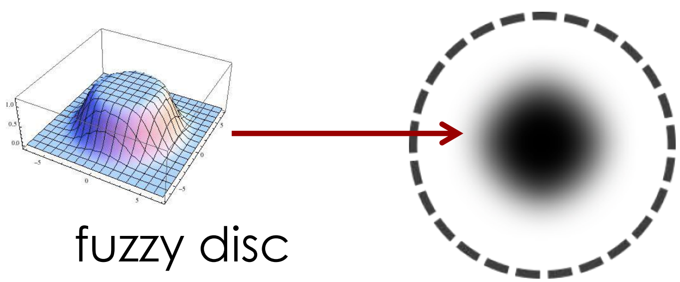
$$\delta_\xi^\star V_\mu = -\xi^\rho \star (\partial_\rho V_\mu) - (\partial_\mu \xi^\rho) \star V_\rho, \quad \text{and so on}$$

- Deformation in source terms [Nicolini, Banerjee, Mukherjee, Rahaman, SK, ...]

$$\delta(r) \rightarrow e^{-r^2/\theta} \quad (\text{width} \sim \theta)$$

Fuzzy object as BH source

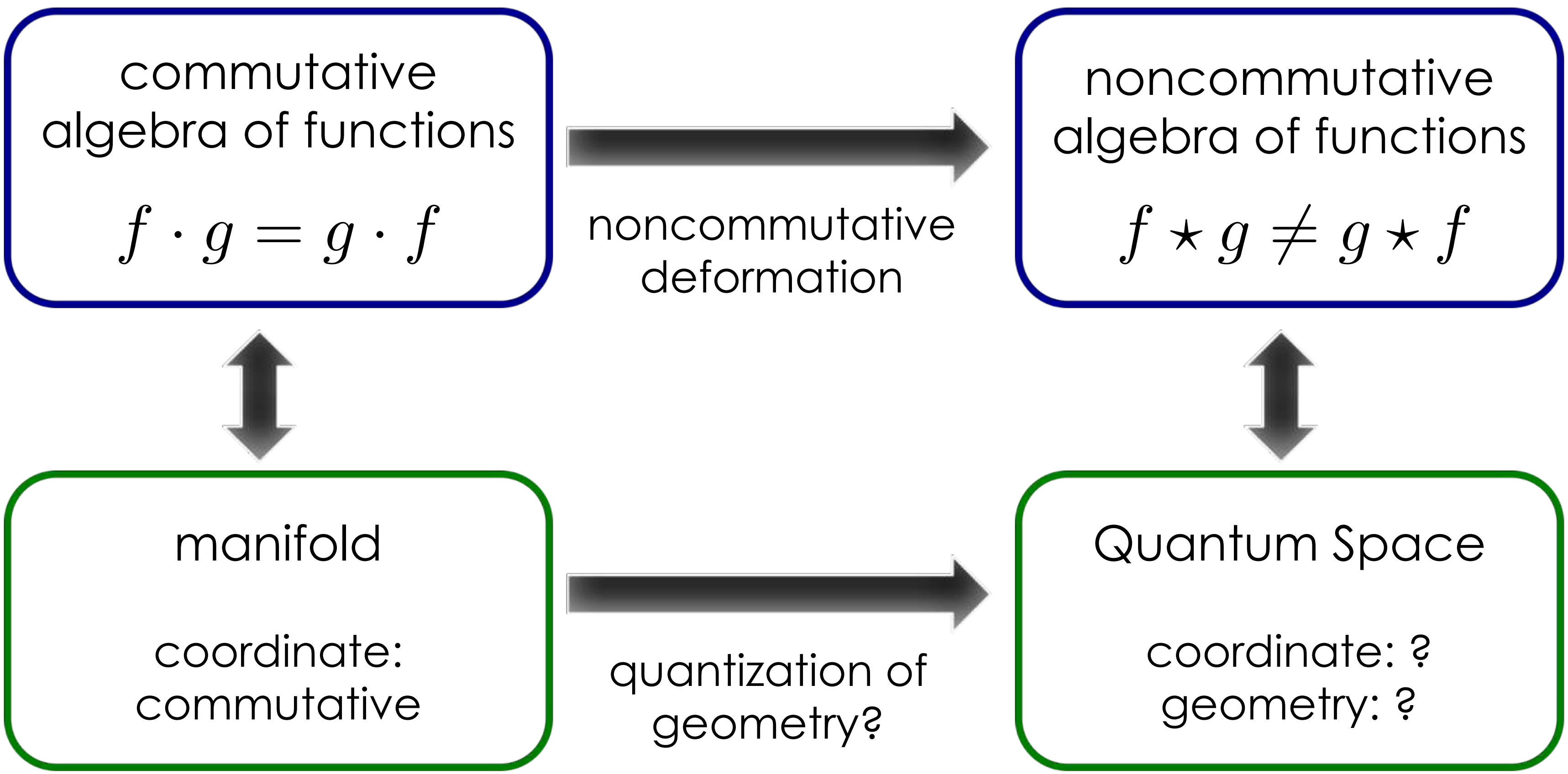
- point mass → Gaussian distribution: $\delta(r) \quad \rightarrow \quad e^{-r^2/2\theta}$
- BH solution with a smeared point mass as a source
  - Schwarzschild-like [Nicolini et al (2006)], RN-like [Ansoldi et al (2007)],
  - Kerr-like [Smailagic et al (2007)], (1+1)-dim. [Mureika et al (2011)]
  - BTZ-like [Rahaman et al (2011), Larranaga et al. (2011), SK (2016)], ...
- BH solution with fuzzy disc or fuzzy annulus as source [SK (2016)]



fuzzy disc

- ordinary gravitational solutions with NC inspired sources

Round way: space & functions on it



Example: Fuzzy Sphere

commutative algebra of spherical harmonics

$$f(\theta,\phi) = \sum a_{lm} Y_{lm}(\theta,\phi)$$

$$f(\theta,\phi)\cdot g(\theta,\phi) = g(\theta,\phi)\cdot f(\theta,\phi)$$

noncommutative deformation

noncommutative algebra of deformed spherical harmonics

$$f(\theta,\phi) = \sum a_{lm} \hat{Y}_{lm}(\theta,\phi)$$

$$f(\theta,\phi)\star g(\theta,\phi) \neq g(\theta,\phi)\star f(\theta,\phi)$$

Two-sphere S^2

a quantization of geometry

Fuzzy Sphere

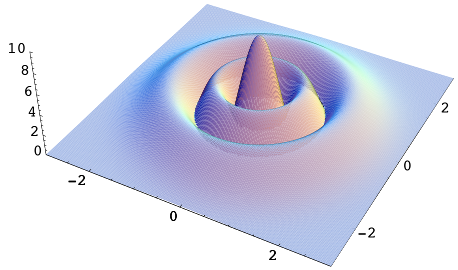
Noncommutative Solitons

- a scalar field theory on NC plane
  [Gopakumar- Minwalla- Strominger, Kraus-Larsen, … (2000)]

$$E = \int_D d^2 z V_\star(\Phi) \qquad \leftarrow \text{no kinetic term}$$

$$V_\star(\Phi) = \frac{b_2}{2}\Phi \star \Phi + \frac{b_3}{3}\Phi \star \Phi \star \Phi + \cdots$$

- nontrivial soliton solutions do exist: GMS solitons



Construction of GMS solitons

- EOM: $0 = \frac{\partial V_\star}{\partial \Phi} = b_2 \Phi + b_3 \Phi \star \Phi + b_4 \Phi \star \Phi \star \Phi + \cdots$

  usually, there is no non-trivial solution (lack of kinetic term)

- if there is $\Phi = \lambda p_n(z, \bar{z})$, where p_n satisfies $p_n \star p_n = p_n$

  EOM $\longrightarrow$ $0 = (b_2 \lambda + b_3 \lambda^2 + b_4 \lambda^3 + \cdots) p_n$

  projection operator-like

- nontrivial solution: $\Phi = \lambda_* p_n(z, \bar{z})$

  λ_* is a solution of the algebraic equation

$$b_2 \lambda + b_3 \lambda^2 + b_4 \lambda^3 + \cdots = 0$$

Functions and Operators: Weyl-Wigner Correspondence

algebra of functions		algebra of operators
function: $f(x,y)$	$\Leftrightarrow$	operator: $\hat{f}(\hat{x},\hat{y})$
star product: $f \star g$		product with an ordering: $\hat{f}\cdot\hat{g}$

$$[x,y] = i\theta$$

$$[\hat{x},\hat{y}] = i\theta \qquad [\hat{z},\hat{z}^{\dagger}] = 1$$

operators acting on
the Fock space
of a harmonic oscillator

Weyl-Wigner Correspondence

"Harmonic oscillator" in operator formalism:

$$\hat{N}\,|\,n\,\rangle = n\,|\,n\,\rangle$$

$$\hat{z} = \frac{\hat{x} + i\hat{y}}{\sqrt{2\theta}} \qquad \hat{z}^\dagger = \frac{\hat{x} - i\hat{y}}{\sqrt{2\theta}} \qquad \hat{N} = \hat{z}^\dagger \hat{z} = \frac{\hat{x}^2 + \hat{y}^2}{2\theta}$$

$$\mathcal{H} = \mathrm{span}\{|0\rangle\,, |1\rangle\,, |2\rangle\,, \cdots\}$$

Weyl-Wigner Correspondence

function $$f(\bar{z}, z) = \sum_{n=0}^{\infty} f_{mn}^{\mathrm{Tay}} \bar{z}^m z^n$$

Weyl projection

operator $$\hat{f}(\hat{z}^\dagger, \hat{z}) = \sum_{n=0}^{\infty} f_{mn}^{\mathrm{Tay}} \hat{z}^{\dagger m} \hat{z}^n$$

Weyl-Wigner Correspondence

function $f(\bar{z}, z) = \langle\, z \,|\, \hat{f} \,|\, z \,\rangle$

inverse Weyl projection

$\hat{z} \,|\, z \,\rangle = z \,|\, z \,\rangle$: coherent state

operator $\hat{f}(\hat{z}^\dagger, \hat{z}) = \sum_{n=0}^{\infty} f_{mn}^{\mathrm{Tay}} \hat{z}^{\dagger m} \hat{z}^n$

Projection operators as NC solitons

- EOM: $0 = \frac{\partial V_\star}{\partial \Phi} = b_2 \Phi + b_3 \Phi \star \Phi + b_4 \Phi \star \Phi \star \Phi + \cdots$
- nontrivial solution: $\Phi = \lambda_* p_n(z, \bar{z})$ with $p_n \star p_n = p_n$

analogy to projection operator

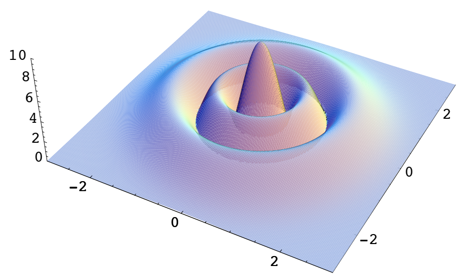
$$\hat{p}_n = |\, n \,\rangle \langle\, n \,|$$

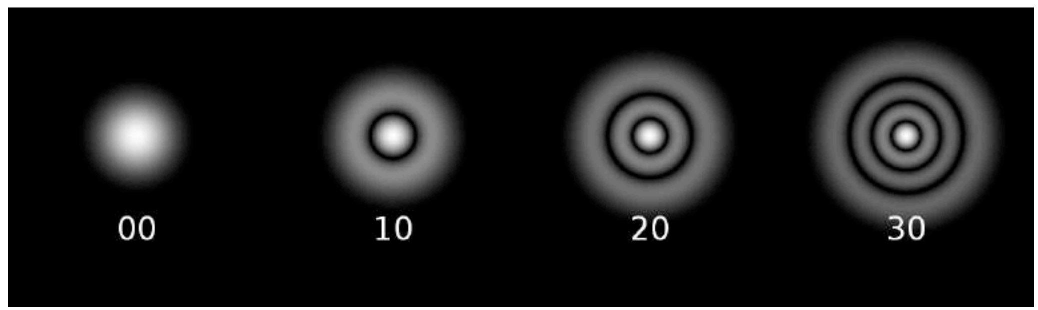
W-W correspondence

$$p_n(r) = \langle\, z \,|\, n \,\rangle \langle\, n \,|\, z \,\rangle = e^{-\frac{r^2}{2\theta}} \frac{r^{2n}}{n!(2\theta)^n}$$

GMS solitons

- circular symmetric solitons





Laguerre-Gausssian function

$$u(r,\phi,z)=\frac{C_{lp}^{LG}}{w(z)}\left(\frac{r\sqrt{2}}{w(z)}\right)^{|l|}\exp\left(-\frac{r^2}{w^2(z)}\right)L_p^{|l|}\left(\frac{2r^2}{w^2(z)}\right)\exp\left(ik\frac{r^2}{2R(z)}\right)\exp(il\phi)\exp\left[-i(2p+|l|+1)\zeta(z)\right],$$

Fuzzy Disc and angle state

a finite disc in the Moyal plane
angular noncommutative solitons

SK-Asakawa, JHEP04(2013)145

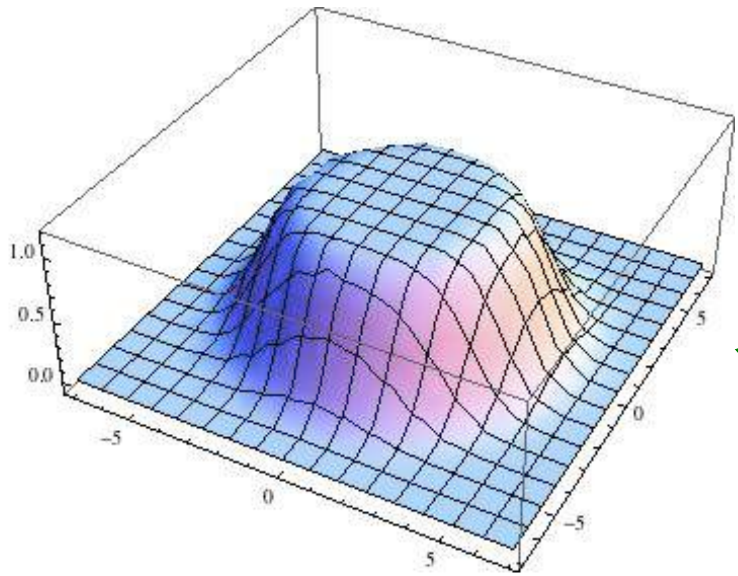
Fuzzy Disc [Lizzi, Vitale, Zampini (2003)]

- Def: finite dim. truncation of a noncommutative plane

$$\mathcal{H}_N = \mathrm{span}\{|\,0\,\rangle\,, |\,1\,\rangle\,, |\,2\,\rangle\,, \cdots\,, |\,N-1\,\rangle\}$$

- Two parameters:
  - noncommutativity: θ
  - fuzzyness : N
- applications:
  - matrix model
  - quantum Hall effect

Shape of a fuzzy disc (N=10, θ =1)



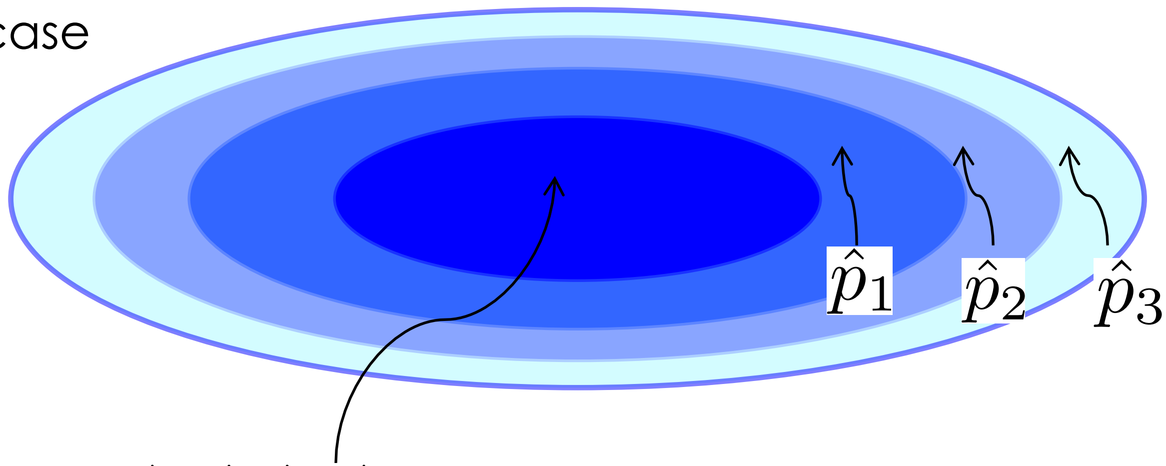
$$\hat{\mathcal{A}}_{10} = \hat{P}_{10}\hat{\mathcal{A}}\hat{P}_{10}$$

$$\hat{P}_{10} = \hat{p}_0 + \hat{p}_1 + \cdots + \hat{p}_9$$

$$p_0 + p_1 + \cdots + p_9 = \sum_{n=0}^{9} e^{-\frac{r^2}{2\theta}} \frac{r^{2n}}{n!(2\theta)^n}$$

GMS soliton and fuzzy disc

N=4 case

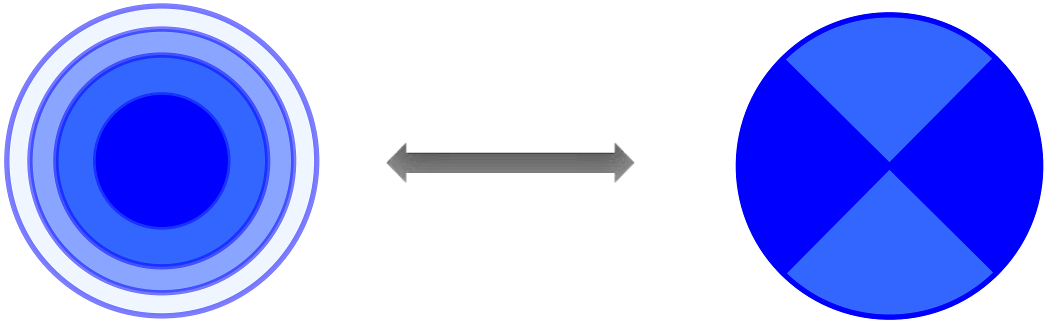


$$\hat{p}_0 = |\,0\,\rangle\,\langle\,0\,|$$

Another orthonormal basis : angle states

[SK-Asakawa, JHEP04(2013)145]

- number basis: concentric cutting of a disc
  - $\hat{N}$ ~ radius operator $(\hat{N} \sim \sqrt{\hat{x}^2 + \hat{y}^2})$
- another basis: radial cutting of disc
  - $\hat{\varphi}$ ~ angle operator



- nontrivial solutions with more general forms
- introduction of the polar coordinate in NC gemoetry

Angle Operator and States

- The angle operator: $\hat{\varphi} = \sum_{m=0}^{N-1} \varphi_m \,|\, \varphi_m \,\rangle \langle\, \varphi_m \,|$

with help of Pegg-Barnett phase operator

- Eigen states of the angle operator: $\hat{\varphi} \,|\, \varphi_m \,\rangle = \varphi_m \,|\, \varphi_m \,\rangle$
- Relation to the number state $|\, \varphi_m \,\rangle = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} e^{in\varphi_m} \,|\, n \,\rangle$
- Orthonormality: $\langle\, \varphi_m \,|\, \varphi_n \,\rangle = \delta_{mn}$
- Angular projection operators:

$$\hat{\pi}_m = |\, \varphi_m \,\rangle \langle\, \varphi_m \,|$$

$\rightarrow$ angular "delta function" peaked at $\varphi_m = \frac{2\pi}{N} m$

Number ⇄ Angle

$$|\varphi_m\rangle = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} e^{in\varphi_m} |n\rangle ,$$

$$\varphi_m = \frac{2\pi}{N} m \quad (m = 0, 1, \cdots, N-1)$$

$$\hat{\varphi} = \sum_{m=0}^{N-1} \varphi_m |\varphi_m\rangle \langle\varphi_m|$$

⟺

$$\hat{N} = \sum_{n=0}^{N-1} n |n\rangle \langle n|$$

Number ⇄ Angle (cont'd)

$$\hat{\varphi} = \left(\varphi_0 + \frac{(N-1)\pi}{N}\right) 1_N + \frac{2\pi}{N} \sum_{n \neq n'} \frac{e^{i(n'-n)\varphi_0}}{e^{2\pi i(n'-n)/N} - 1} |n'\rangle \langle n|$$

$$\hat{U} = \exp(i\hat{\varphi})$$

$$\hat{U} = |0\rangle \langle 1| + |1\rangle \langle 2| + \cdots + |N-2\rangle \langle N-1| + e^{iN\varphi_0} |N-1\rangle \langle 0|$$

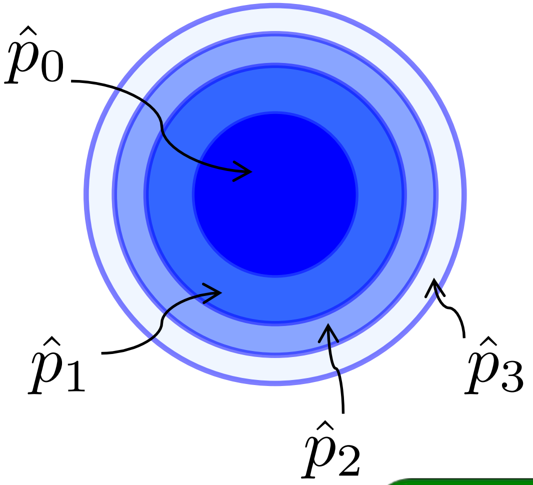
unitary, cyclic operator

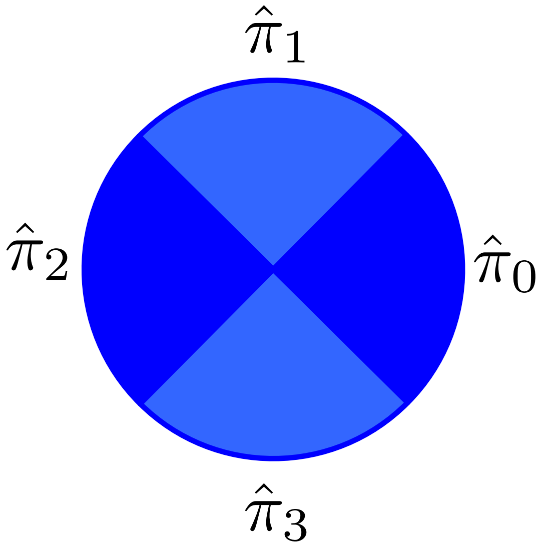
Pegg-Barnett formalism in quantum optics

Two descriptions for fuzzy disc

- baum-kuchen vs shortcake

$$N = 4,\ \mathcal{H}_4 = \{|\,0\,\rangle\,,|\,1\,\rangle\,,|\,2\,\rangle\,,|\,3\,\rangle\}$$





$$\sum_{n=0}^{N-1} \hat{p}_n = \sum_{m=0}^{N-1} \hat{\pi}_m$$

Algebraic definition of fuzzy disc

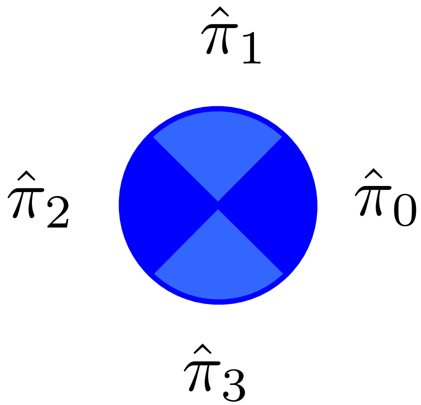
$$\hat{V} := e^{i\frac{2\pi}{N}\hat{N}} = \sum_{n=0}^{N-1} e^{i\frac{2\pi}{N}n} \left|n\right\rangle \left\langle n\right| = \sum_{m=0}^{N-1} \left|\varphi_{m+1}\right\rangle \left\langle \varphi_m\right|,$$

$$\hat{U} := e^{i\hat{\varphi}} = \sum_{m=0}^{N-1} e^{i\varphi_m} \left|\varphi_m\right\rangle \left\langle \varphi_m\right| = \sum_{n=0}^{N-1} \left|n-1\right\rangle \left\langle n\right|,$$

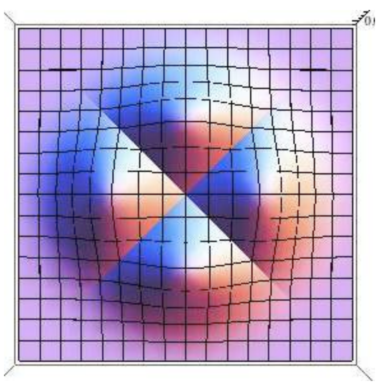
$$[\hat{N},\hat{U}] = -\hat{U} + N\hat{U}\hat{p}_0, \quad [\hat{\varphi},\hat{V}] = \frac{2\pi}{N}\hat{V}, \quad \hat{U}\hat{V} = e^{\frac{2\pi i}{N}}\hat{V}\hat{U}.$$

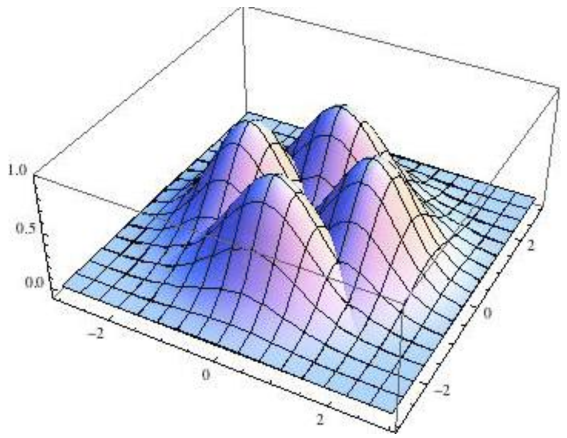
Function counterparts of angular projection operators

$$\pi_k^{(N)}(r,\varphi) = \frac{1}{N}\sum_{m,n=0}^{N-1} e^{-\frac{r^2}{2\theta}} \frac{r^{m+n}}{\sqrt{m!n!(2\theta)^{m+n}}} e^{-i(m-n)(\varphi-\varphi_k)}$$



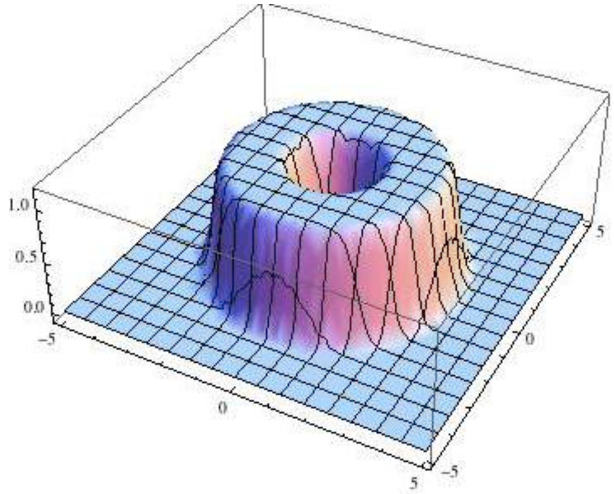
N=4 case





not concentric, but fan-shaped, like pieces of cake

Other fuzzy objects: e.g.) fuzzy Annulus



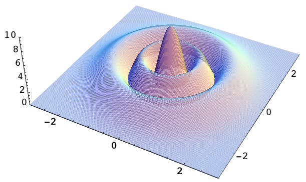
$$\hat{P}_N^M := \hat{p}_M + \hat{p}_{M+1} + \cdots + \hat{p}_{M+N-1}$$

any set of N orthonormal operators is allowed for truncation

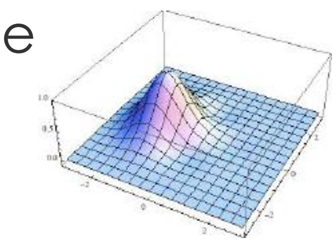
Angular NC solitons as D0-branes?

- scalar field on the NC plane
  = tachyon filed on a non-BPS D2-brane

- The solution $\Phi = \lambda_* \hat{p}_n$
  = a D0-brane (rank $\hat{p}_n = 1 \rightarrow$ same tension)



- Same thing can be said:
  the solution $\Phi = \lambda_* \hat{\pi}_m$ also can be seen as a D0-brane



- Commutative limit (with Nθ fixed),
  angular NC soliton becomes thinner and thinner

NC gravity of cosmological constant

[Asakawa-SK, CQG27(2010)105014]

- We propose a 3 dim. model with C.C. term only

$$S = -\frac{\Lambda}{\kappa^2}\int dt\, d^2x\, \det{}_\star E\ ,$$
$$\det{}_\star E = \frac{1}{3!}\epsilon_{abc}\epsilon^{\mu\nu\rho}E^a_\mu \star E^b_\nu \star E^c_\rho .$$

- Infinitely many nontrivial solutions

NC gravity of cosmological constant (cont'd)

- Other quantities (metric, determinant, Ricci tensor etc.) are viewed as composites of vielbeins

  - NC metric $$G_{\mu\nu} = \frac{1}{2}\left(E^a_\mu \star E^b_\nu + E^b_\nu \star E^a_\mu\right)\eta_{ab}$$
  - "Commutative" metric $$g_{\mu\nu} = E^a_\mu \cdot E^b_\nu \; \eta_{ab}$$
  - Several kinds of determinants $$\det{}_\star G, \quad \det G$$

Equation of motion

- EOM: $\epsilon^{\mu\nu\rho}\epsilon_{abc}\{E^b_\nu, E^c_\rho\}_\star = 0$
- an example of solutions: diagonal ansatz

$$E^a_\mu = \begin{pmatrix} E^0_0 & 0 & 0 \\ 0 & E^1_1 & 0 \\ 0 & 0 & E^2_2 \end{pmatrix} \Longrightarrow \begin{array}{ll} E^0_0 : & \{E^1_1, E^2_2\}_\star = 0, \\ E^1_1 : & \{E^2_2, E^0_0\}_\star = 0, \\ E^2_2 : & \{E^0_0, E^1_1\}_\star = 0. \end{array}$$

- Mutually anti-commuting $\Rightarrow$ realized by

1) Projection operators [Asakawa-SK (2010)]

2) Dirac gamma matrices (Clifford algebra)

Solution 1: Diagonal Solution

- Simplest ansatz: diagonal

$$E^a_\mu = \begin{pmatrix} E^0_0 & 0 & 0 \\ 0 & E^1_1 & 0 \\ 0 & 0 & E^2_2 \end{pmatrix}$$

- EOMs:

$$\begin{aligned} E^0_0 &: \quad \{E^1_1, E^2_2\}_\star = 0, \\ E^1_1 &: \quad \{E^2_2, E^0_0\}_\star = 0, \\ E^2_2 &: \quad \{E^0_0, E^1_1\}_\star = 0. \end{aligned}$$

mutually anti-commuting
$\rightarrow$ realized by
the orthogonality of ϕ_i

- Solution (simplest one)

$$E^b_\nu = \begin{pmatrix} \alpha_0\phi_0 & 0 & 0 \\ 0 & \alpha_1\phi_1 & 0 \\ 0 & 0 & \alpha_2\phi_2 \end{pmatrix},$$

where $\alpha_0, \alpha_1, \alpha_2$: arbitrary constants

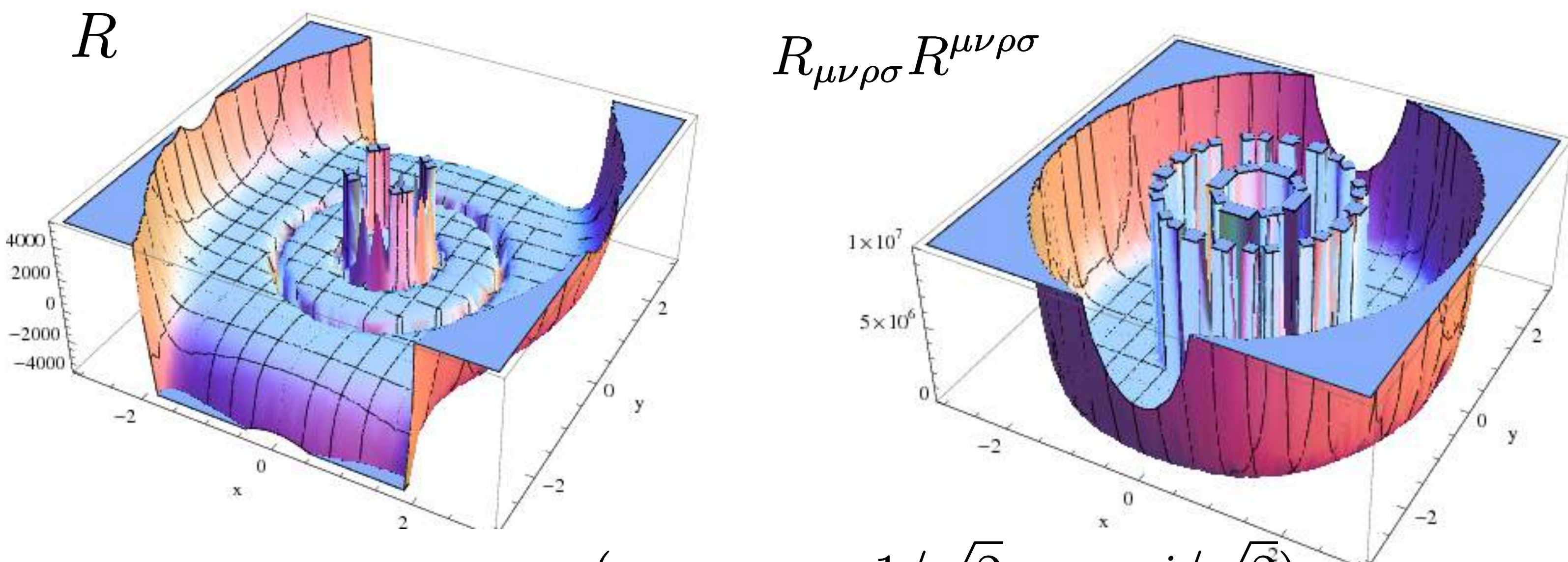
- Line element

$$\begin{aligned} ds^2 &= -\alpha_0^2\phi_0 dt^2 + \alpha_1^2\phi_1 dx^2 + \alpha_2^2\phi_2 dy^2 \\ &= 2e^{-r^2/\theta}\Bigg(-\alpha_0^2 dt^2 - \alpha_1^2\left(1 - \frac{2r^2}{\theta}\right)dx^2 \\ &\qquad + \alpha_2^2\left(1 - \frac{4r^2}{\theta} + \frac{2r^4}{\theta^2}\right)dy^2\Bigg), \end{aligned}$$

- Diverges when θ→0: particular to NC gravity
- Does not have the star inverse,
  but has the ordinary inverse of itself.

$$(\det{}_\star G_{\mu\nu} = 0 \quad \text{but} \quad \det G_{\mu\nu} \neq 0)$$

- Scalar invariants can be defined on commutative space
  (we switched to the metric formalism here)



$$(\alpha_0 = \alpha_2 = 1/\sqrt{2}, \alpha_1 = i/\sqrt{2})$$

solution 2: nondiagonal

- Ansatz

$$E^a_\mu = \begin{pmatrix} E^0_0 & 0 & 0 \\ 0 & E^1_1 & E^2_1 \\ 0 & E^1_2 & E^2_2 \end{pmatrix}.$$

- EOMs

$$0 = \{E^1_1, E^2_2\}_\star - \{E^2_1, E^1_2\}_\star,$$
$$0 = \{E^0_0, E^a_\mu\}_\star \quad (a, \mu = 1, 2).$$

- Solution (simplest one)

$$E^b_\nu = \begin{pmatrix} \alpha_0\phi_0 & 0 & 0 \\ 0 & \alpha_1\phi_1 & \alpha_1\phi_1 \\ 0 & \alpha_1\phi_1 & \alpha_1\phi_1 \end{pmatrix},$$

- Line element

$$\begin{aligned} ds^2 &= -\alpha_0^2\phi_0 dt^2 + 2\alpha_1^2\phi_1\big(dx^2 + 2dxdy + dy^2\big) \\ &= 2e^{-r^2/\theta}\left(-\alpha_0^2 dt^2 - 2\alpha_1^2\left(1 - \frac{2r^2}{\theta}\right)(dx+dy)^2\right). \end{aligned}$$

- Effectively two-dimensional
  $\rightarrow$ Discrepancy b/w manifold & metrical dim.
  (typical feature of quantum gravity?)

Solution3: Clifford algebra-like

- EOM: $\epsilon^{\mu\nu\rho}\epsilon_{abc}\{E_\nu^b, E_\rho^c\}_\star = 0,$

  $\rightarrow$ vielbein should be mutually anti-commuting

- The vielbein obeying the Clifford algebra solves the EOM, e.g.,

$$E_\mu^a = \begin{pmatrix} \gamma^0 & 0 & 0 \\ 0 & \gamma^1 & 0 \\ 0 & 0 & \gamma^2 \end{pmatrix} \quad \text{and} \quad \{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}\mathbf{1}_2$$

- Representation of the matrix elements in the harmonic oscillator basis

$$\gamma^0 = \sigma^3 = |\,0\,\rangle\,\langle\,0\,| - |\,1\,\rangle\,\langle\,1\,|\,,$$
$$\gamma^1 = \sigma^1 = |\,1\,\rangle\,\langle\,0\,| + |\,0\,\rangle\,\langle\,1\,|\,,$$
$$\gamma^2 = \sigma^2 = i\,|\,1\,\rangle\,\langle\,0\,| - i\,|\,0\,\rangle\,\langle\,1\,|\,.$$

- Metric:

$$\begin{aligned} G_{\mu\nu} &= \eta_{\mu\nu}\,(|\,0\,\rangle\,\langle\,0\,| + |\,1\,\rangle\,\langle\,1\,|) \\ &= \eta_{\mu\nu}\,(\phi_0 + \phi_1) \\ &= \frac{4r^2}{\theta} e^{-r^2/\theta} \eta_{\mu\nu} \end{aligned}$$

$\leftarrow$ propotional to
the Minkowski spacetime

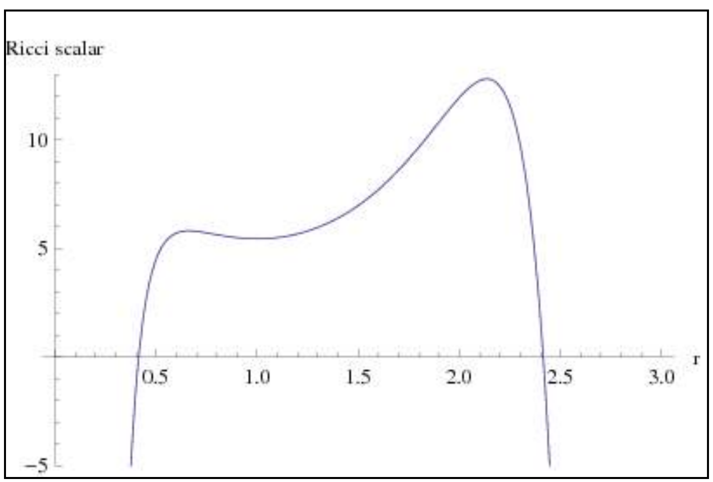
- Interpolates two vacua, $G_{\mu\nu} = 0$ and $G_{\mu\nu} = \eta_{\mu\nu}$
$\rightarrow$ same as the noncommutative scalar solitons

- Scalar invariants
  note: defined on commutative space (not on NC space)

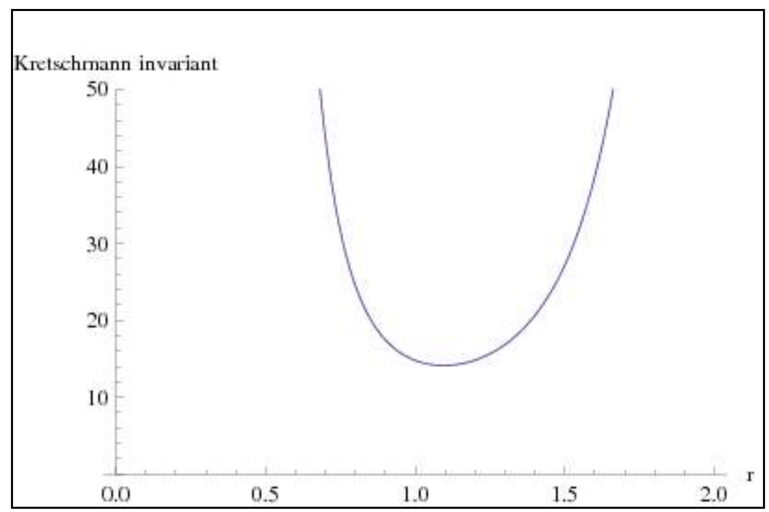
$$R = -\frac{e^{r^2/\theta}}{2r^4\theta}(\theta^2 - 6r^2\theta + r^4),$$

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{e^{2r^2}}{4r^8\theta^2}(5\theta^4 - 10r^2\theta^3 + 18r^4\theta^2 - 6r^6\theta + r^8).$$

R



$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$



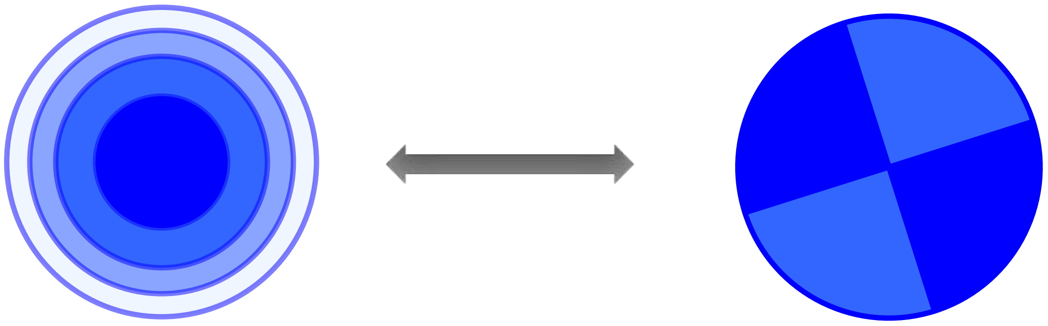
Two descriptions for fuzzy disc

- number basis: concentric cutting of a disc

  $\hat{N}$: radius operator $(\hat{N} \sim \sqrt{\hat{x}^2 + \hat{y}^2})$

- angle basis: radial cutting of disc [SK-Asakawa, 2013]

  $\hat{\varphi}$: angle operator $\leftarrow$ with aid of phase state in quantum optics [Pegg-Barnett]



Radius ⇄ Angle

$$|\varphi_m\rangle = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} e^{in\varphi_m} |n\rangle,$$

$$\varphi_m = \varphi_0 + \frac{2\pi}{N} m \quad (m = 0, 1, \cdots, N-1)$$

$$\hat{N} = \sum_{n=0}^{N-1} n|n\rangle\langle n|$$

conjugate ⟺

$$\hat{\varphi} = \sum_{m=0}^{N-1} \varphi_m |\varphi_m\rangle\langle\varphi_m|$$

- introduction of the polar coordinate in NC geometry
- how about fuzzy sphere? (extension to "3D")

Angular NC solitons in gravity

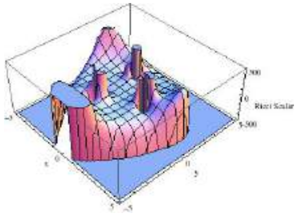
$$S = -\frac{\Lambda}{\kappa^2}\int dtd^2z\ E^\star \qquad E^\star = \det{}_\star E = \frac{1}{3!}\epsilon^{\mu\nu\rho}\epsilon_{abc}E^a_\mu \star E^b_\nu \star E^c_\rho$$

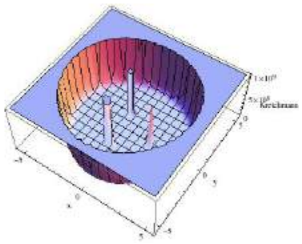
$$\epsilon^{\mu\nu\rho}\epsilon_{abc}\{E^b_\nu, E^c_\rho\}_\star = 0$$

$$E^a_\mu = \begin{pmatrix} E^0_0 & 0 & 0 \\ 0 & E^1_1 & 0 \\ 0 & 0 & E^2_2 \end{pmatrix} = \begin{pmatrix} \alpha_0\pi_0^{(N)} & 0 & 0 \\ 0 & \alpha_1\pi_1^{(N)} & 0 \\ 0 & 0 & \alpha_2\pi_2^{(N)} \end{pmatrix}$$

$$ds^2 = -\alpha_0^2\pi_0^{(3)}dt^2 + \alpha_1^2\pi_1^{(3)}dx^2 + \alpha_2^2\pi_2^{(3)}dy^2$$

$$\pi_k^{(3)}(r,\varphi) = \frac{1}{3}e^{-r^2/\theta}\left[1 + \frac{2r}{\theta^{1/2}}\cos(\varphi - \varphi_k^{(3)}) + \frac{r^2}{\theta}\left\{1 + \sqrt{2}\cos[2(\varphi - \varphi_k^{(3)})]\right\} + \frac{\sqrt{2}r^3}{\theta^{3/2}}\cos(\varphi - \varphi_k^{(3)}) + \frac{r^4}{\theta^2}\right]$$

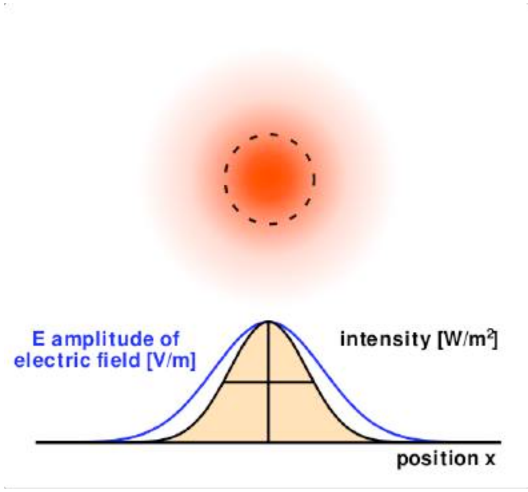


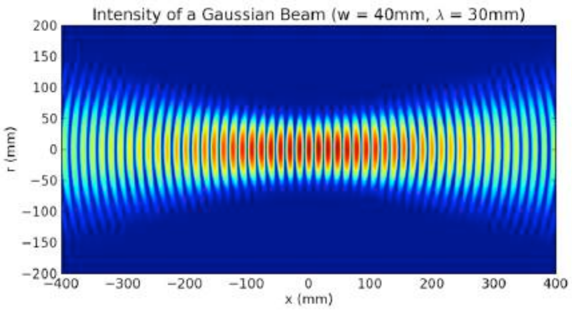


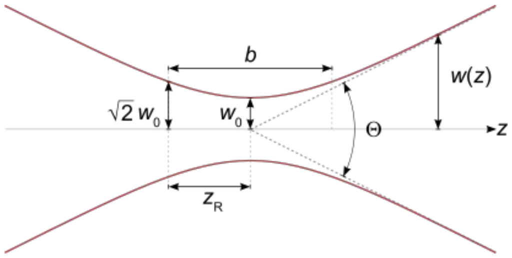
Analogy to Gaussian beam

- Gaussian beam

$$I(r,z) = \frac{|E(r,z)|^2}{2\eta} = I_0 \left(\frac{w_0}{w(z)}\right)^2 \exp\left(\frac{-2r^2}{w^2(z)}\right) ,$$



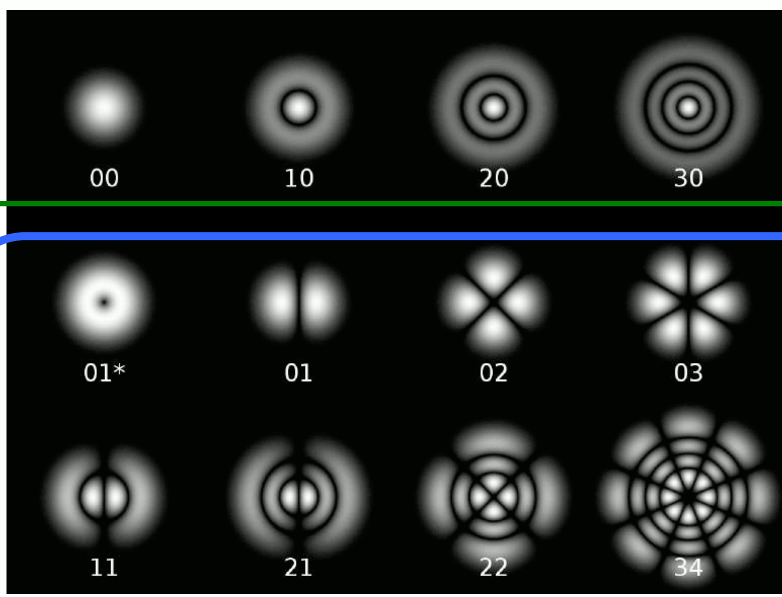




Analogy to Gaussian beam

- Laguerre-Gaussian beam

$$u(r,\phi,z) = \frac{C_{lp}^{LG}}{w(z)}\left(\frac{r\sqrt{2}}{w(z)}\right)^{|l|}\exp\left(-\frac{r^2}{w^2(z)}\right)L_p^{|l|}\left(\frac{2r^2}{w^2(z)}\right)\exp\left(ik\frac{r^2}{2R(z)}\right)\exp(il\phi)\exp\left[-i(2p+|l|+1)\zeta(z)\right],$$



GMS solitons

angular NC solitons

Example: Fuzzy Sphere

commutative algebra of spherical harmonics

$$f(\theta,\phi) = \sum a_{lm} Y_{lm}(\theta,\phi)$$

$$f(\theta,\phi)\cdot g(\theta,\phi) = g(\theta,\phi)\cdot f(\theta,\phi)$$

→ noncommutative deformation →

noncommutative algebra of deformed spherical harmonics

$$f(\theta,\phi) = \sum a_{lm} \hat{Y}_{lm}(\theta,\phi)$$

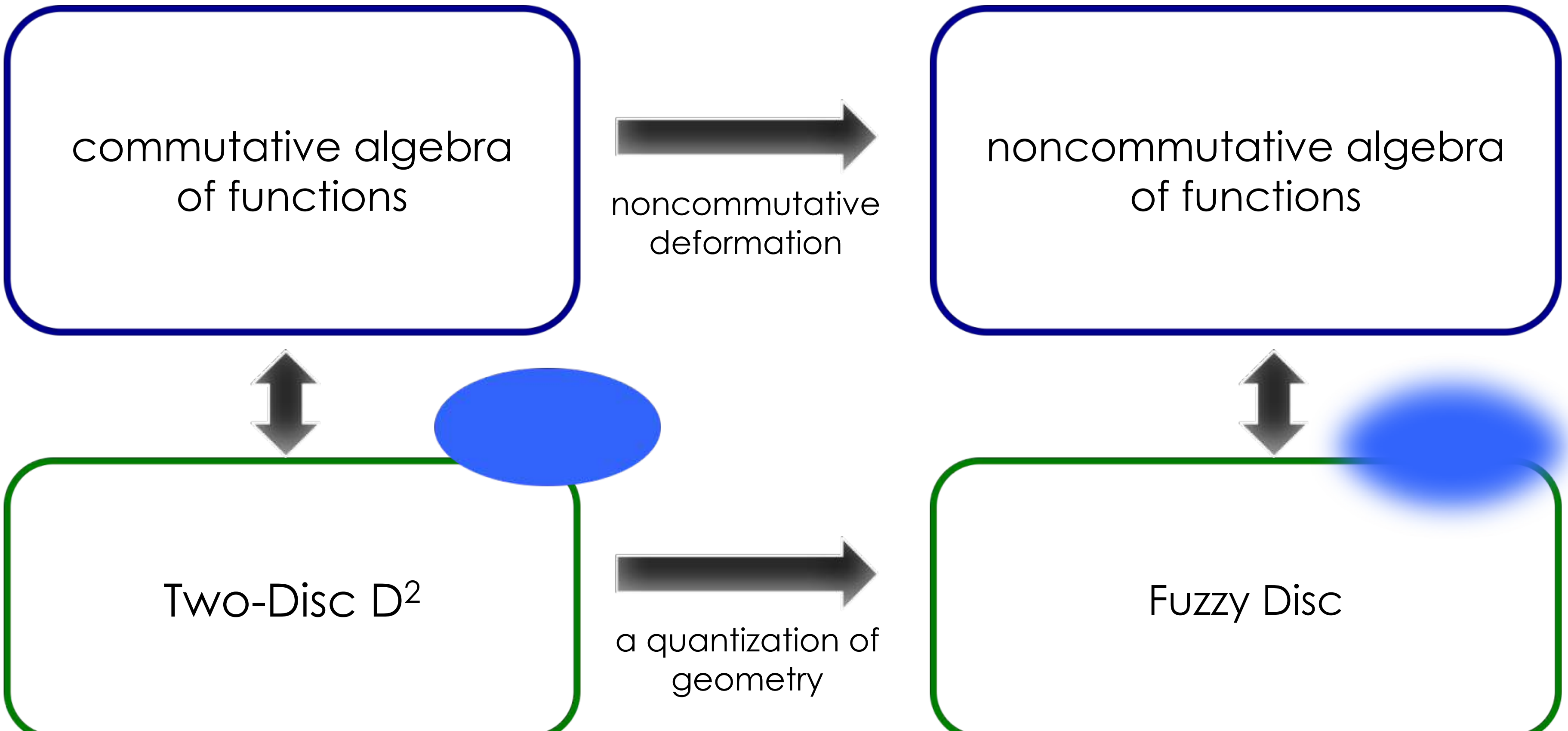
$$f(\theta,\phi)\star g(\theta,\phi) \neq g(\theta,\phi)\star f(\theta,\phi)$$

Two-sphere S^2

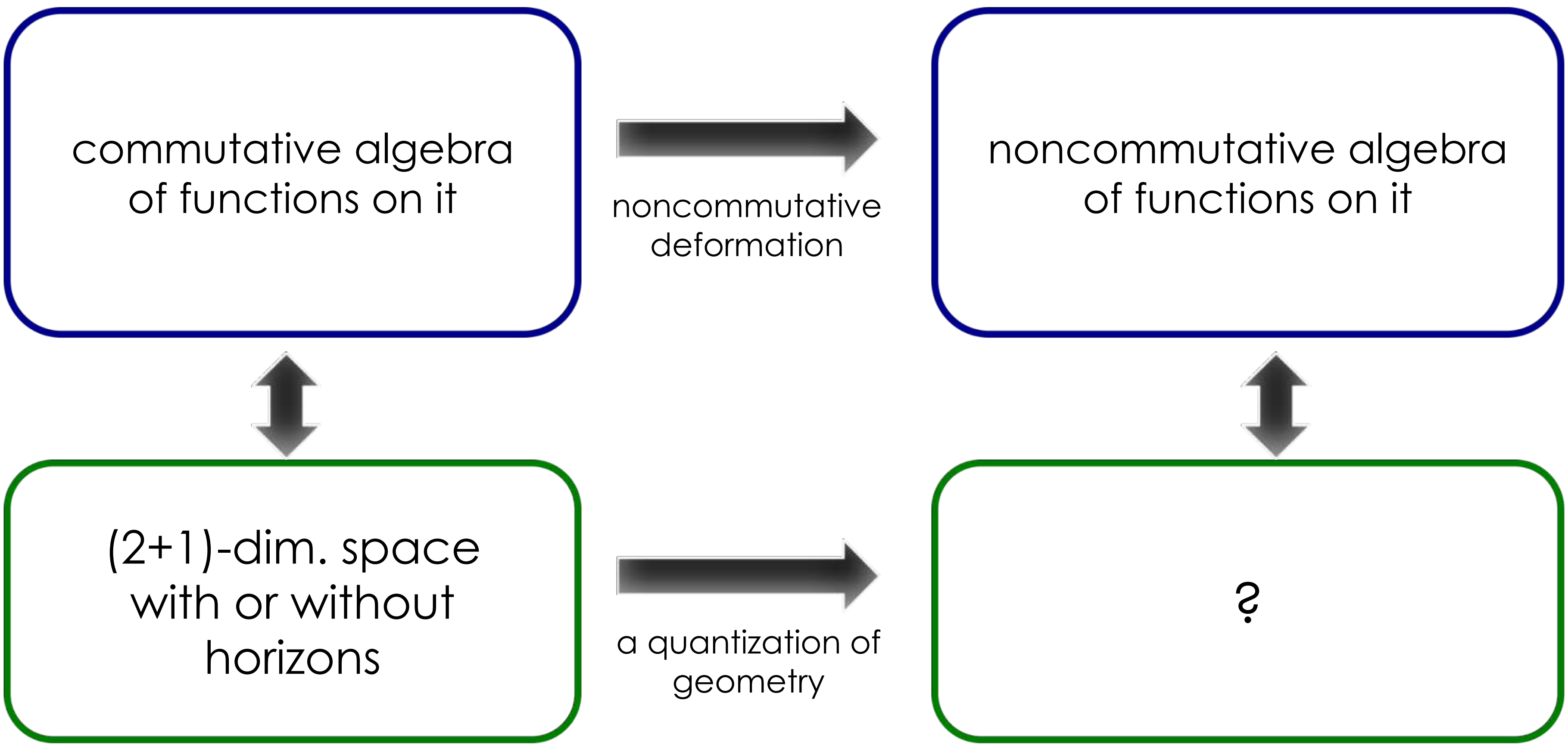
→ a quantization of geometry →

Fuzzy Sphere

Example: Fuzzy Disc



Toward quantum black holes



(2+1)-dim. spacetime with conical singularity

[Deser et al., Deser-Jackiw (1984)]

$$\sqrt{-g}T^{tt} = \alpha m \delta^{(2)}(r), \quad T^{ta} = T^{ab} = 0 \quad (a, b = r, \theta)$$

$$\Downarrow$$

$$ds^2 = -dt^2 + r^{-8m}(dr^2 + r^2 d\varphi^2)$$

(2+1)-dim. spacetime with conical singularity

[Deser et al., Deser-Jackiw (1984)]

$$ds^2 = -dt^2 + r^{-8m}(dr^2 + r^2 d\varphi^2)$$

$$\downarrow \quad \rho = \frac{r^p}{p}, \quad \phi = p\varphi, \quad p = 1 - 4m$$

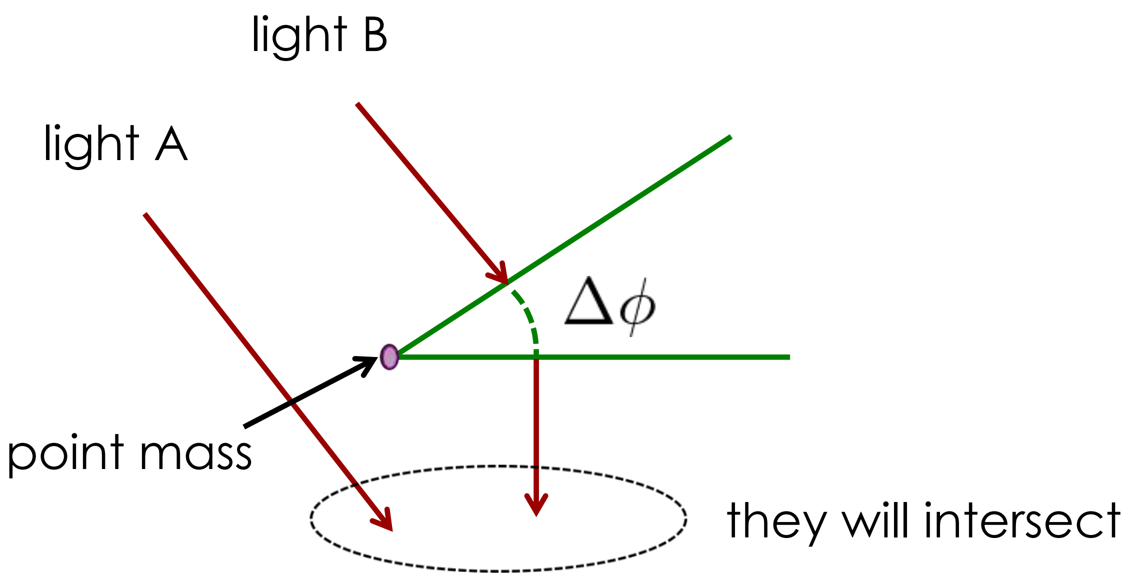
$$ds^2 = -dt^2 + d\rho^2 + \rho^2 d\phi^2$$

locally flat, but there is a conical singularity

(2+1)-dim. spacetime with conical singularity

[Deser et al., Deser-Jackiw (1984)]

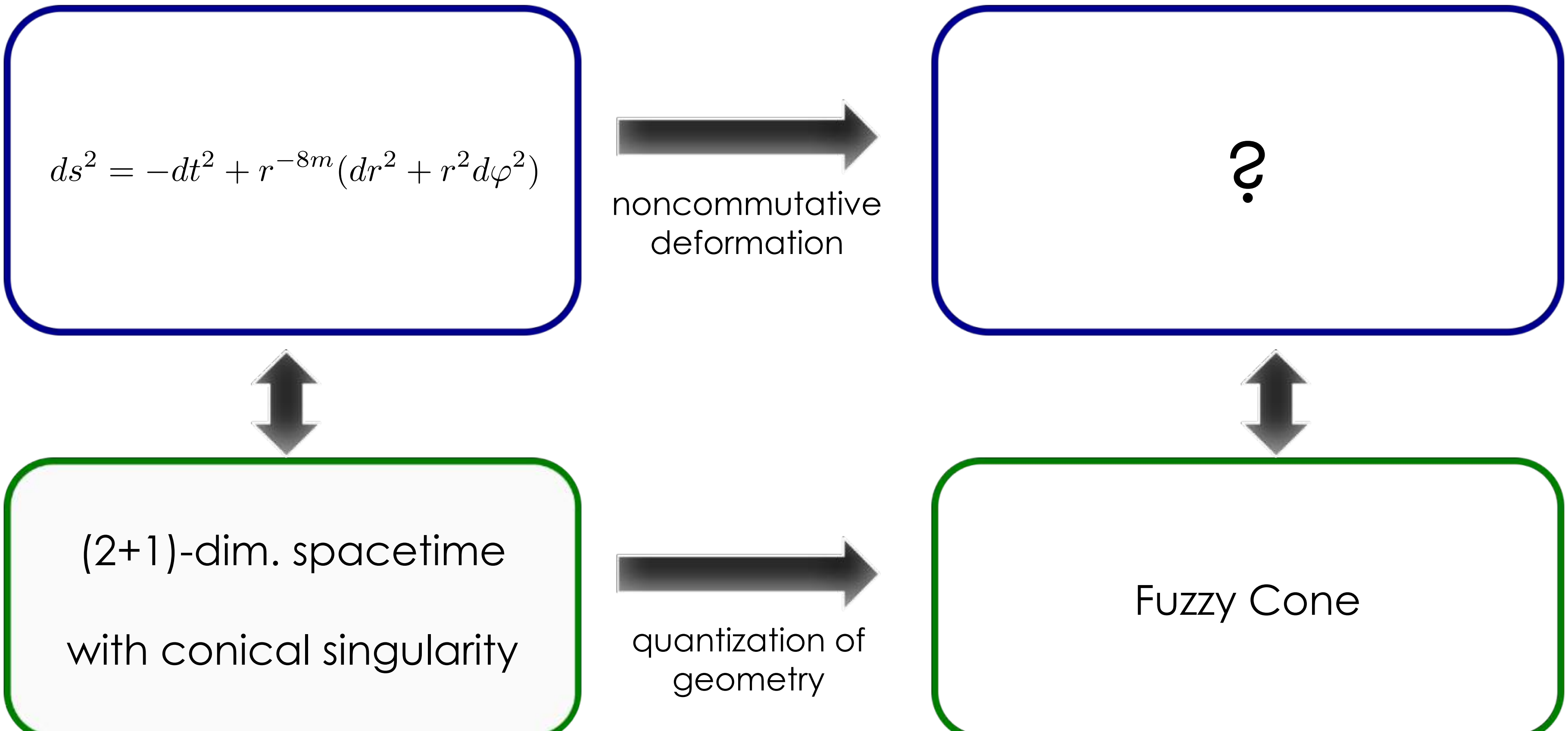
deficit angle $$\Delta\phi = 2\pi - 2\pi p = 8\pi m$$



conical singularity

geometry of cone

Fuzzy Cone


$ds^2 = -dt^2 + r^{-8m}(dr^2 + r^2 d\varphi^2)$
noncommutative deformation
?
(2+1)-dim. spacetime with conical singularity
quantization of geometry
Fuzzy Cone

Effective theory

- deformation of source distribution [Nicolini et al,...]

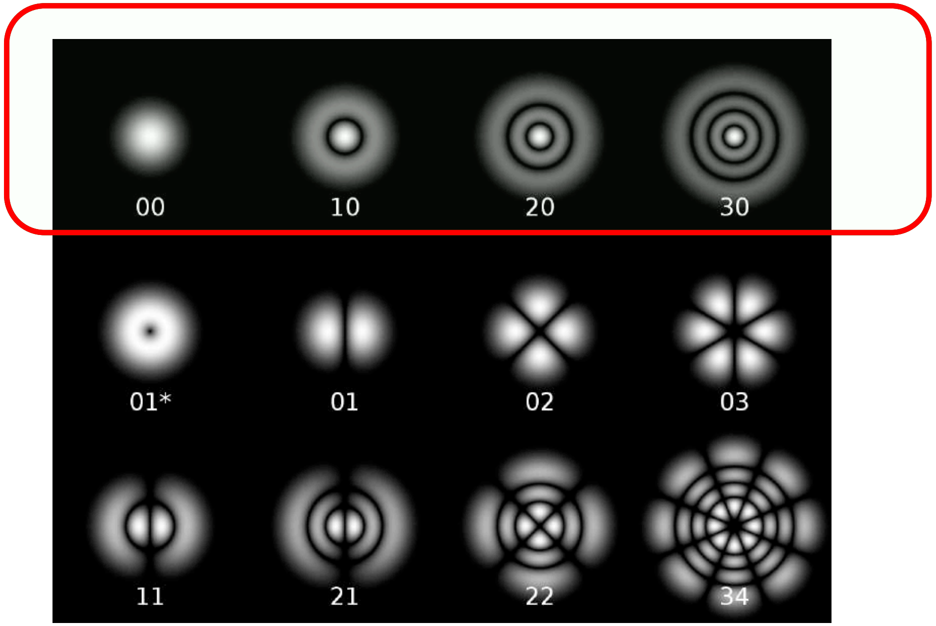
$$M\delta(\vec{r}) \to \alpha e^{-\frac{r^2}{2\theta}}$$

- effective theory of (2+1)D NC graivity [SK (2016), Sadohara (2016卒研)]
- BTZ-like BH with Gaussian density distribution
- (of course,) we cannot see "thickness" of horizon
- full quantum treatment is needed

Density distribution

- Laguerre-Gaussian beam

$$u(r,\phi,z) = \frac{C_{lp}^{LG}}{w(z)} \left(\frac{r\sqrt{2}}{w(z)}\right)^{|l|} \exp\left(-\frac{r^2}{w^2(z)}\right) L_p^{|l|}\left(\frac{2r^2}{w^2(z)}\right) \exp\left(ik\frac{r^2}{2R(z)}\right) \exp(il\phi) \exp\left[-i(2p+|l|+1)\zeta(z)\right],$$



solution?

Summary

- Noncommutative geometry
  - one of the trials to know quantum gravity/geometry
  - algebraic structure of functions on a NC space
    $\longrightarrow$ equivalent to know the NC space, intuitively easier
- Applications to gravity
  - a trial: NC gravity without Ricci scalar
  - horizon with "thickness"
- There are many ways to introduce noncommutativity
  criterion?
- noncommutative/discrete
  - discrete integrable geometry $\longrightarrow$ fuzzball?

阪上さんに言われたこと

- 「小林君はアホやからなあ」
  「物理で『アホ』は褒め言葉やで」

- 「小林君はミーハーやからいつも不安やと思うで」
  「僕もそうやからわかる」

- 「小林君は何かごちゃごちゃ言ってるだけや」
- 早田さん「小林君の議論は地に足がついてない」
- Myers "For example?"

- 「そんなのはもう終わってるんや」