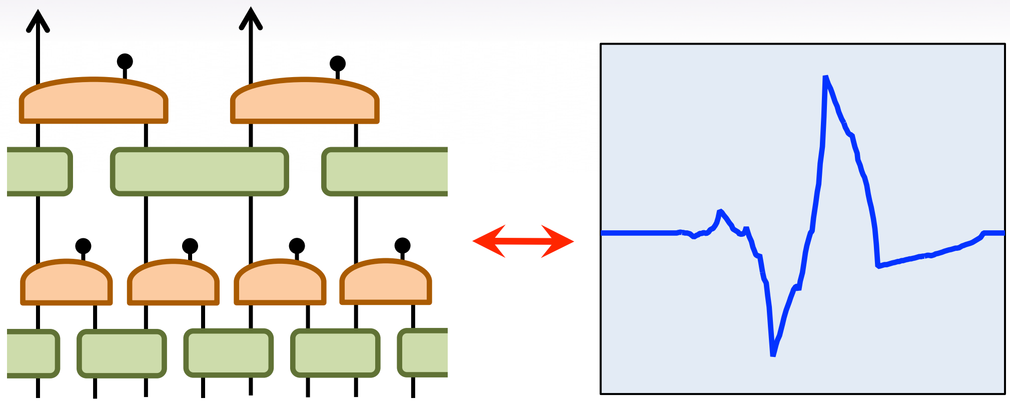
YITP, June 2016

Entanglement Renormalization and Wavelets



Glen Evenbly

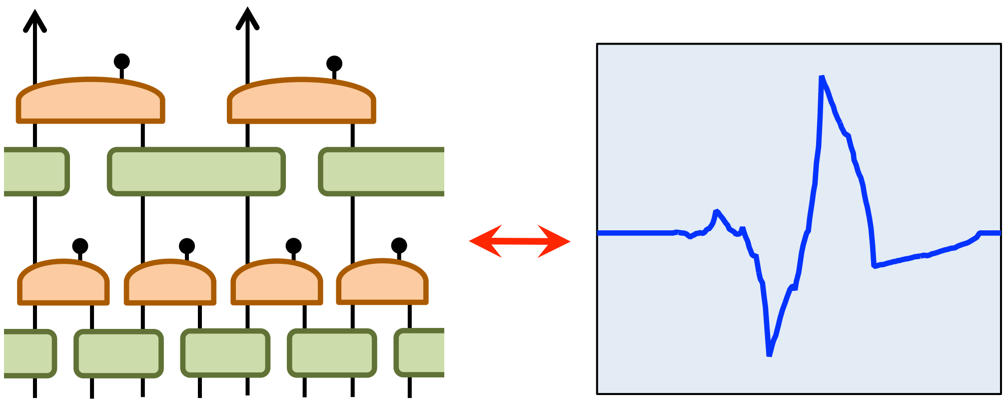
G.E., Steven. R. White, Phys. Rev. Lett 116. 140403 (April `16).

G.E., Steven. R. White, arXiv: 1605.07312 (May `16).

Entanglement renormalization and wavelets

- real-space renormalization
- quantum circuits
- tensor networks (MERA)

- compact, orthogonal wavelets



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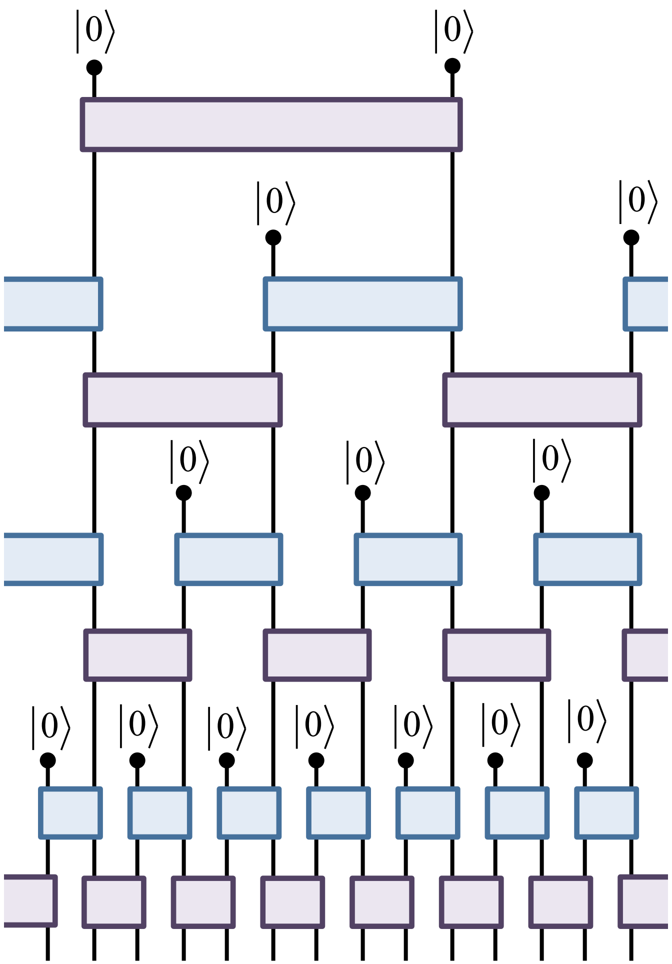
Introduction: MERA

Multi-scale Entanglement Renormalization Ansatz (MERA):

proposed by Vidal to represent ground states of local Hamiltonians



Can be formulated as:

(i) a quantum circuit

(ii) resulting from coarse-graining **(entanglement renormalization)**

product state

$$|0\rangle|0\rangle|0\rangle|0\rangle|0\rangle|0\rangle|0\rangle|0\rangle|0\rangle|0\rangle|0\rangle|0\rangle|0\rangle|0\rangle$$

↓ circuit

$$|\psi\rangle$$

entangled state

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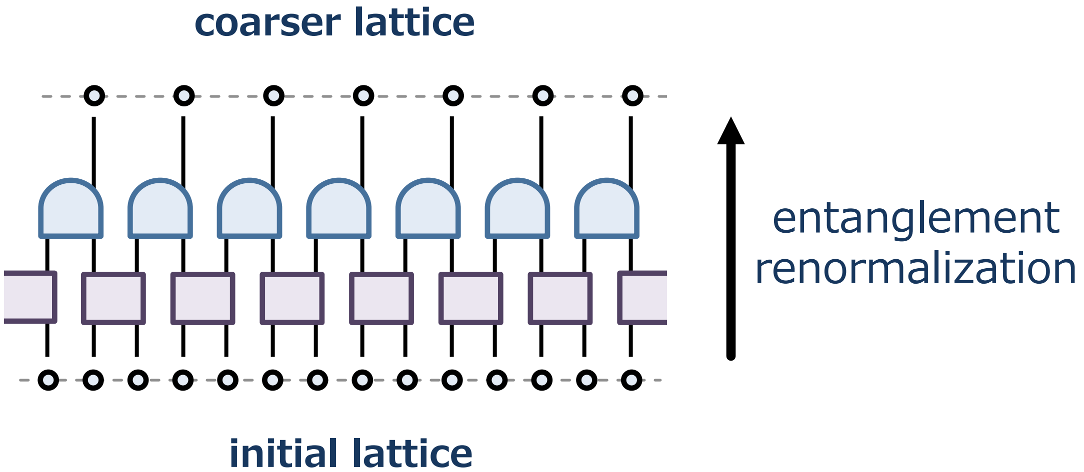

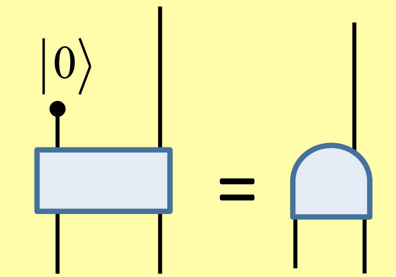

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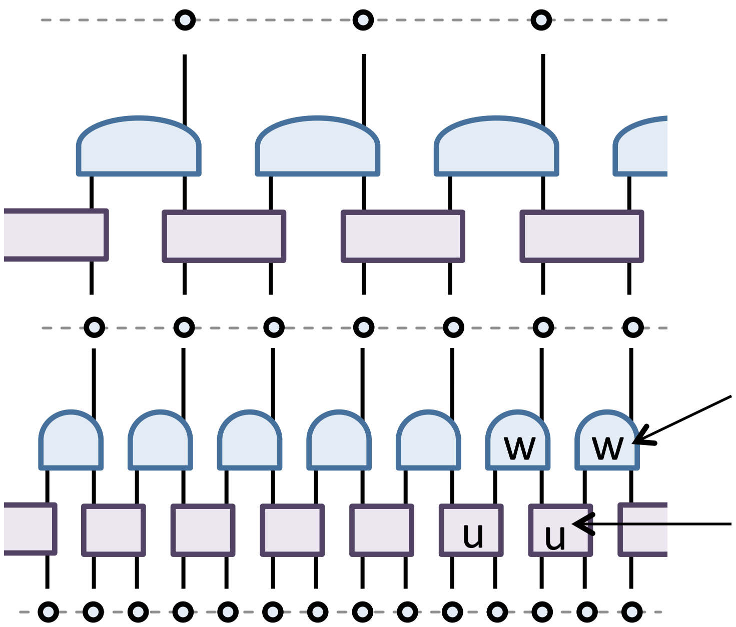
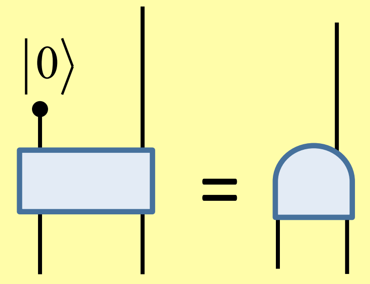


isometries
(truncate blocks)

disentanglers
(mediate entanglement between blocks)



fixed index can be omitted

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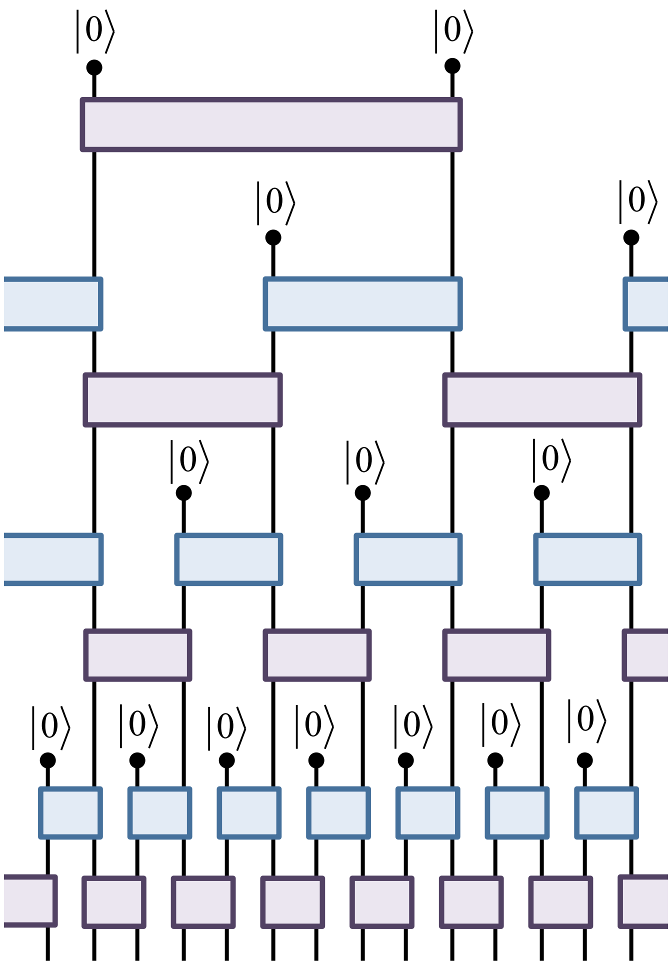

Can be formulated as:

(i) a quantum circuit

(ii) resulting from coarse-graining **(entanglement renormalization)**

Key properties:

(i) **efficiently contractible** (for local observables, correlators, etc)

(ii) reproduce **logarithmic correction** to the area law (for 1D quantum systems)

$$S_L \sim \log(L)$$

(iii) reproduce **polynomial** decay of correlations

(iv) can capture **scale-invariance**

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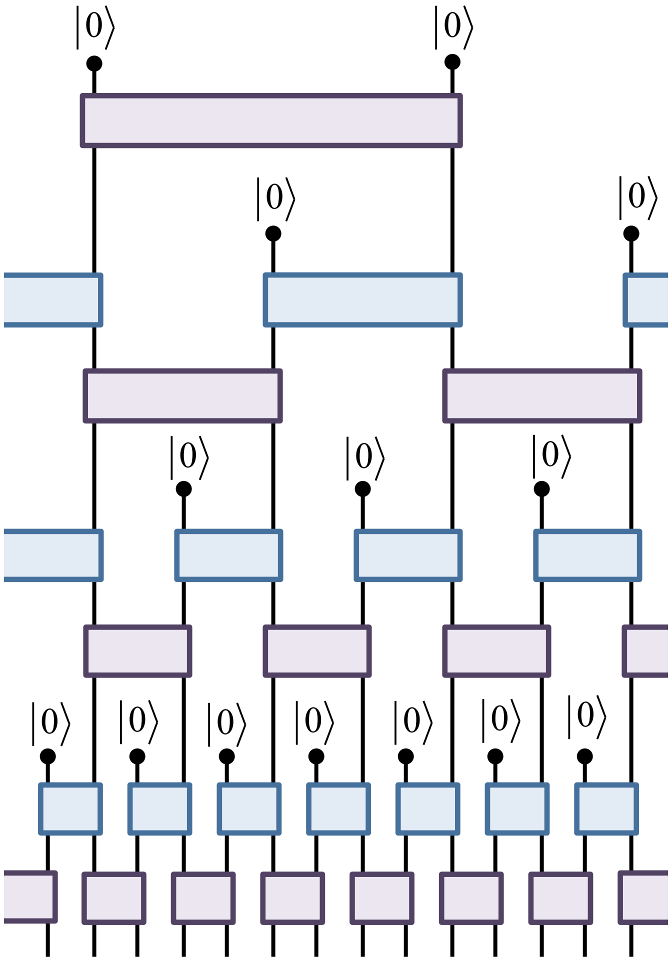

Applications:

- numeric study of quantum critical systems
- error-correcting codes (e.g. holographic codes) and topologically ordered systems
- machine learning (convolutional neural networks)
- data compression (multi-resolution analysis and wavelets)
- holography?

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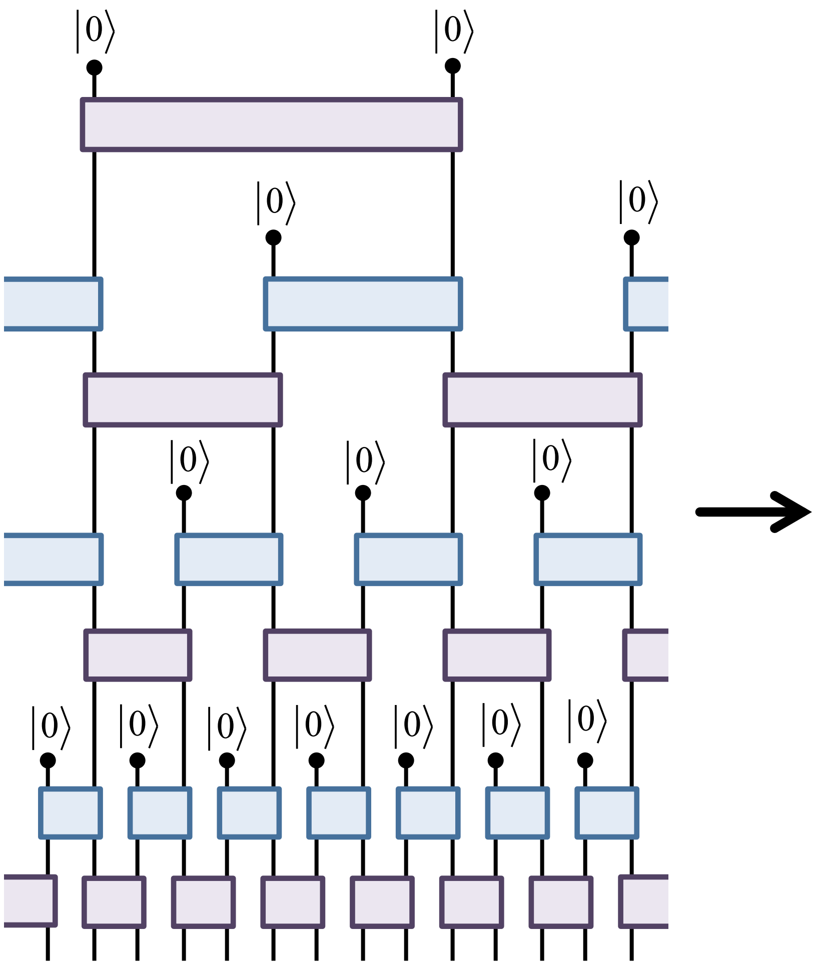
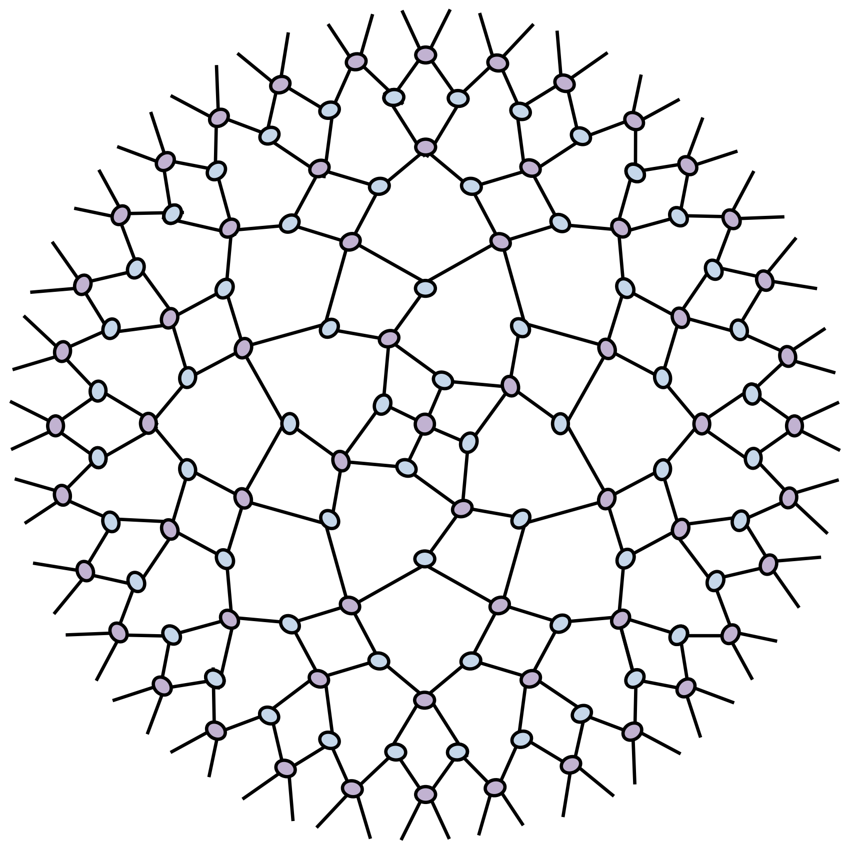


MERA can be viewed as a tiling of the hyperbolic disk:



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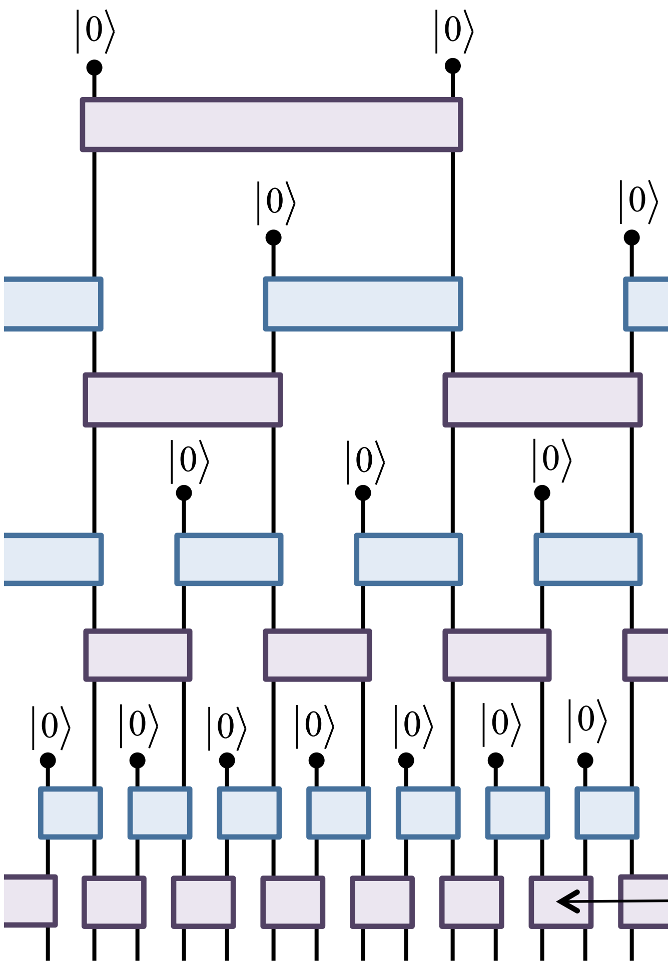


MERA are useful as a numeric tool for studying ground states of many-body systems...

...but we lack a deeper conceptual understanding

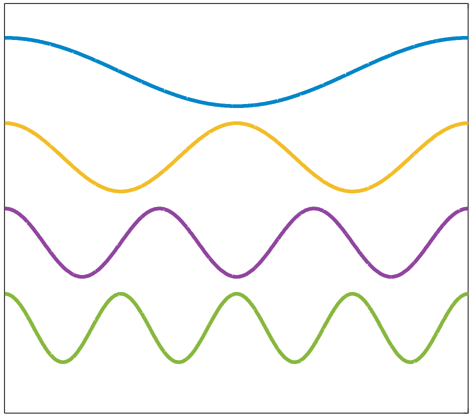
input: Hamiltonian → Numerical magic: **Optimise tensors** (i.e. energy minimization) → output: Ground state (approximate)

parameters defining disentangler, *u*:

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ u_{21} & u_{22} & u_{23} & u_{24} \\ u_{31} & u_{32} & u_{33} & u_{34} \\ u_{41} & u_{42} & u_{43} & u_{44} \end{bmatrix}$$

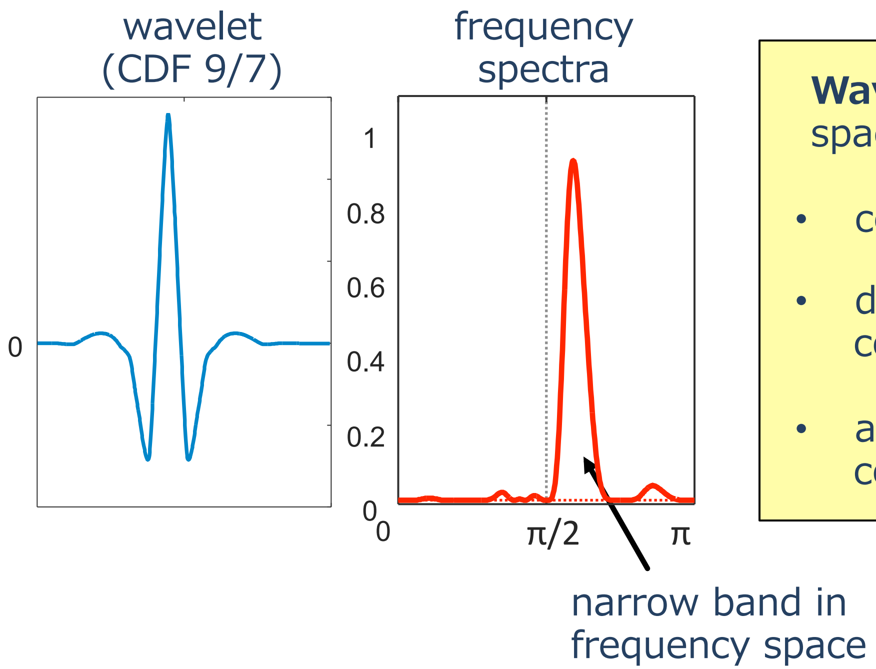
Can we understand how MERA can represent the ground state of a lattice CFT? (even if only in a simple example...)

Introduction: Wavelets



Fourier expansions are ubiquitous in math, science and engineering

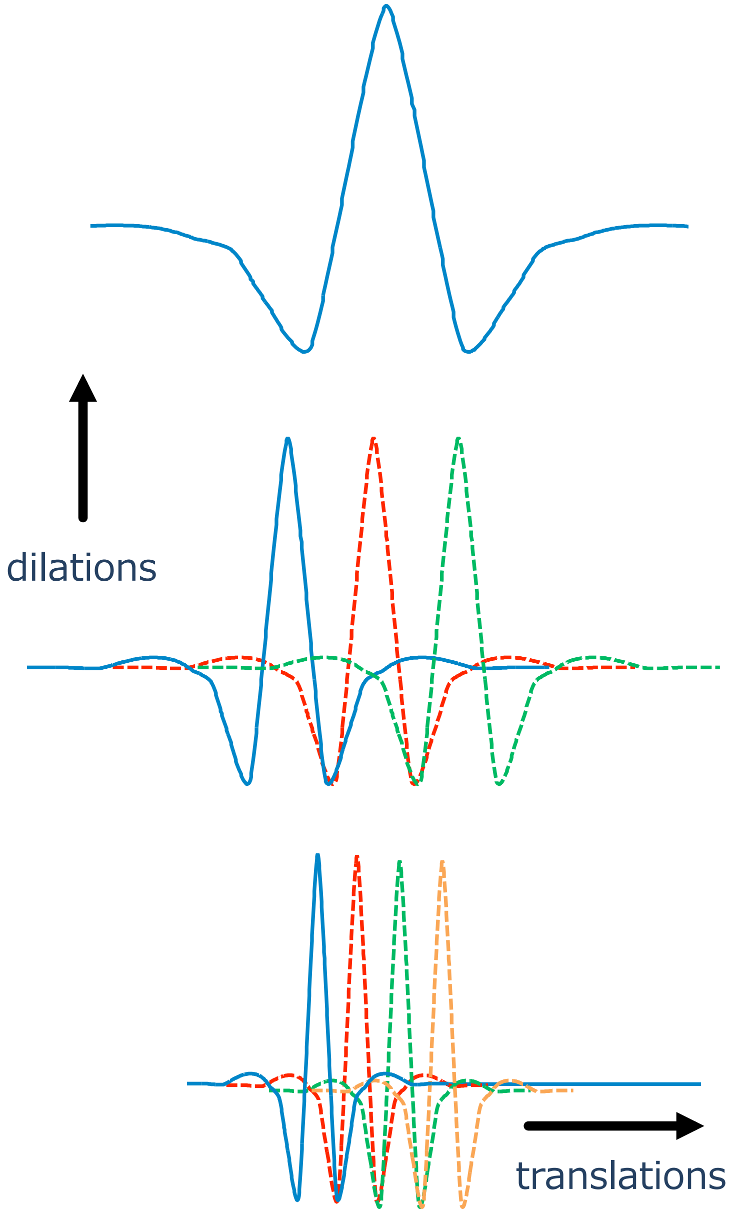
- many problems are simplified by expanding in Fourier modes
- smooth functions can be approximated by only a few non-zero Fourier coefficients



Wavelets are a good compromise between real-space and Fourier-space representations

- compact in real-space and in frequency-space
- developed by math and signal processing communities in late 80's
- applications in signal and image processing, data compression (e.g. JPEG2000 image format)

Introduction: Wavelets



Wavelet basis consists of translations and dilations of the wavelet function

- is a complete, orthonormal basis
- is a multi-resolution analysis (MRA)

Wavelets are a good compromise between real-space and Fourier-space representations

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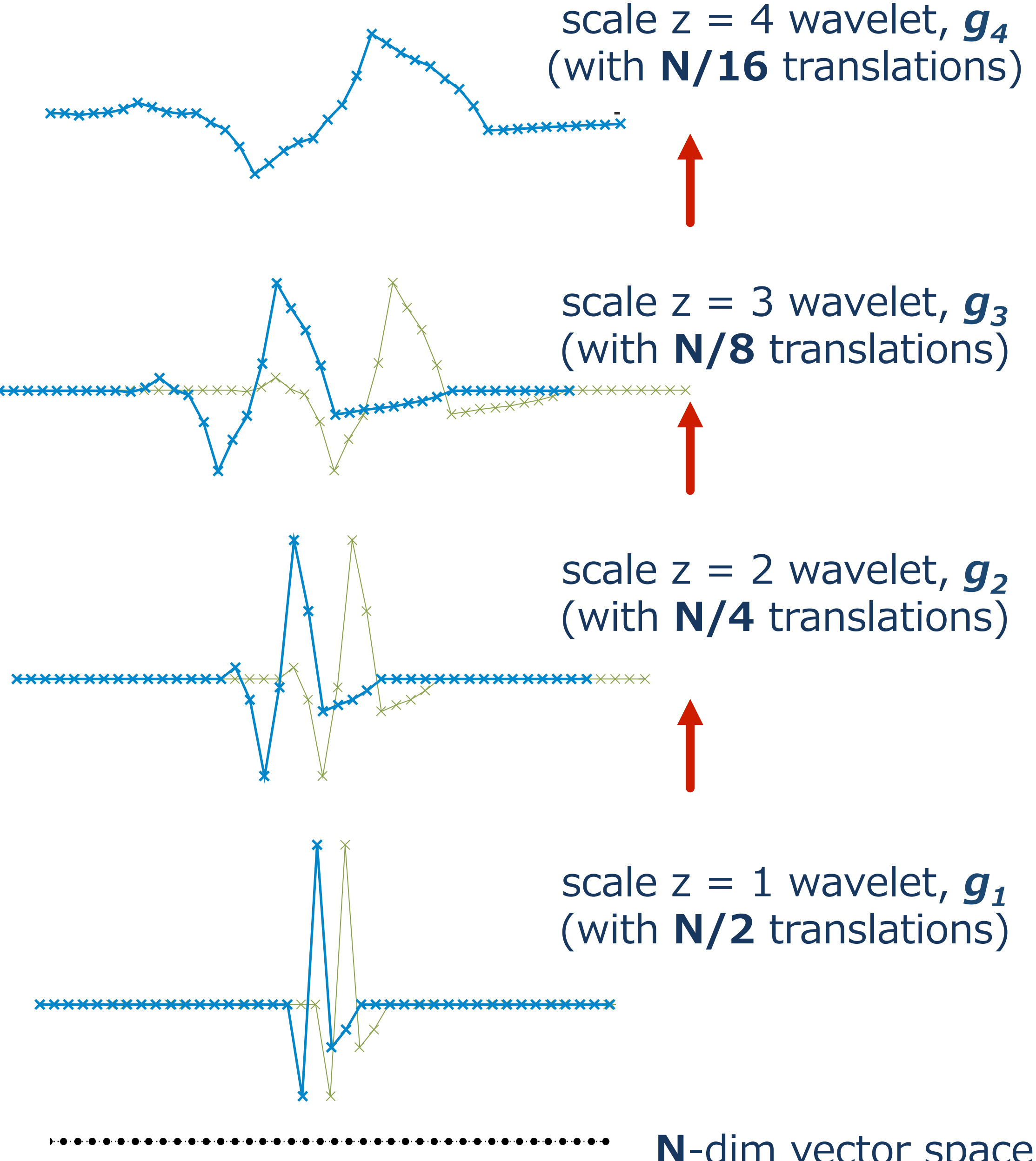
Daubechies wavelets



Daubechies D4 wavelets

- complete, orthonormal basis
- have 2 vanishing moments (orthogonal to constant + linear functions)
- useful for resolving information at different scales

large scale wavelets encode long range (**low-frequency**) information

↑

small scale wavelets encode short range (**high-frequency**) information

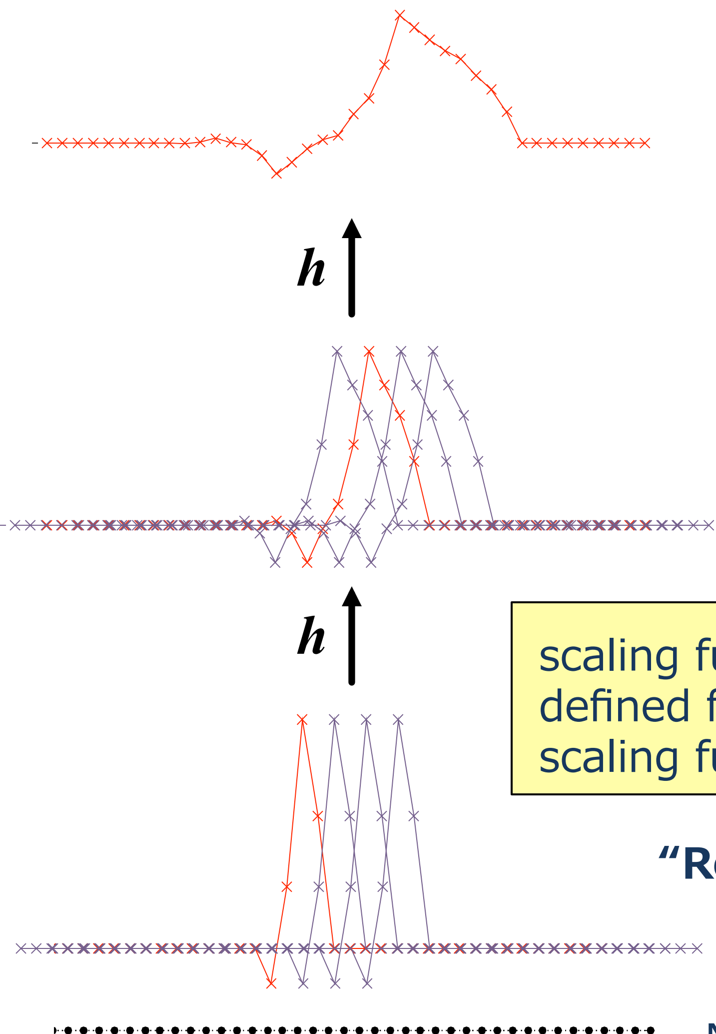
Daubechies wavelets

How can we construct wavelets?

- first construct scaling functions (allows recursive construction of functions at different scales)

D4 scaling sequence

$$\boldsymbol{h} = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{bmatrix} = \begin{bmatrix} -0.1294 \\ 0.2241 \\ 0.8365 \\ 0.4830 \end{bmatrix}$$

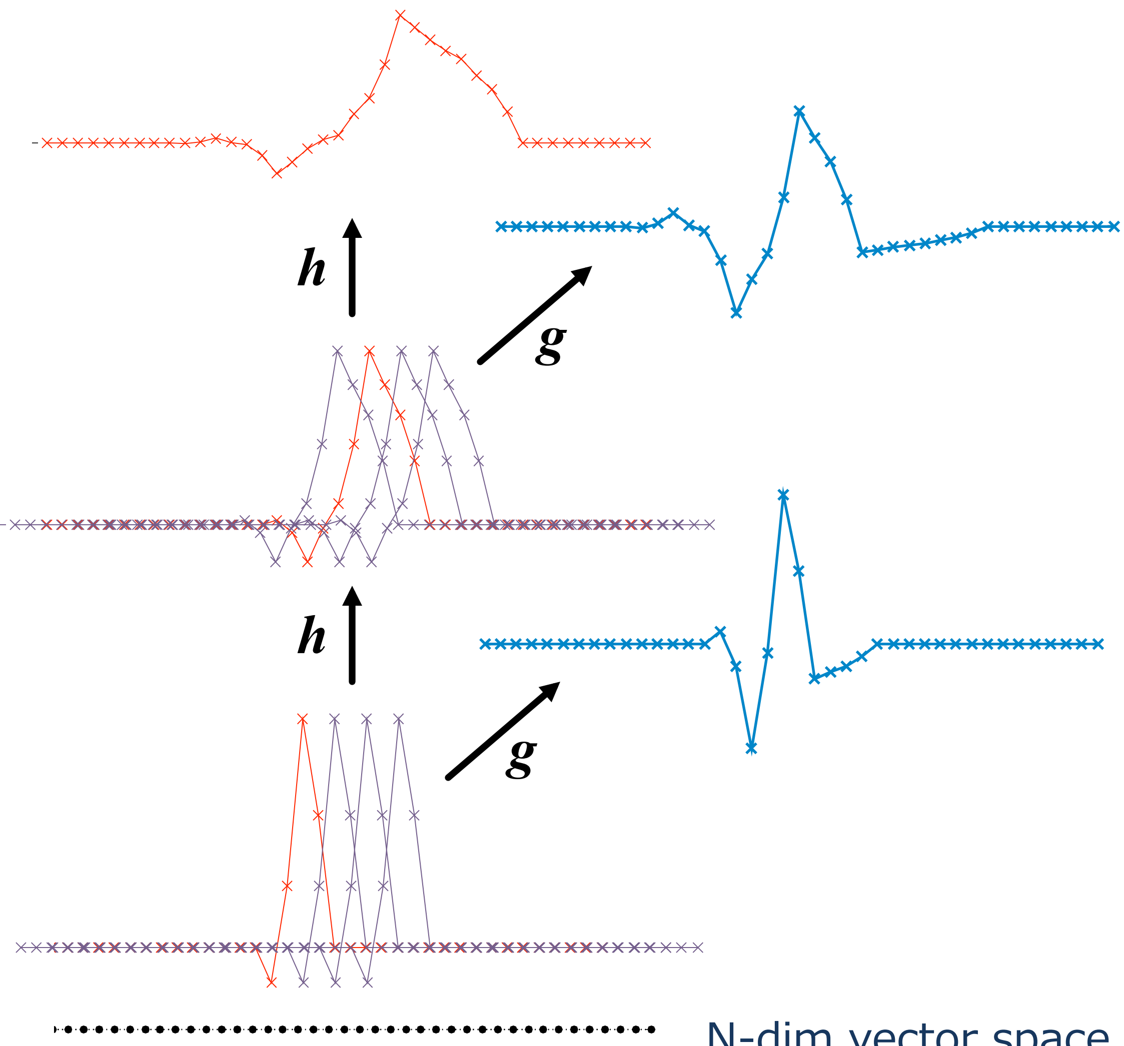


scaling function at larger scale defined from a linear combination of scaling functions at previous scale

"Refinement equation"

N-dim vector space

Daubechies wavelets



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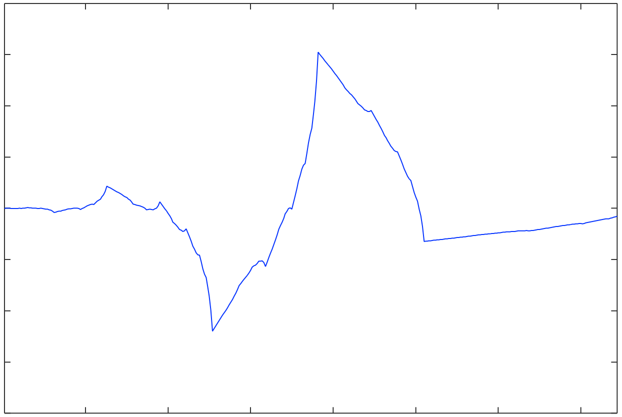
- wavelets then defined from scaling functions using wavelet sequence

D4 wavelet sequence

$$\boldsymbol{g} = \begin{bmatrix} -h_4 \\ h_3 \\ -h_2 \\ h_1 \end{bmatrix} = \begin{bmatrix} -0.4830 \\ 0.8365 \\ -0.2241 \\ -0.1294 \end{bmatrix}$$

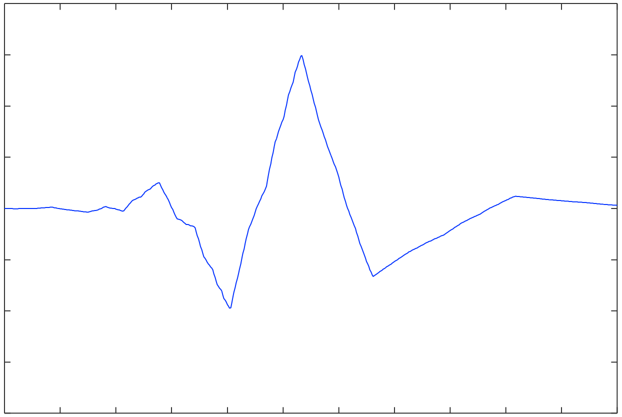
Daubechies wavelets

D4 Daubechies wavelets
(large scale limit)



orthogonal to **constant** +
linear functions

D6 Daubechies wavelets
(large scale limit)



orthogonal to **constant** +
linear + **quadratic** functions

- higher-order wavelets have more vanishing moments (**D2N** Daubechies have **N** vanishing moments)
- higher order may achieve better compression ratios
- many other wavelet families (e.g. Coiflets, Symlets...)

Introduction

G.E., Steven. R. White, Phys. Rev. Lett 116. 140403 (April `16).
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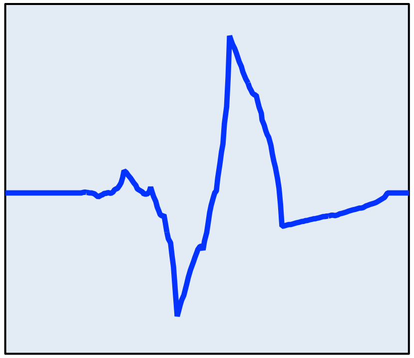
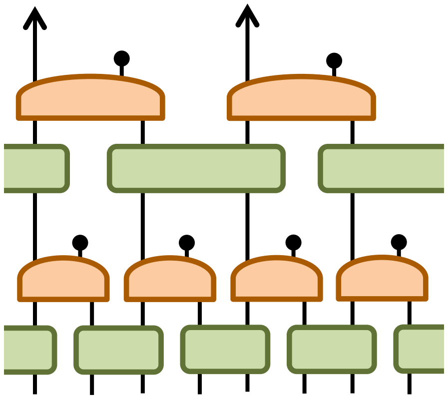
Real-space renormalization and wavelets have many conceptual similarities...
... but can one establish a precise connection?





classical multi-scale methods (wavelets) ⇔ **quantum** multi-scale methods (renormalization group and MERA tensor networks)

Free fermion systems:

Wavelet transform of fermionic modes precisely corresponds to **Gaussian MERA**

More generally:

MERA can be interpreted as the **generalization of wavelets** from ordinary functions to many-body wavefunctions

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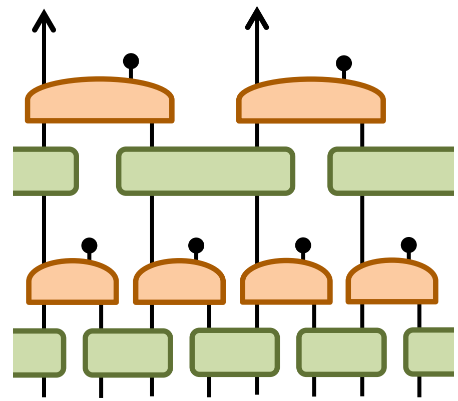

wavelets ⇄ **MERA**

Applications: →

- better understanding of MERA
- construction of analytic examples of MERA (e.g. for Ising CFT)
- analytic error bounds for MERA?

Applications: ←

- design of better wavelets (e.g. for image compression)

Outline: Entanglement renormalization and Wavelets

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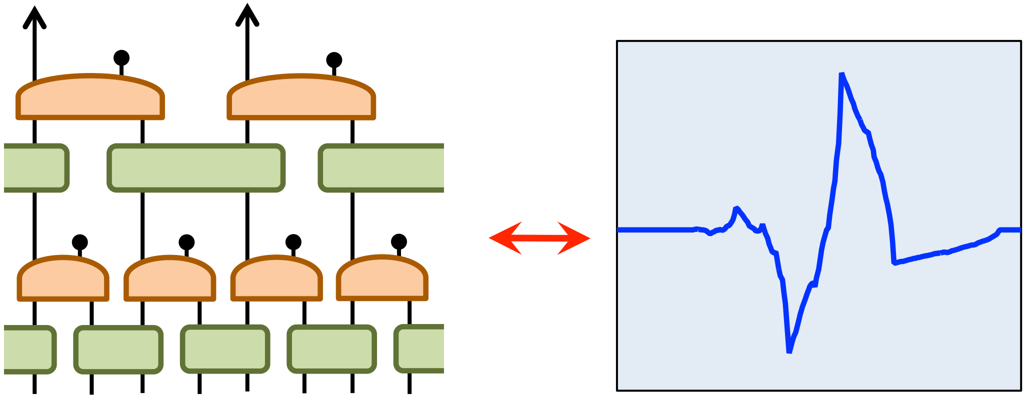
Introduction

Wavelet solution to free fermion model

Representation of wavelets as unitary circuits

Benchmark calculations from wavelet based MERA

Further application of wavelet – MERA connection



Wavelets for free fermions

Can we expand the ground state of free spinless fermions as wavelets?

$$H_{\mathrm{FF}} = \frac{1}{2}\sum_r \left(\underbrace{\hat{a}_r^\dagger \hat{a}_{r+1}}_{\text{hopping term}} + \mathrm{h.c.} \right)$$

dispersion relation: $\Lambda_k = \cos\left(2\pi k / N\right)$

$$H_{\mathrm{FF}} = \int_{-\pi}^{\pi} \Lambda_k \hat{c}_k^\dagger \hat{c}_k \, dk$$

first consider plane waves:

$\hat{a}_r$

spatial modes

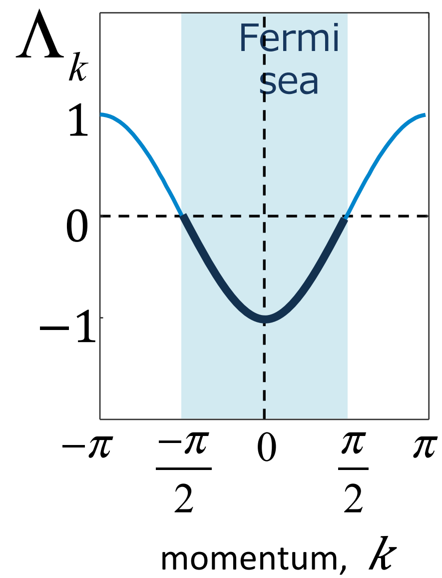
Fourier Transform

$$\hat{c}_k = \frac{1}{\sqrt{N}} \sum_r \hat{a}_r e^{-i2\pi kr/N}$$

fourier modes

ground state is given by filling in negative energy states (fermi-sea):

$$\langle \psi_{GS} | \hat{c}_k^\dagger c_k | \psi_{GS} \rangle = \begin{cases} 0 & \Lambda_k > 0 \\ 1 & \Lambda_k < 0 \end{cases}$$



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$\hat{a}_r$

spatial modes

Wavelet transform

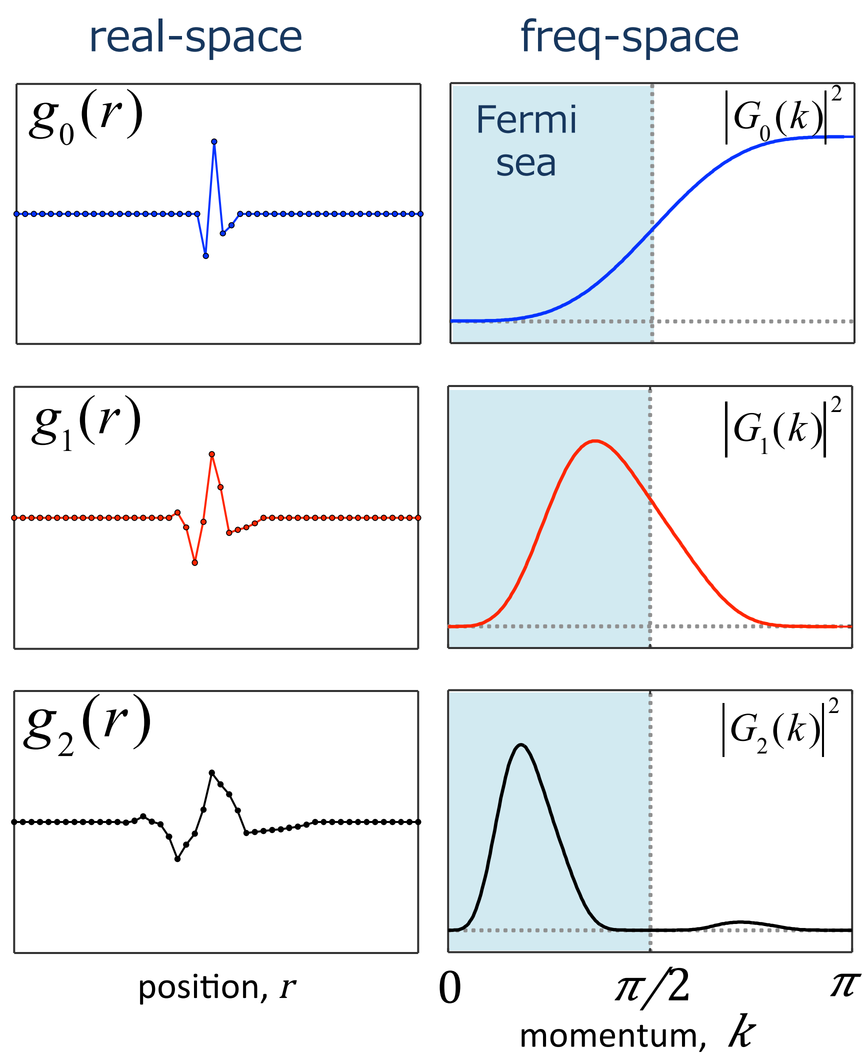
$$\hat{b}_z = \sum_r g_z \hat{a}_r$$

wavelet modes

Not suitable!

- standard wavelets target $\mathrm{k} = 0$
- want wavelets that target $\mathrm{k} = \pm\pi/2$

D4 daubechies wavelets



Wavelets for free fermions

$\hat{a}_r$

spatial modes

Wavelet transform

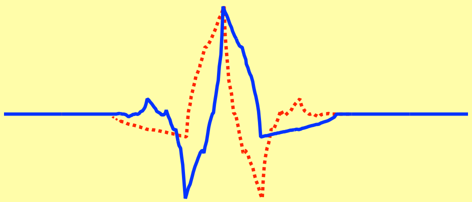
$$\hat{b}_z^+ = \sum_r l_z \hat{a}_r$$

symmetric wavelets

$$\hat{b}_z^- = \sum_r h_z \hat{a}_r$$

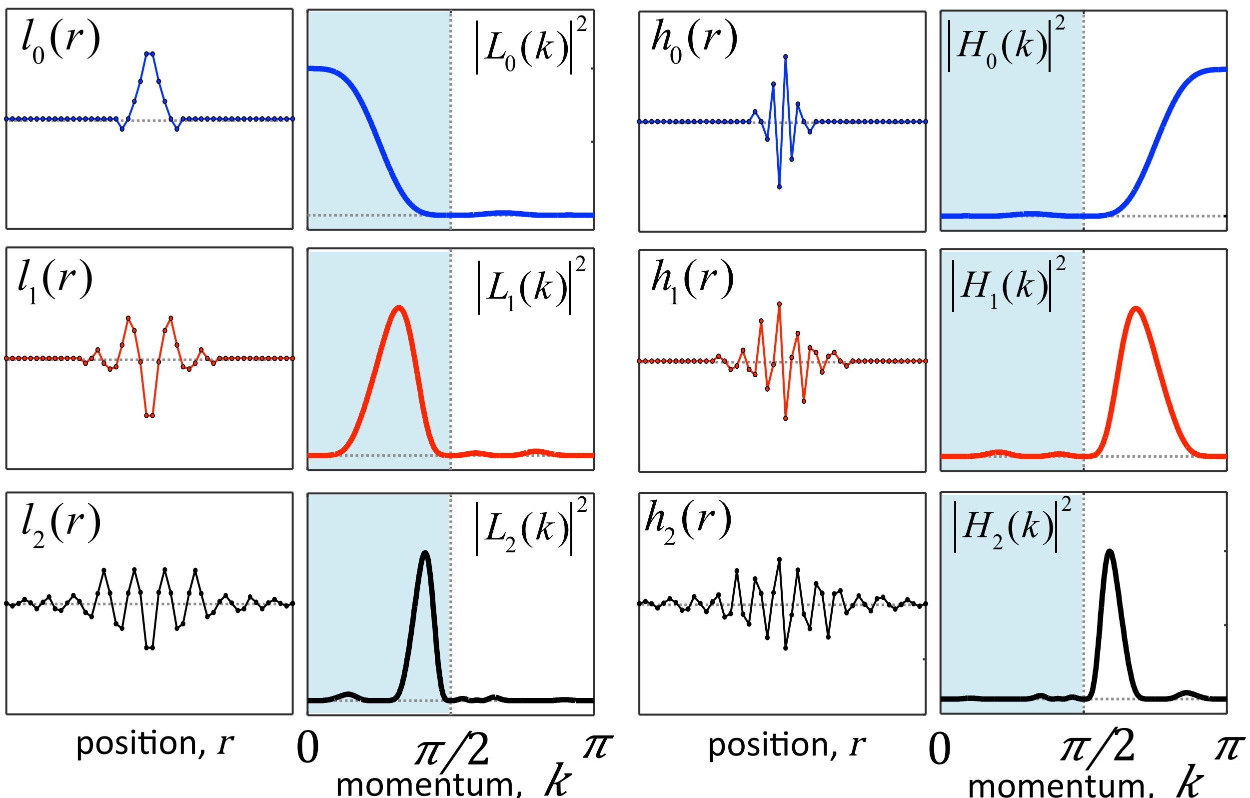
antisymmetric wavelets

Solution: take symmetric and antisymmetric combination of two copies of D4 daubechies wavelets:



symmetric (low freq)

antisymmetric (high freq)



Wavelets for free fermions

$\hat{a}_r$

spatial modes

ground state is approximated by filling in symmetric (low freq.) wavelet modes: $|\psi_{GS}\rangle = \prod_z \hat{b}_z^+ |0\rangle$

- how accurate is this approximation?
- can this be improved? Later!

Wavelet transform

$$\hat{b}_z^+ = \sum_r l_z \hat{a}_r$$

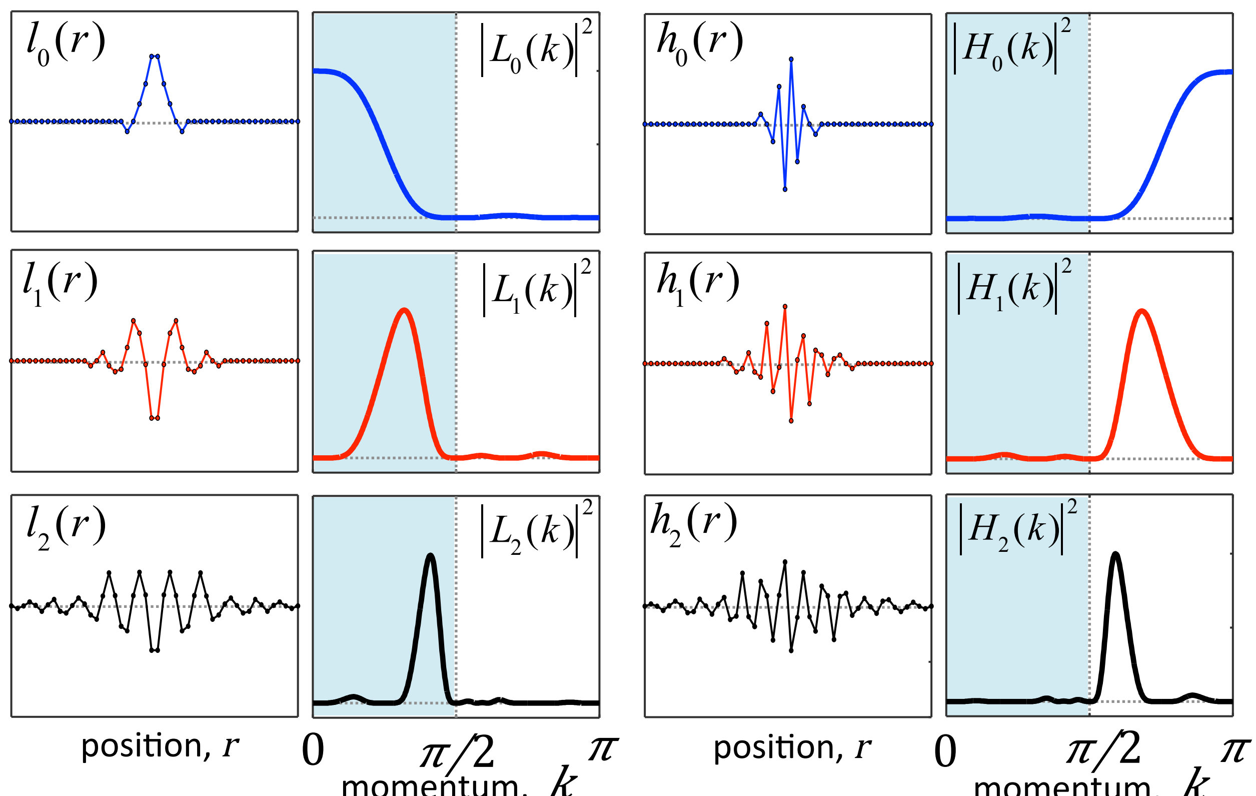
symmetric wavelets

$$\hat{b}_z^- = \sum_r h_z \hat{a}_r$$

antisymmetric wavelets

symmetric (low freq)

antisymmetric (high freq)



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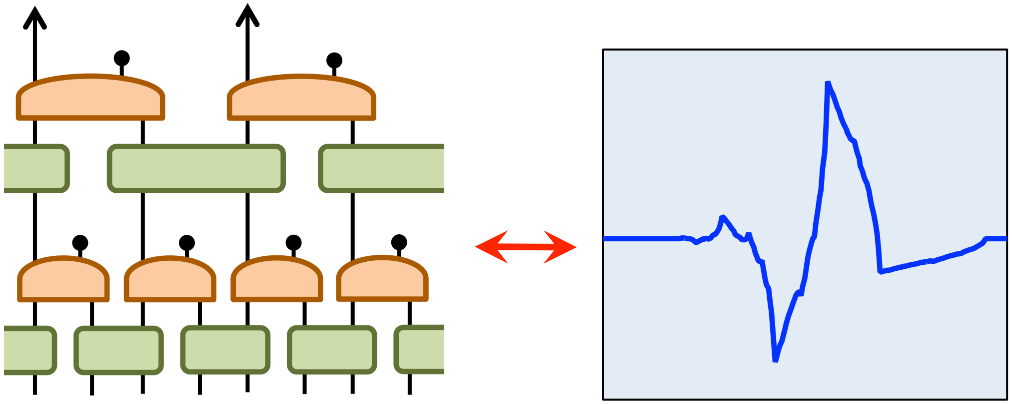
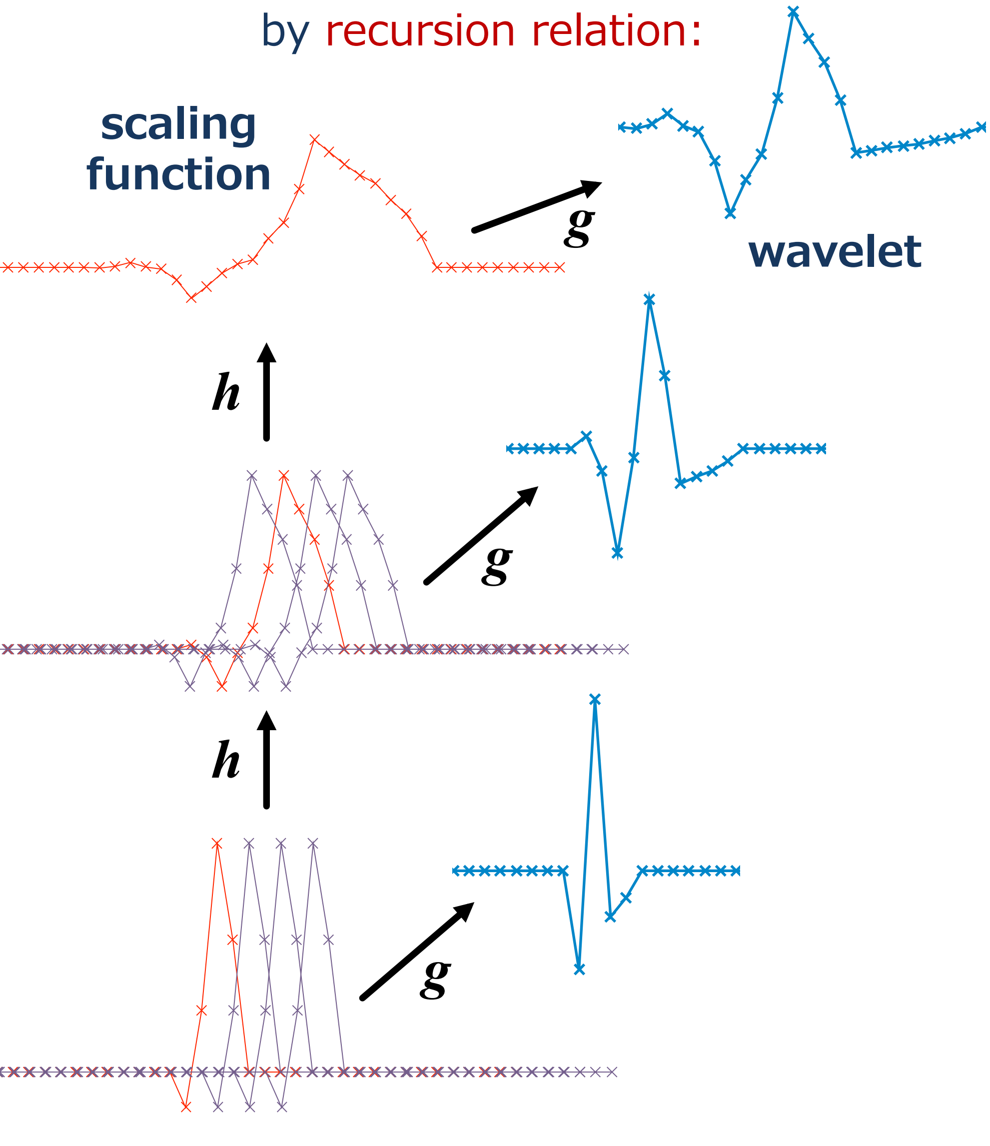
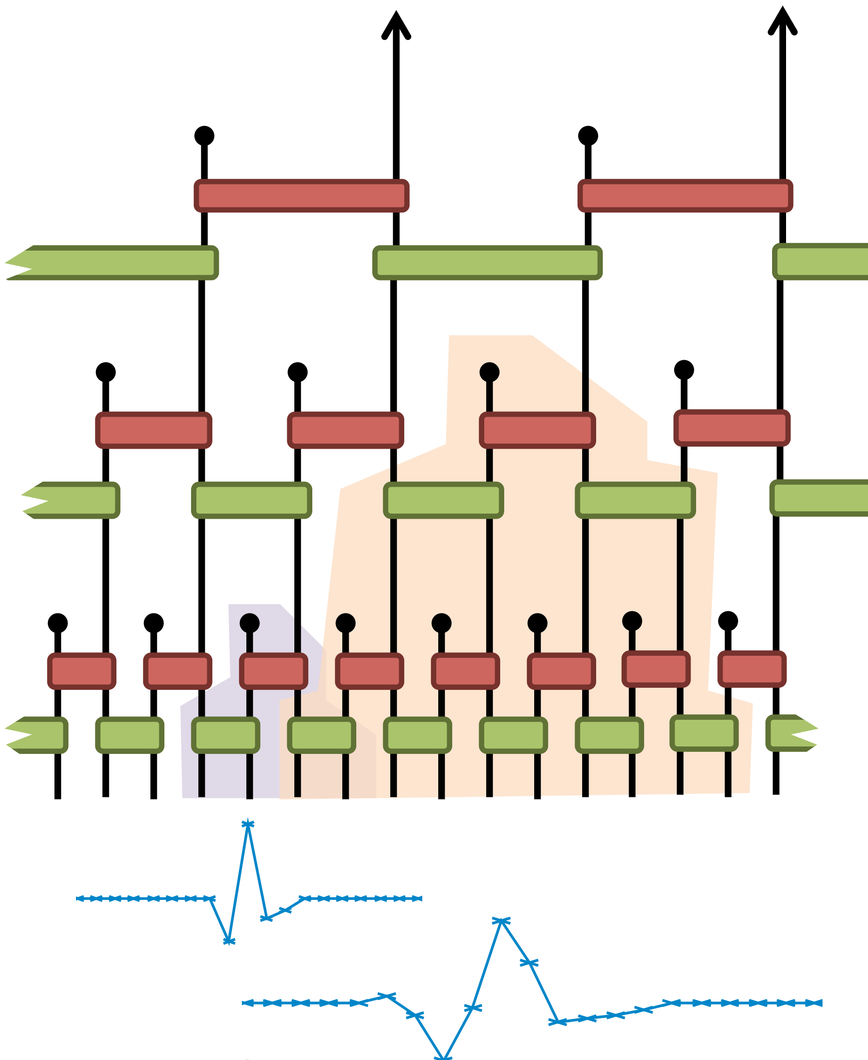


Circuit representation of wavelets

Wavelet transform described by recursion relation:



Recursion relation can be encoded as a (classical) unitary circuit:



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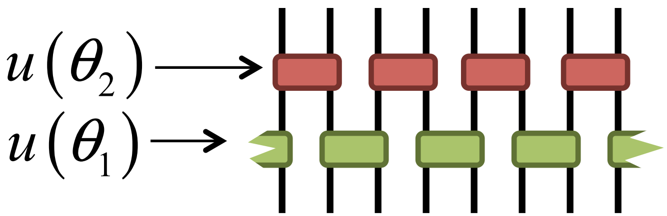
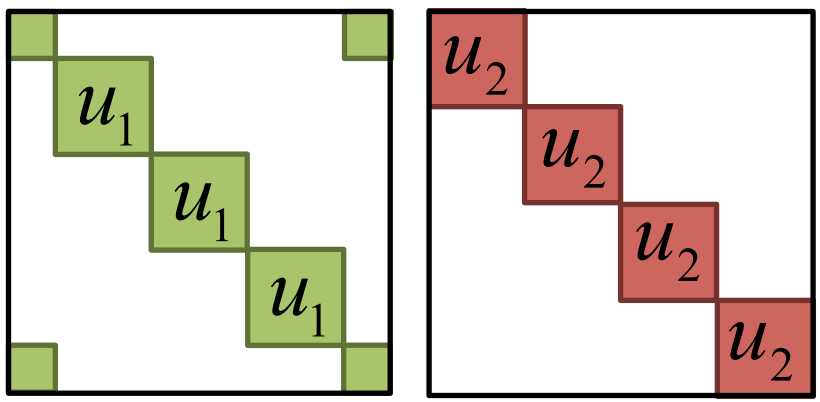
Circuit representation of wavelets

Diagrammatic notation:



$$u(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$



Wavelet transform maps from vector of **N scalars** to vector of **N scalars**!

Classical circuit here represents **direct sum** of unitaries (not **tensor product!**)

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Circuit representation of wavelets

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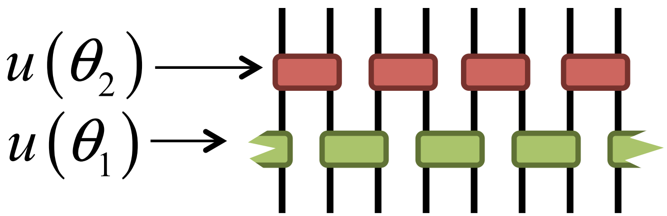

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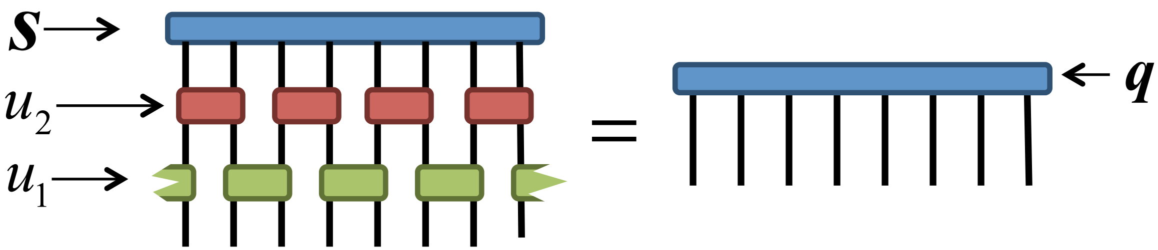

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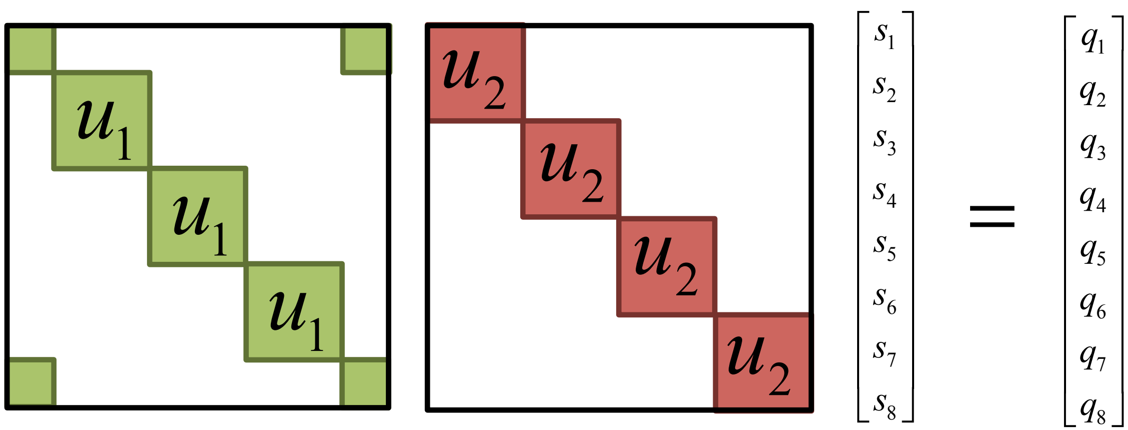

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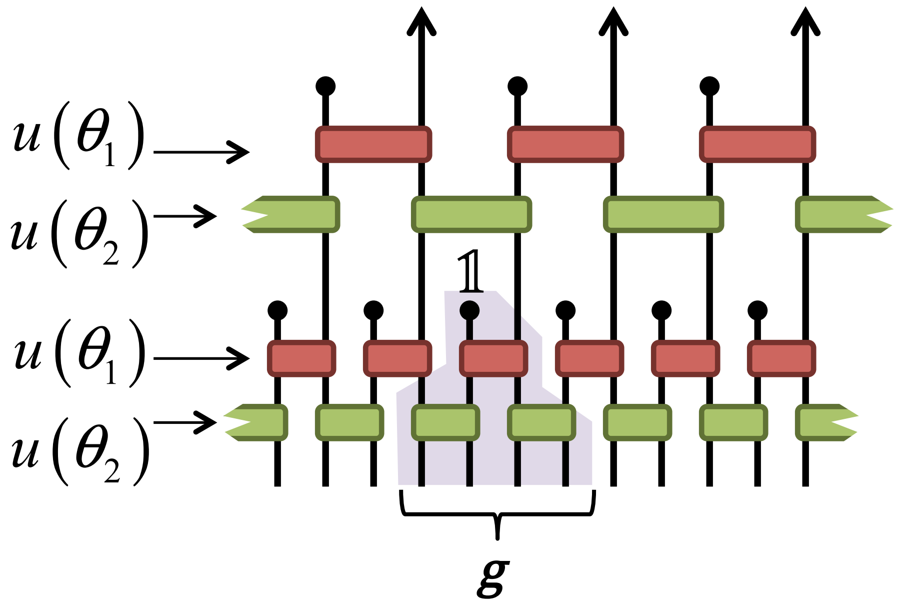
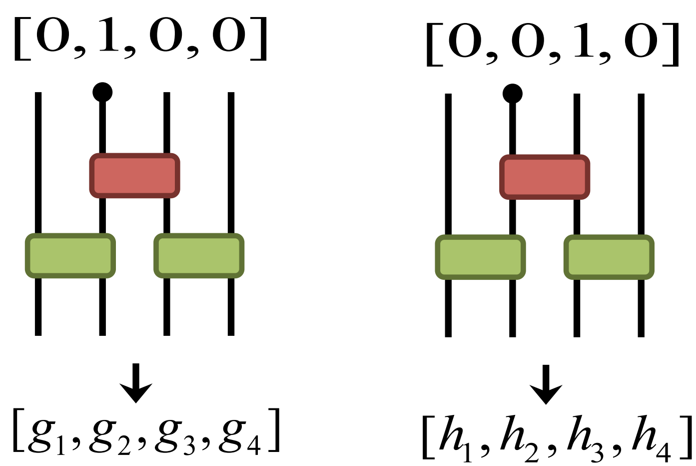
Circuit representation of wavelets



wavelet sequence
associated to inverse transforming unit vector (odd sublattice)

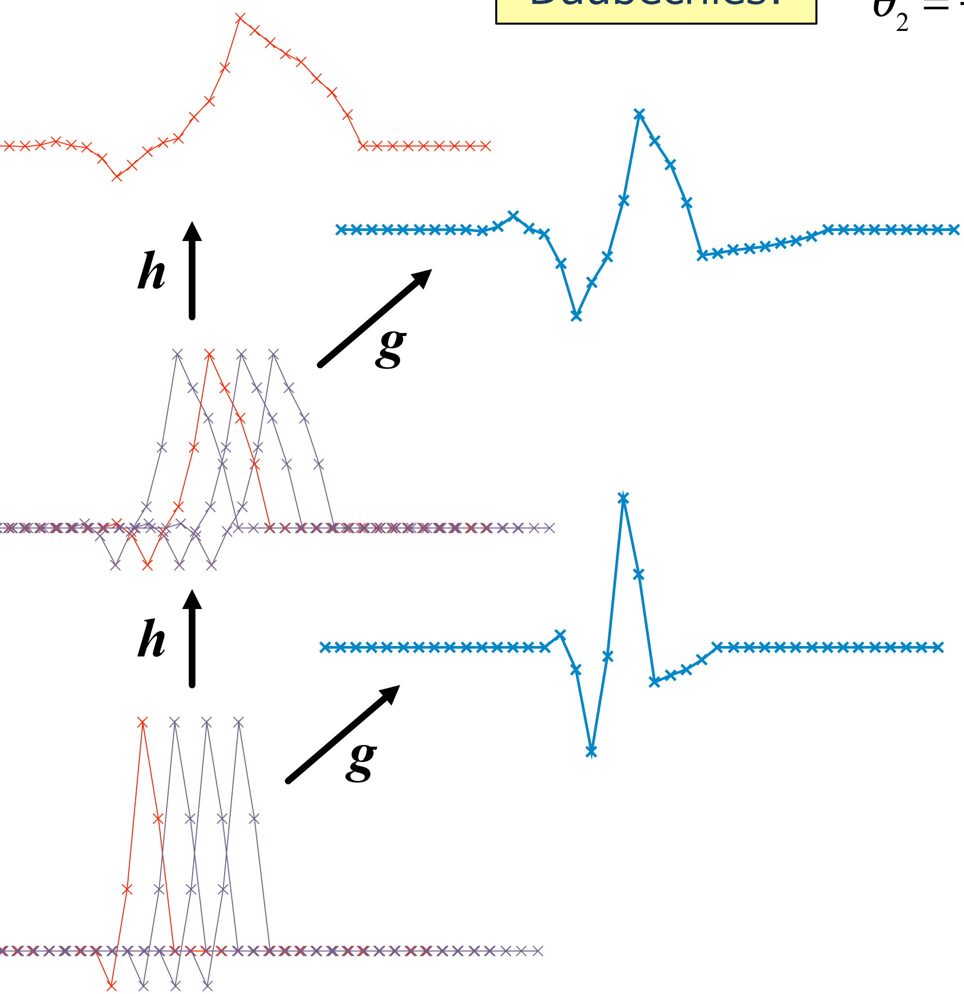


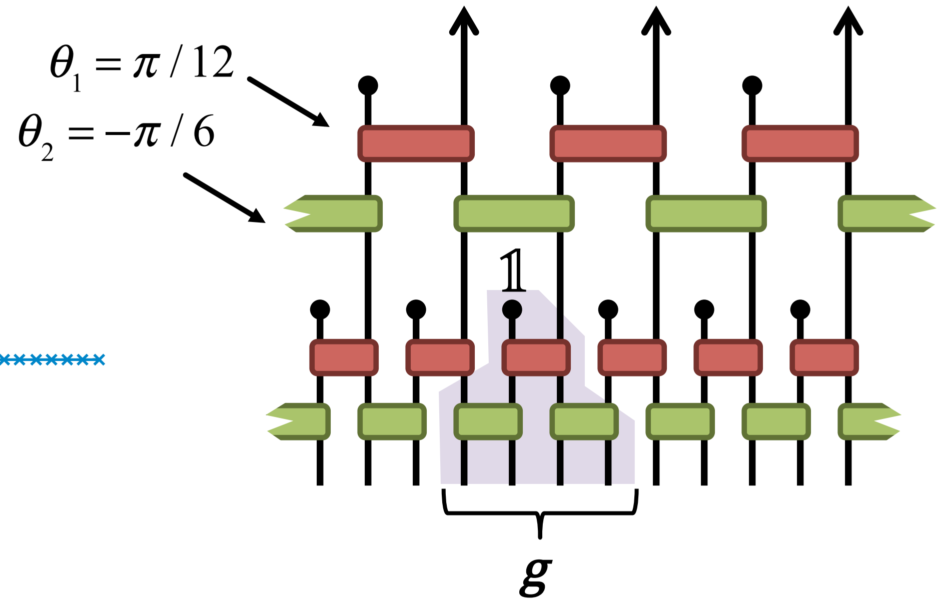
scaling sequence
associated to inverse transforming unit vector (even sublattice)

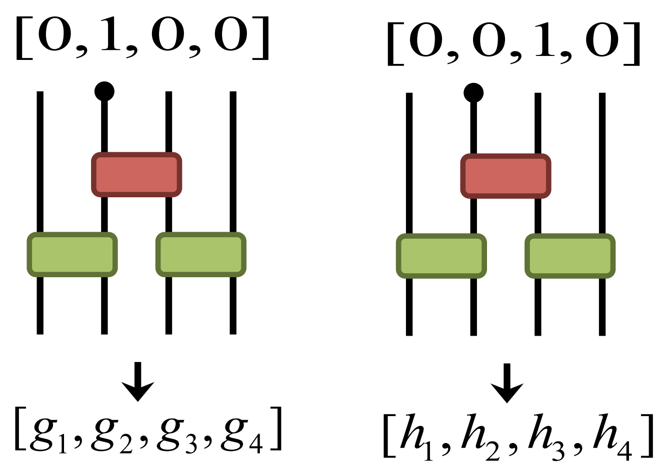
G.E., Steven. R. White, arXiv: 1605.07312 (May `16).

Circuit representation of wavelets

Reproduce D4 Daubechies:







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Circuit representation of wavelets

$$\begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \\ s_7 \\ s_8 \end{bmatrix} \xrightarrow{\text{wavelet transform}} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \end{bmatrix}$$

wavelet transform

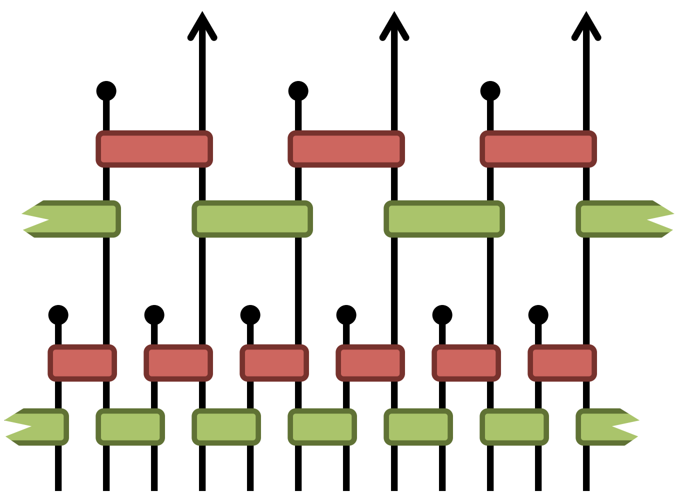
(can be realised as a classical unitary circuit)

N scalars → **N scalars**

$$\begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \\ \hat{a}_4 \\ \hat{a}_5 \\ \hat{a}_6 \\ \hat{a}_7 \\ \hat{a}_8 \end{bmatrix} \xrightarrow{\text{wavelet transform}} \begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \\ \hat{b}_4 \\ \hat{b}_5 \\ \hat{b}_6 \\ \hat{b}_7 \\ \hat{b}_8 \end{bmatrix}$$

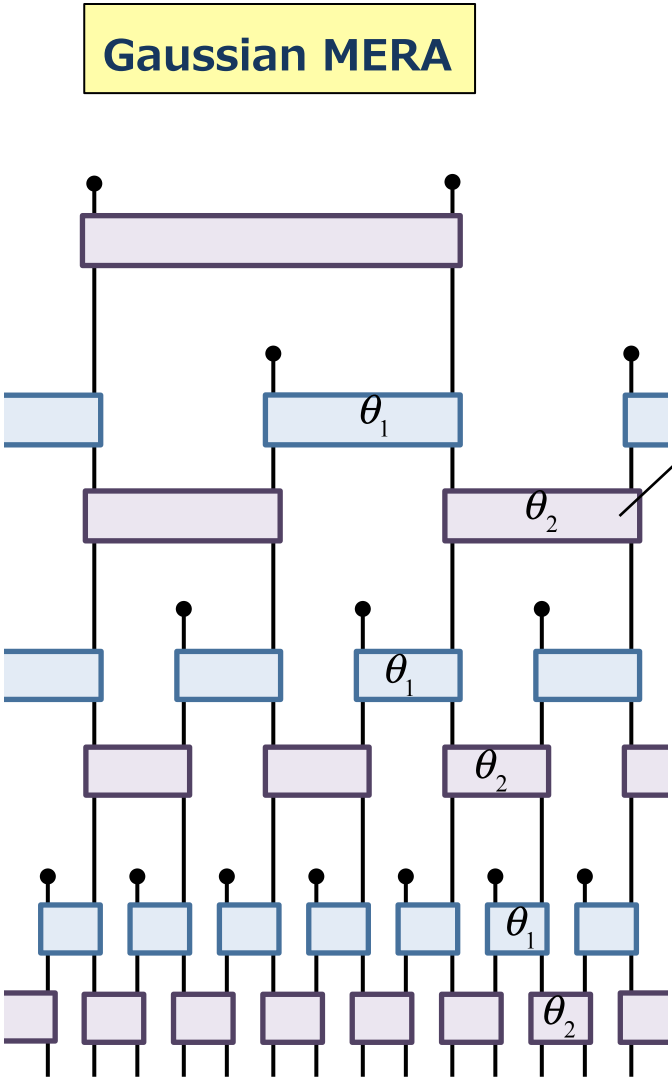
wavelet transform

N fermionic modes → **N fermionic modes**



Gaussian MERA!

MERA for free fermions



- MERA where unitary gates map fermionic modes linearly:

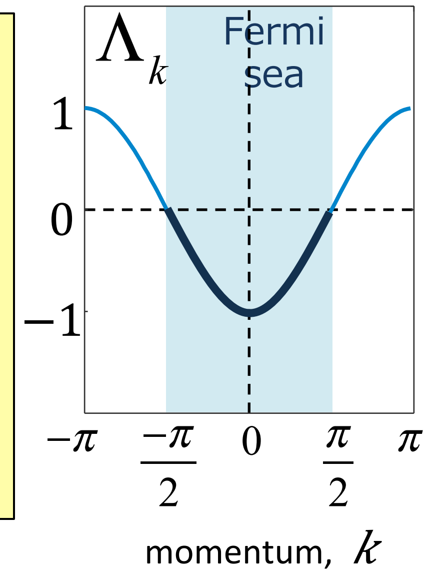
$$\begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \end{bmatrix}$$

- two sites gates are parameterised by a **single angle**

Free fermions in 1D:

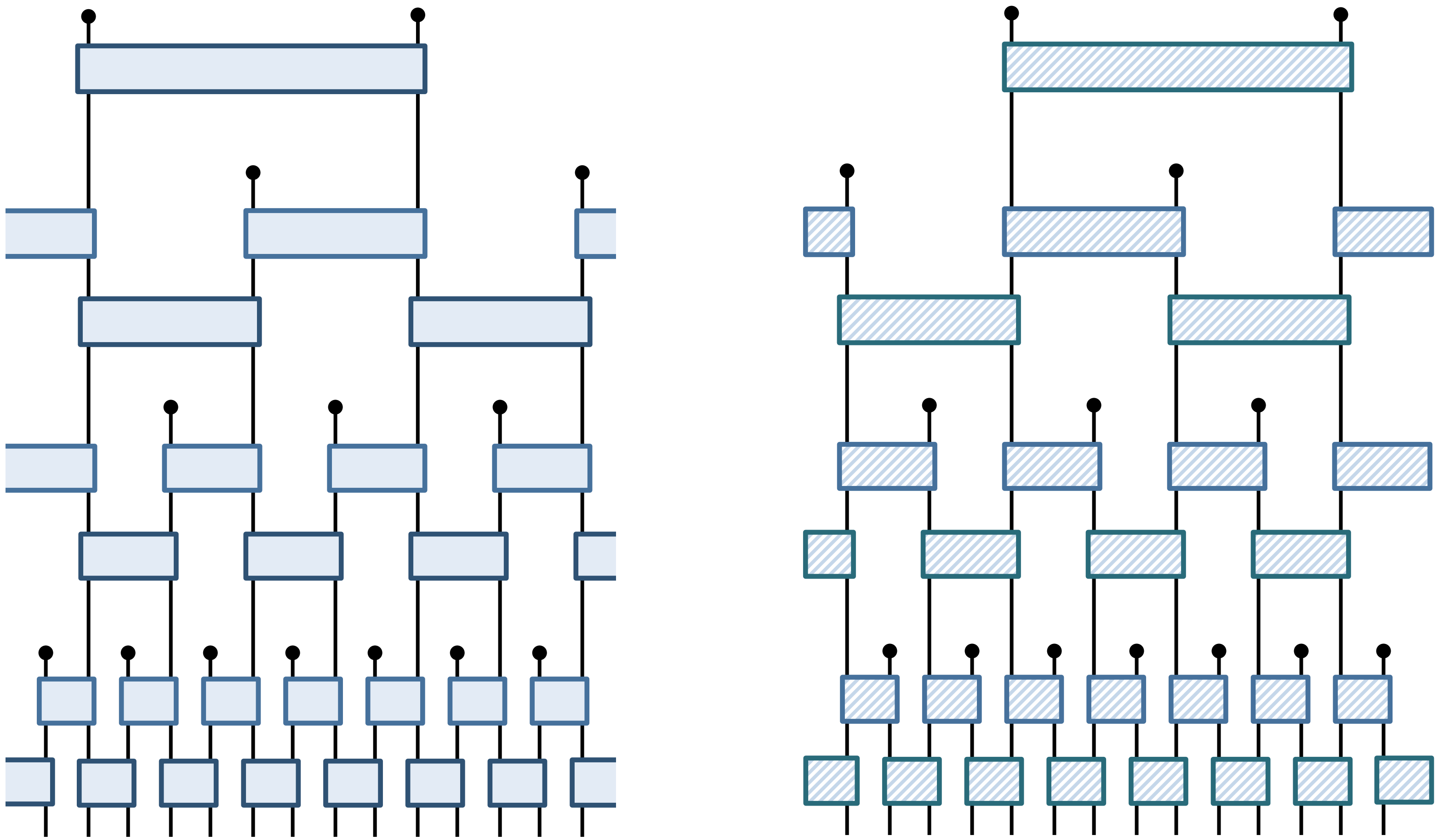
$$H_{\text{FF}} = \frac{1}{2} \sum_r \left(\hat{a}_r^\dagger \hat{a}_{r+1} + \text{h.c.} \right)$$

Can we express the wavelet solution for the ground state as a MERA?

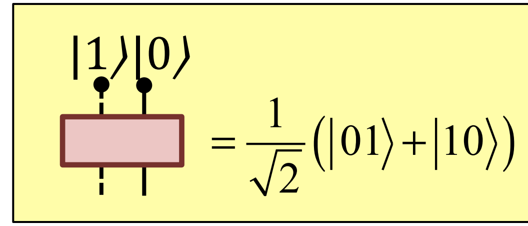


MERA for free fermions

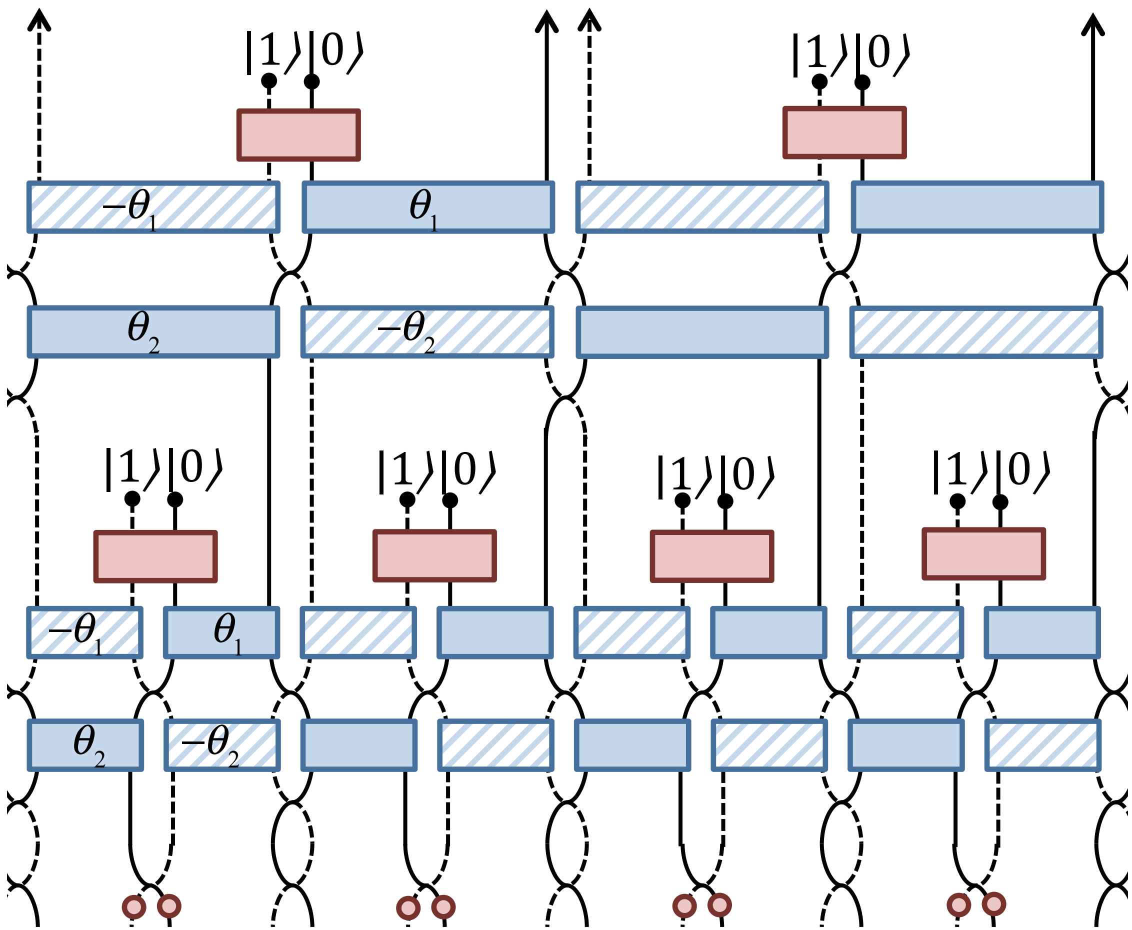
- Take two copies of Gaussian MERA that implement the **D4 Daubechies wavelet** transform
- Combine and then symmetrise



MERA for free fermions



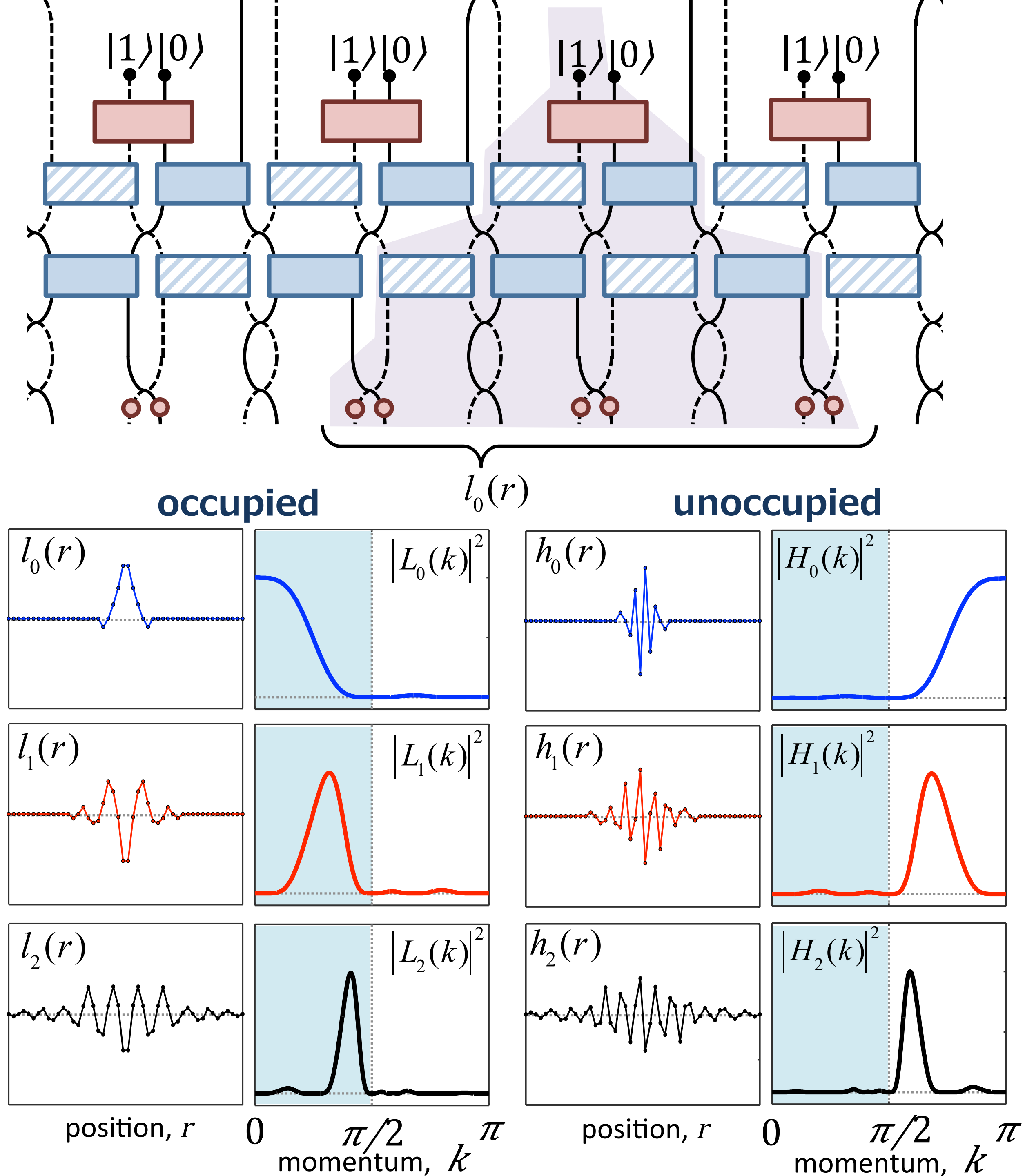
Quantum circuit which (approximately) prepares the ground state of 1D free fermions:



$$\theta_1 = \pi / 12$$

$$\theta_2 = -\pi / 6$$

MERA for free fermions



MERA for free fermions

gate angles: $\theta_1 = \pi/12$

$\theta_2 = -\pi/6$

Free fermions at half-filling:

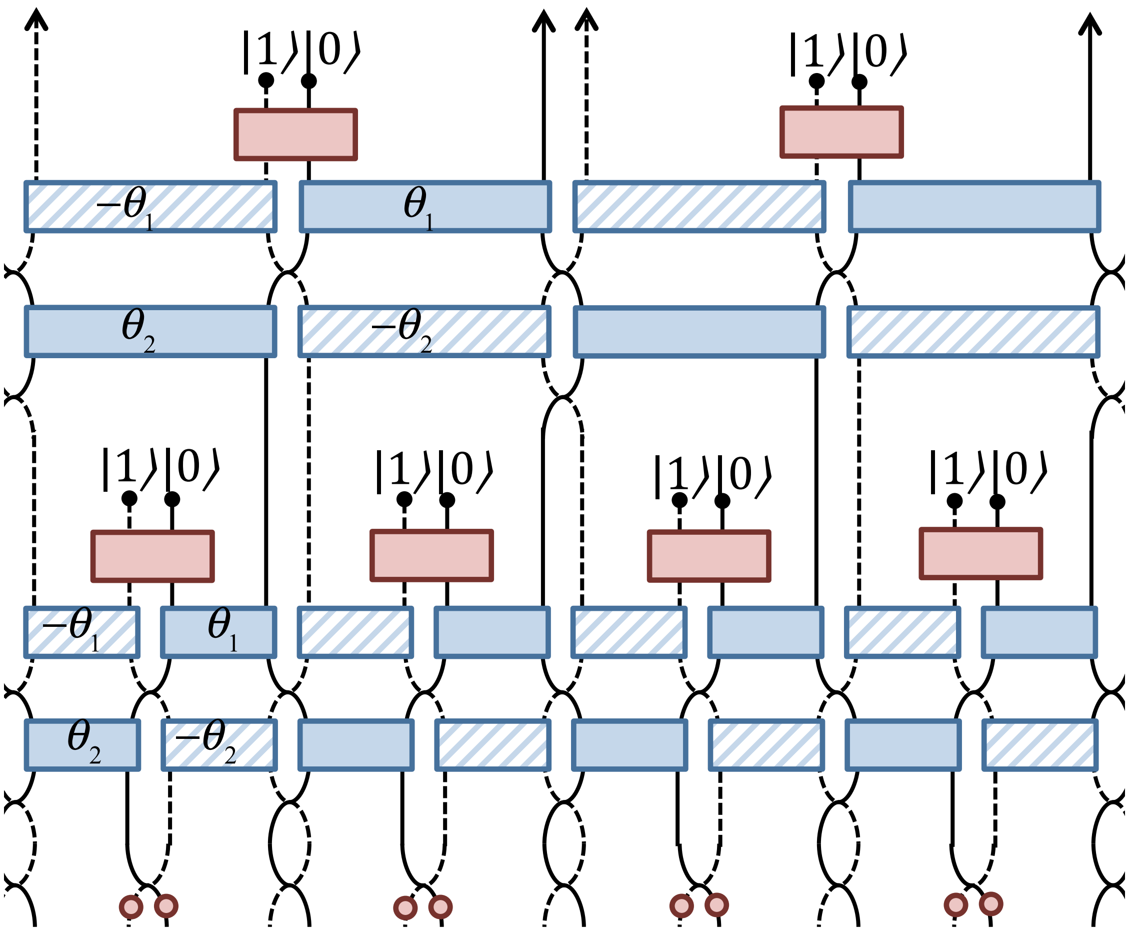
$$H_{\text{FF}} = \frac{1}{2}\sum_r \left(\hat{a}^\dagger_{r+1}\hat{a}_r + h.c.\right) - \mu \sum_r \hat{a}^\dagger_r \hat{a}_r$$

unitary circuit offers accurate (**real-space**) approximation to the ground state $|\psi_{GS}\rangle$ in terms of:

- ground energy and local observables
- entanglement entropy

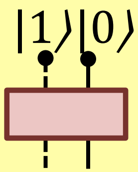
$$S_L = \frac{c}{3}\log(L) + \text{const.}$$

- conformal data (scaling dimensions, OPE coefficients, central charge)
- RG flow of the Hamiltonian (flows to gapless fixed point)

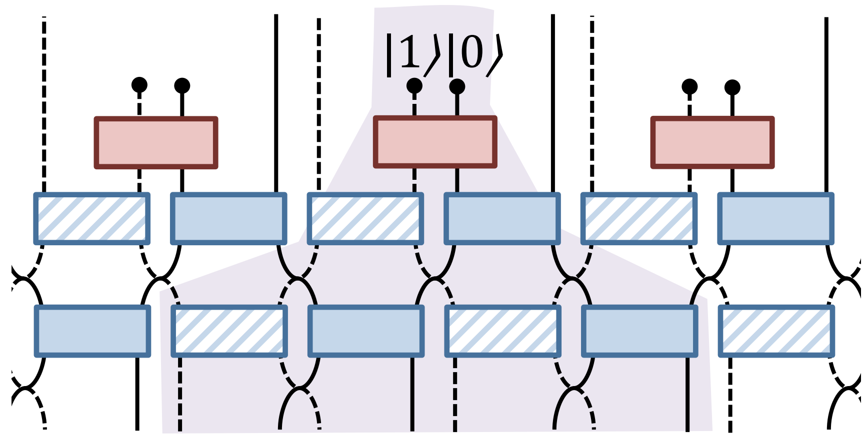


MERA for free fermions

$\pi/4$ gate creates
entangled state
in the bulk:

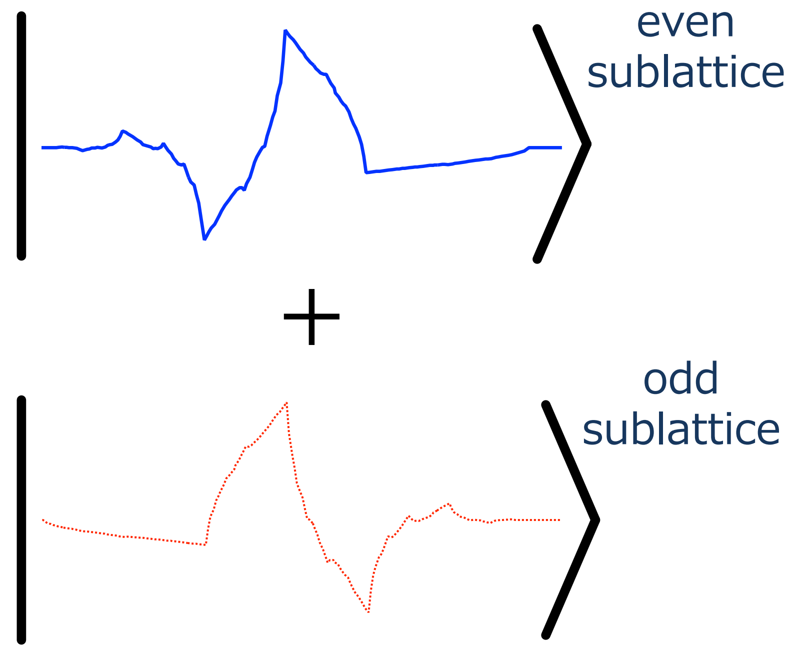


$$= \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right)$$



Unitary circuit then
'smears out' particles
on the boundary

single particle
wavefunction:

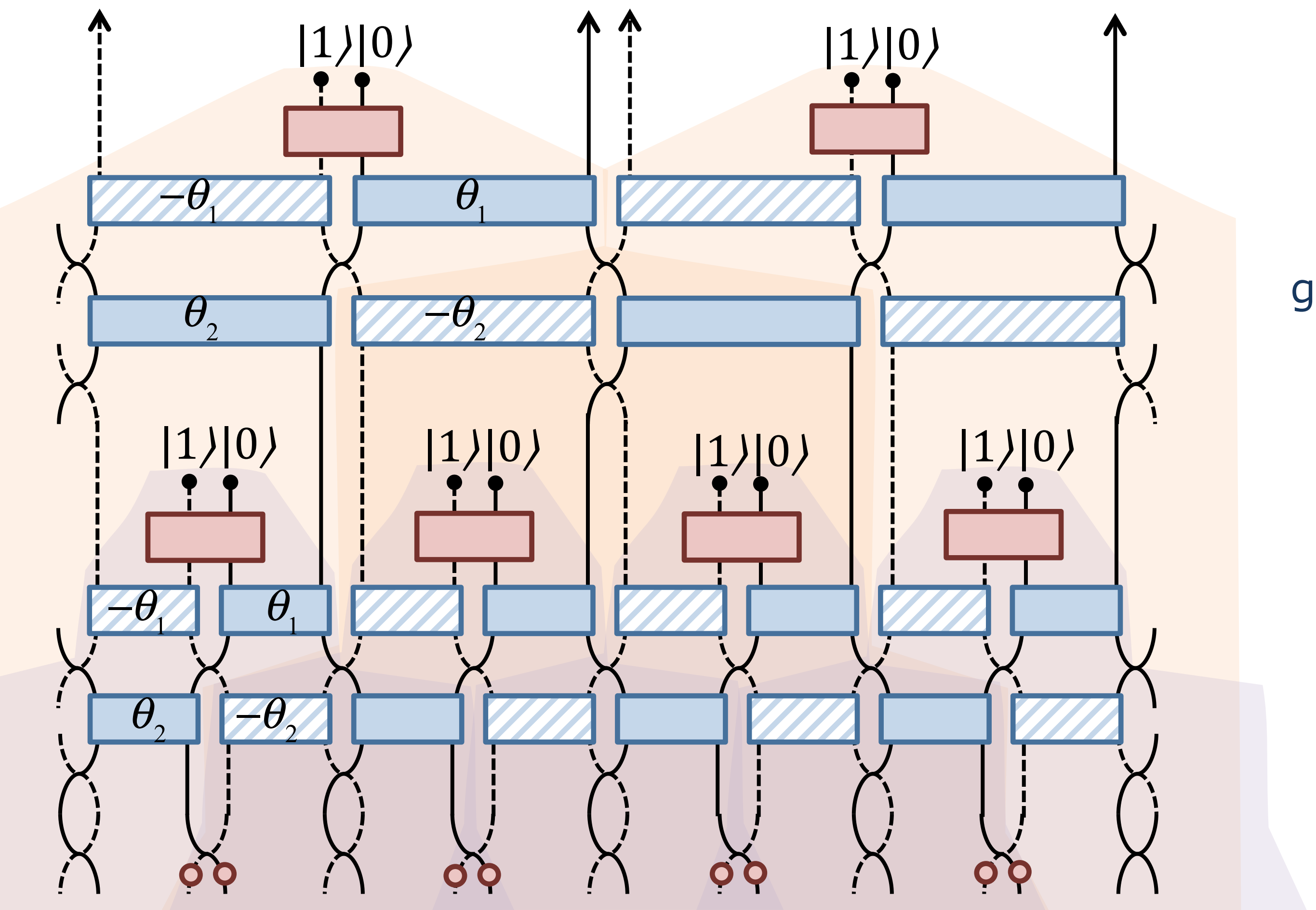


MERA for free fermions

Free fermions at half-filling:

$$H_{\mathrm{FF}} = \frac{1}{2}\sum_r \left(\hat{a}^\dagger_{r+1}\hat{a}_r + h.c.\right) - \mu \sum_r \hat{a}^\dagger_r \hat{a}_r$$

gate angles:	$\theta_1 = \pi / 12$ $\theta_2 = -\pi / 6$



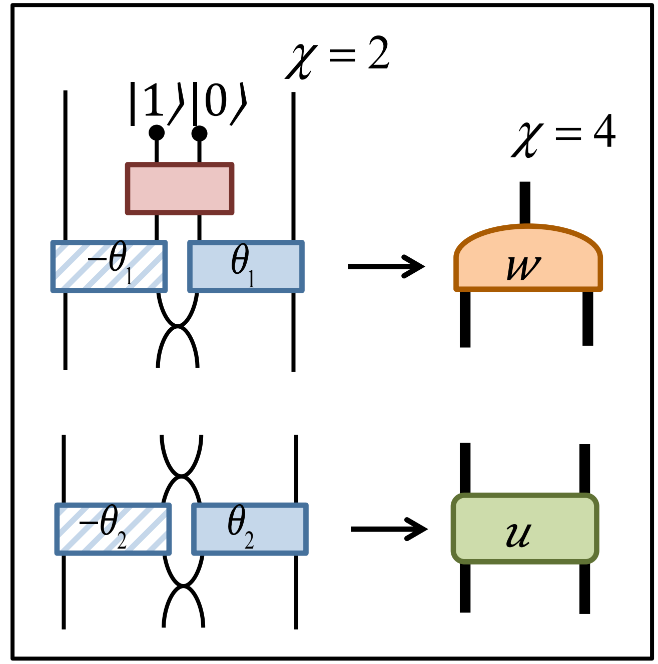
higher-level tensors generate **longer**-ranged entanglement

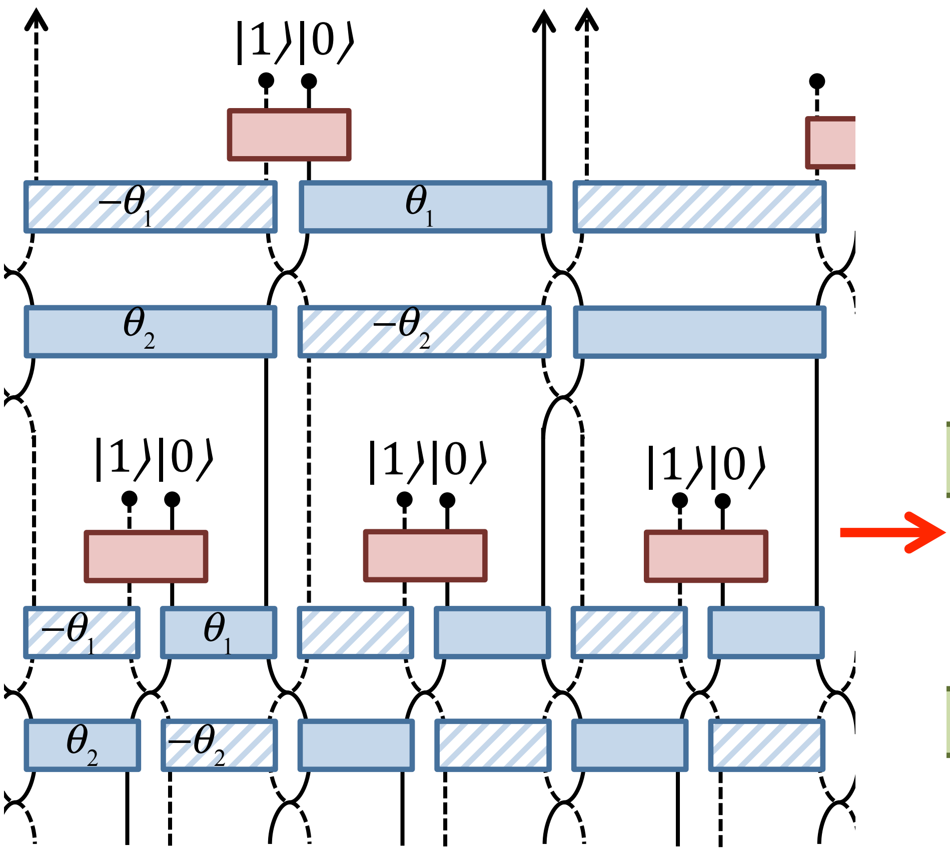
low-level tensors generate **short**-ranged entanglement

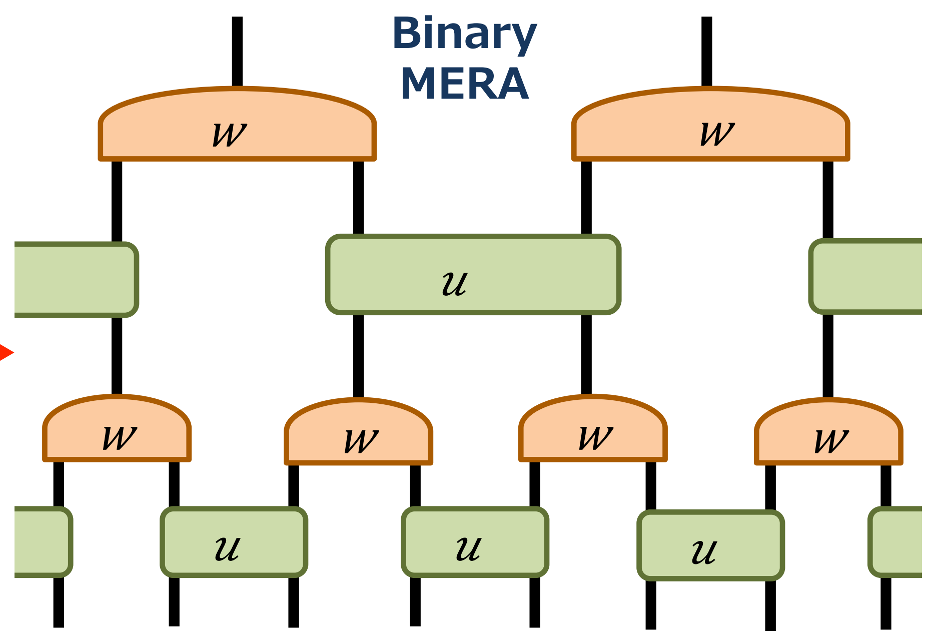
MERA for free fermions

Is this circuit related to the standard (binary) MERA?

Yes! Just group gates together







MERA for critical Ising

Free fermions at half-filling

$$H_{\mathrm{FF}} = \frac{1}{2}\sum_r \left(\hat{a}^\dagger_{r+1}\hat{a}_r + h.c.\right)$$

Express as 2N majorana fermions

$$H_{\mathrm{FF}} = \sum_r i\left(\breve{d}_{2r}\breve{d}_{2r+1} - \breve{d}_{2r-1}\breve{d}_{2r+2}\right)$$

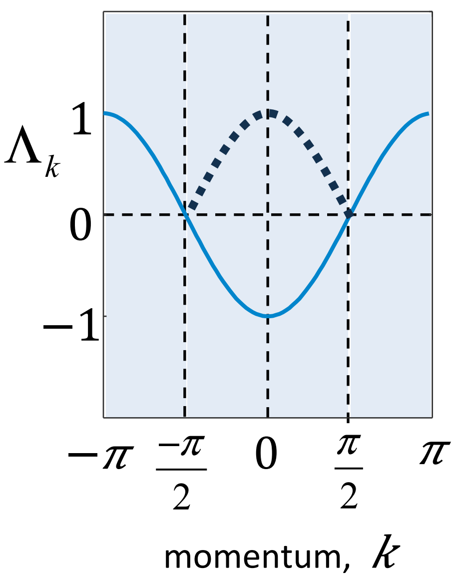
Decouple (via local unitaries) into 2 copies of free majorana fermions

$$H_{\mathrm{FM}} = \sum_r i\left(\breve{d}_r\breve{d}_{r+1} - \breve{d}_{r+1}\breve{d}_r\right)$$

Jordan-Wigner

$$H_{\mathrm{Ising}} = \sum_r \left(-X_r X_{r+1} + Z_r\right)$$

Quantum critical Ising model



Can one get a representation of the ground state of the quantum **critical Ising Model?**

MERA for critical Ising

Free fermions at half-filling

$$H_{\mathrm{FF}} = \frac{1}{2}\sum_r \left(\hat{a}_{r+1}^{\dagger}\hat{a}_r + h.c. \right)$$

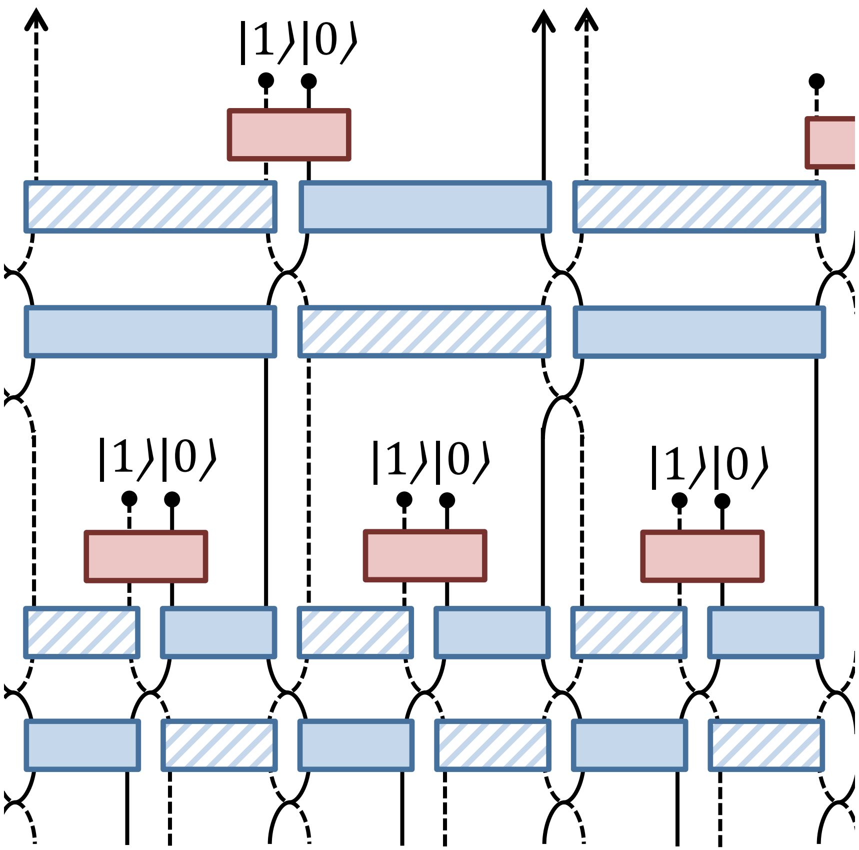
Express as 2N majorana fermions

$$H_{\mathrm{FF}} = \sum_r i\left(\breve{d}_{2r}\breve{d}_{2r+1} - \breve{d}_{2r-1}\breve{d}_{2r+2} \right)$$

Decouple (via local unitaries) into 2 copies of free majorana fermions

$$H_{\mathrm{FM}} = \sum_r i\left(\breve{d}_r\breve{d}_{r+1} - \breve{d}_{r+1}\breve{d}_r \right)$$

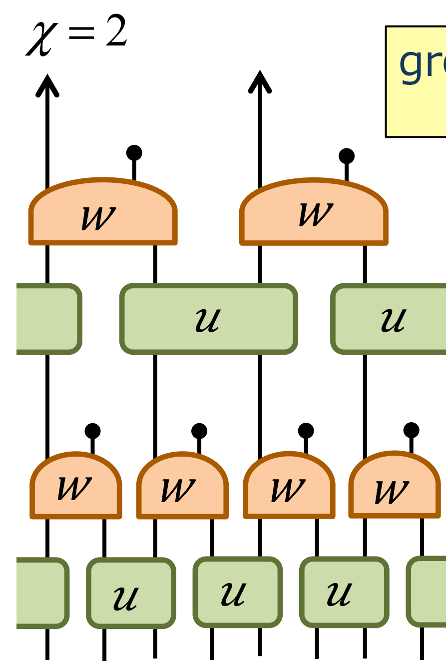
ground state



Jordan-Wigner

$$H_{\mathrm{Ising}} = \sum_r \left(-X_r X_{r+1} + Z_r \right)$$

Quantum critical Ising model



ground state MERA

MERA for critical Ising

Quantum critical Ising model

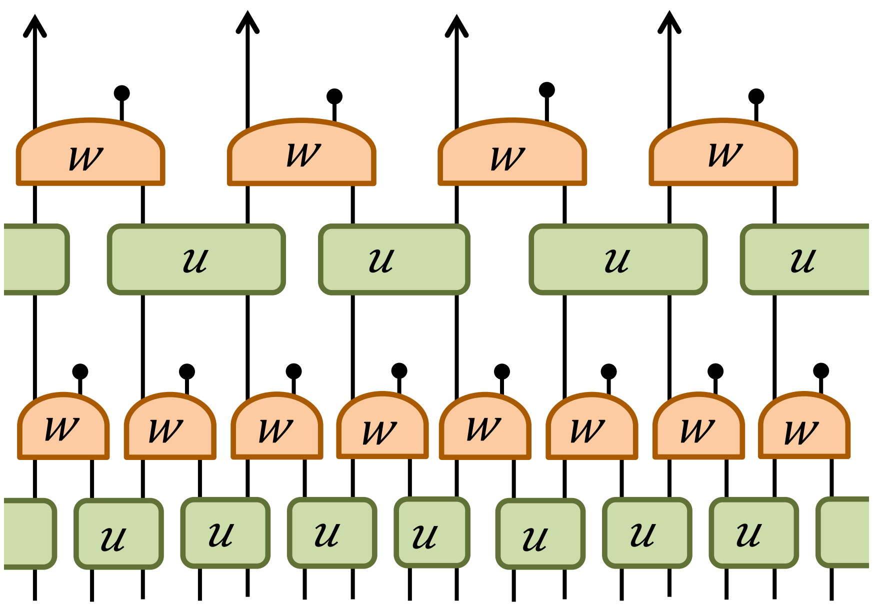
$$H_{\text{Ising}} = \sum_r \left(-X_r X_{r+1} + Z_r\right)$$

Expressed in Pauli matrices:

Isometries: $$w_{r,r+1} = \left(\tfrac{\sqrt{3}+\sqrt{2}}{4}\right) I_r I_{r+1} + \left(\tfrac{\sqrt{3}-\sqrt{2}}{4}\right) Z_r Z_{r+1} + i\left(\tfrac{1+\sqrt{2}}{4}\right) X_r Y_{r+1} + i\left(\tfrac{1-\sqrt{2}}{4}\right) Y_r X_{r+1}$$

Disentanglers: $$u_{r,r+1} = \left(\tfrac{\sqrt{3}+2}{4}\right) I_r I_{r+1} + \left(\tfrac{\sqrt{3}-2}{4}\right) Z_r Z_{r+1} + \left(\tfrac{i}{4}\right) X_r Y_{r+1} + \left(\tfrac{i}{4}\right) Y_r X_{r+1}$$

Higher order wavelet solutions can also be expressed as a MERA!



Recap:

1. Ground state of 1D free fermions (or critical Ising model) can be approximated as wavelets

2. Wavelet solution precisely corresponds to a MERA

Outline: Entanglement renormalization and Wavelets

G.E., Steven. R. White, **Phys. Rev. Lett 116.** 140403 (April `16).

G.E., Steven. R. White, arXiv: 1605.07312 (May `16).

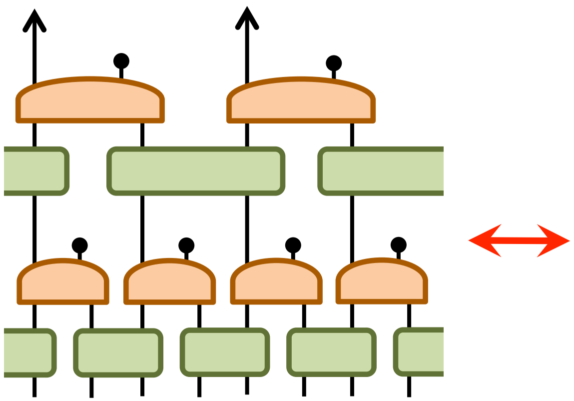
Overview

Wavelet solution to free fermion model

Representation of wavelets as unitary circuits

Benchmark calculations from wavelet based MERA

Further application of wavelet – MERA connection



MERA for critical Ising

Quantum critical Ising model

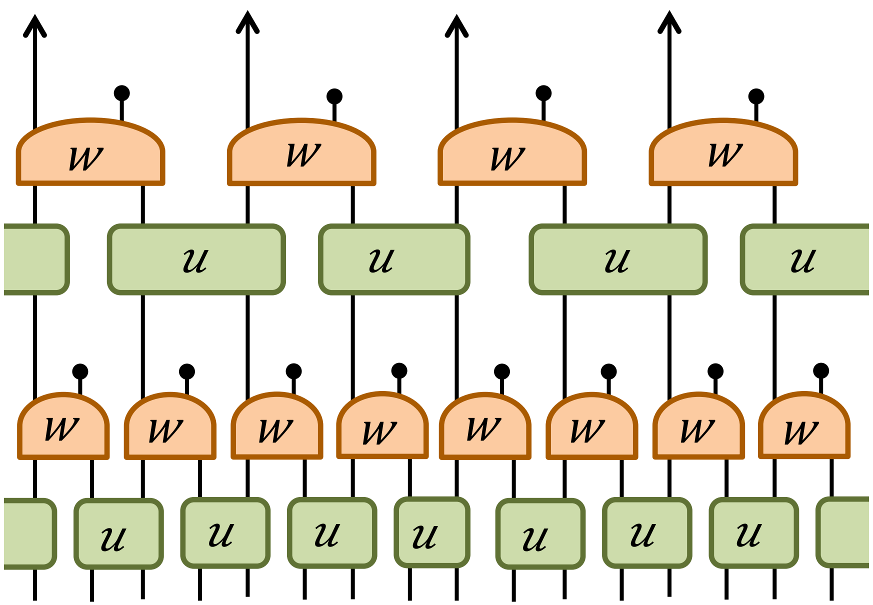
$$H_{\text{Ising}} = \sum_r \left(-X_r X_{r+1} + Z_r\right)$$

Expressed in Pauli matrices:

Isometries: $w_{r,r+1} = \left(\frac{\sqrt{3}+\sqrt{2}}{4}\right) I_r I_{r+1} + \left(\frac{\sqrt{3}-\sqrt{2}}{4}\right) Z_r Z_{r+1} + i\left(\frac{1+\sqrt{2}}{4}\right) X_r Y_{r+1} + i\left(\frac{1-\sqrt{2}}{4}\right) Y_r X_{r+1}$

Disentanglers: $u_{r,r+1} = \left(\frac{\sqrt{3}+2}{4}\right) I_r I_{r+1} + \left(\frac{\sqrt{3}-2}{4}\right) Z_r Z_{r+1} + \left(\frac{i}{4}\right) X_r Y_{r+1} + \left(\frac{i}{4}\right) Y_r X_{r+1}$

How accurate is the wavelet-based ground state MERA?



ground energy

exact:	-1.27323…	
MERA: D4 wavelets	-1.24211	rel. err. 2.4%

Conformal data from MERA

Quantum critical Ising model

$$H_{\text{Ising}} = \sum_r \left(-X_r X_{r+1} + Z_r\right)$$

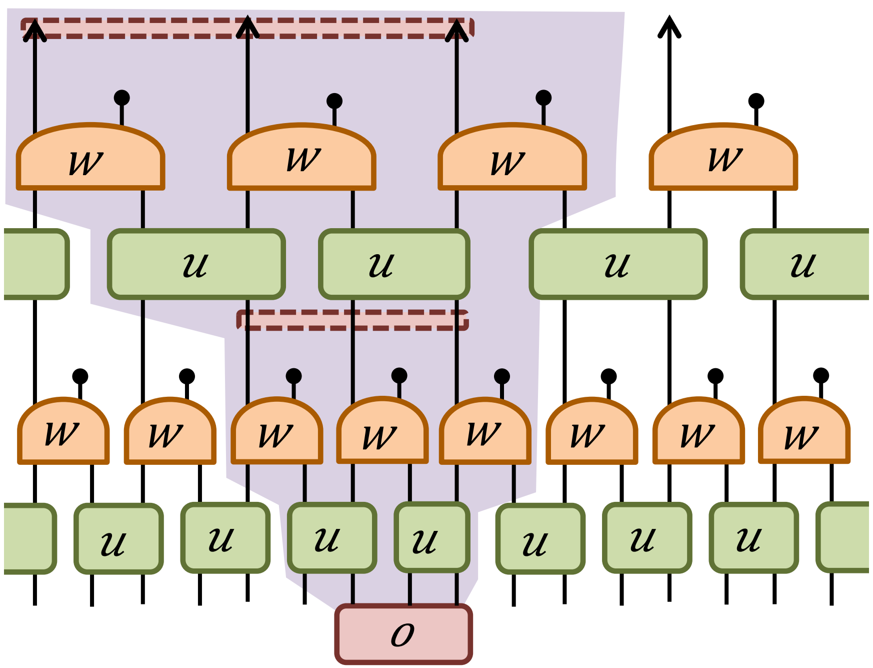
Expressed in Pauli matrices:

Isometries: $w_{r,r+1} = \left(\frac{\sqrt{3}+\sqrt{2}}{4}\right) I_r I_{r+1} + \left(\frac{\sqrt{3}-\sqrt{2}}{4}\right) Z_r Z_{r+1} + i\left(\frac{1+\sqrt{2}}{4}\right) X_r Y_{r+1} + i\left(\frac{1-\sqrt{2}}{4}\right) Y_r X_{r+1}$

Disentanglers: $u_{r,r+1} = \left(\frac{\sqrt{3}+2}{4}\right) I_r I_{r+1} + \left(\frac{\sqrt{3}-2}{4}\right) Z_r Z_{r+1} + \left(\frac{i}{4}\right) X_r Y_{r+1} + \left(\frac{i}{4}\right) Y_r X_{r+1}$

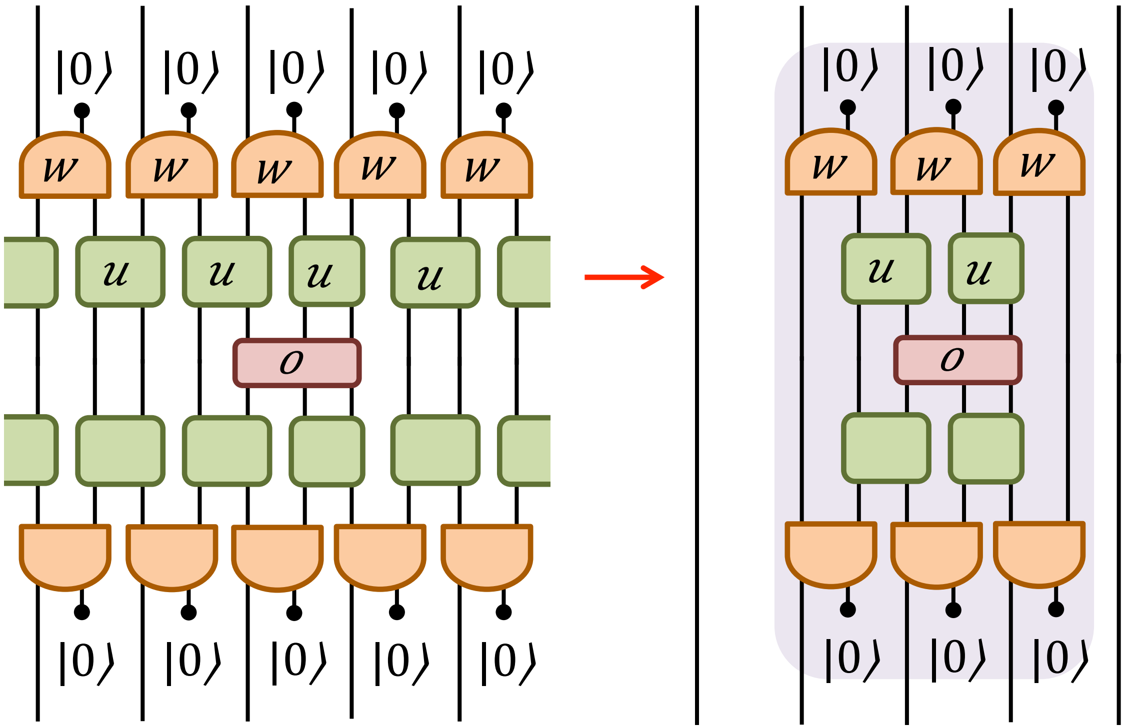
Conformal data from MERA?

Consider coarse-graining local operators...

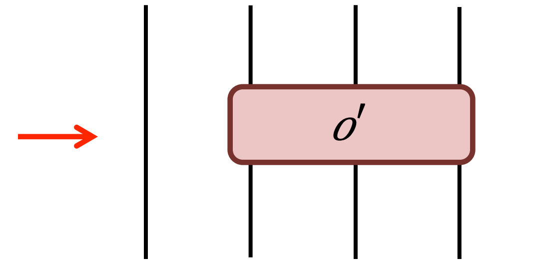


- MERA has bounded causal width (3 sites for binary MERA)
- Local operators coarse-grained through the causal cone

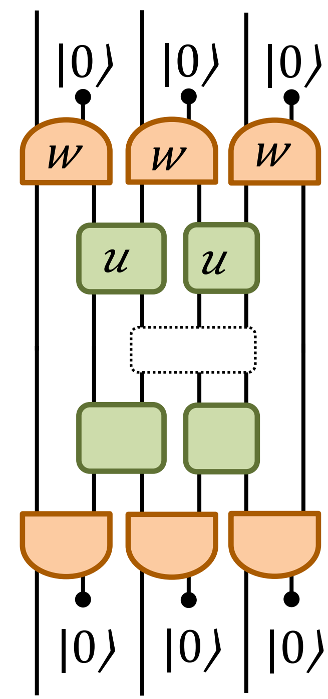
Conformal data from MERA



Local operator is coarse-grained into new local operator



Scaling superoperator, ***S***



Scaling operators are eigen-operators of ***S***

$$S\left(\phi_\alpha\right) = 2^{-\Delta_\alpha}\phi_\alpha$$

ϕ_α scaling operators

Δ_α scaling dimensions

Conformal data from MERA

Lowest order solution, $\chi=2$ MERA

Isometries: $w_{r,r+1} = \left(\frac{\sqrt{3}+\sqrt{2}}{4}\right) I_r I_{r+1} + \left(\frac{\sqrt{3}-\sqrt{2}}{4}\right) Z_r Z_{r+1} + i\left(\frac{1+\sqrt{2}}{4}\right) X_r Y_{r+1} + i\left(\frac{1-\sqrt{2}}{4}\right) Y_r X_{r+1}$

Disentanglers: $u_{r,r+1} = \left(\frac{\sqrt{3}+2}{4}\right) I_r I_{r+1} + \left(\frac{\sqrt{3}-2}{4}\right) Z_r Z_{r+1} + \left(\frac{i}{4}\right) X_r Y_{r+1} + \left(\frac{i}{4}\right) Y_r X_{r+1}$

	exact	**MERA** D4 wavelets
I	0	0
σ	0.125	0.140
ε	1	1
	1.125	1.136
	1.125	1.150
	2	2
	2	2
	2	2
	2	2
	2.125	2.113
	2.125	2.113
	2.125	2.131

- Scaling dimensions of primary fields and some descendants are reproduced
- Integer scaling dimensions reproduced exactly

Conformal data from MERA

Quantum critical Ising model

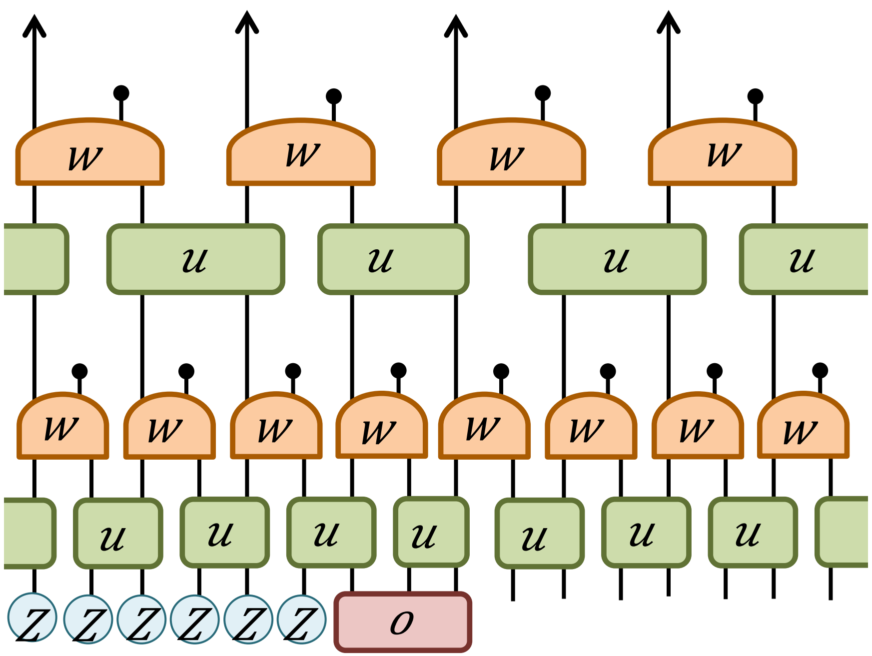
$$H_{\text{Ising}} = \sum_r \left(-X_r X_{r+1} + Z_r\right)$$

Expressed in Pauli matrices:

Isometries: $$w_{r,r+1} = \left(\tfrac{\sqrt{3}+\sqrt{2}}{4}\right) I_r I_{r+1} + \left(\tfrac{\sqrt{3}-\sqrt{2}}{4}\right) Z_r Z_{r+1} + i\left(\tfrac{1+\sqrt{2}}{4}\right) X_r Y_{r+1} + i\left(\tfrac{1-\sqrt{2}}{4}\right) Y_r X_{r+1}$$

Disentanglers: $$u_{r,r+1} = \left(\tfrac{\sqrt{3}+2}{4}\right) I_r I_{r+1} + \left(\tfrac{\sqrt{3}-2}{4}\right) Z_r Z_{r+1} + \left(\tfrac{i}{4}\right) X_r Y_{r+1} + \left(\tfrac{i}{4}\right) Y_r X_{r+1}$$

Conformal data from MERA?
Consider coarse-graining local operators...

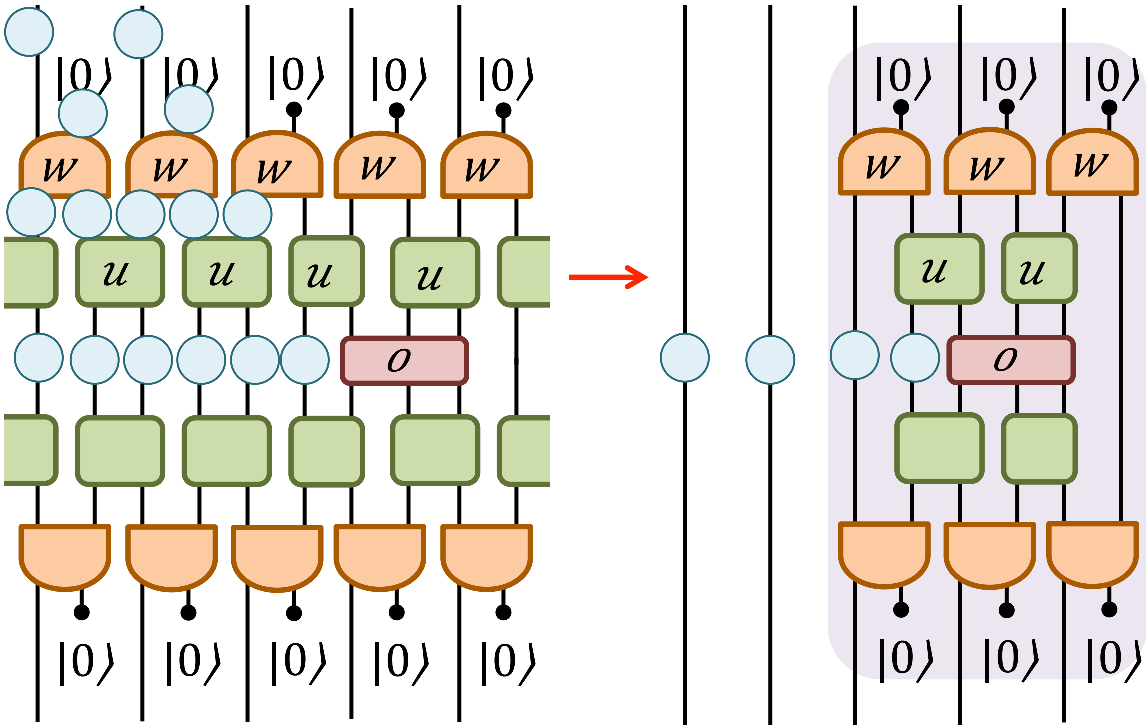


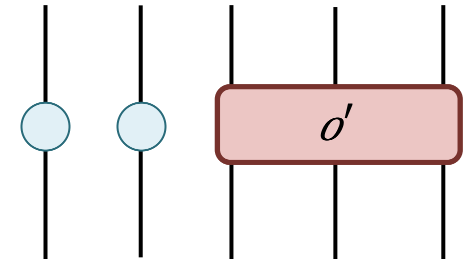
What about **non-local** scaling operators?

- Specifically those that come with a string of Z's (correspond to fermionic operators)

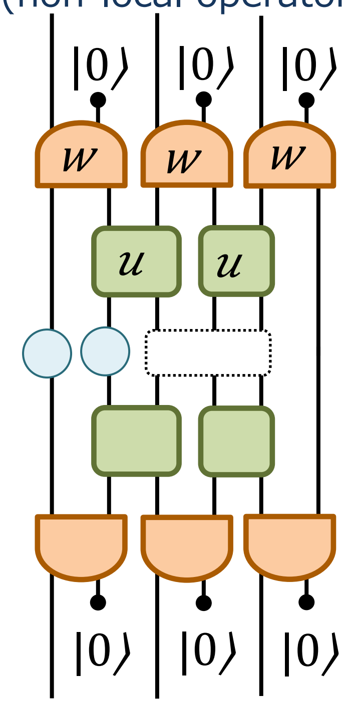
Conformal data from MERA

Local operator (with string) is coarse-grained into new local operator (with string)





Scaling superoperator, $\mathcal{S}$
(non-local operators)



(non-local)
Scaling superoperator

$$\tilde{\mathcal{S}}\left(\tilde{\phi}_\alpha\right) = 2^{-\Delta_\alpha}\,\tilde{\phi}_\alpha$$

$\tilde{\phi}_\alpha$ (non-local) scaling operators

Δ_α scaling dimensions

Conformal data from MERA

Lowest order solution, $\chi = 2$ MERA

Isometries: $$w_{r,r+1} = \left(\tfrac{\sqrt{3}+\sqrt{2}}{4}\right) I_r I_{r+1} + \left(\tfrac{\sqrt{3}-\sqrt{2}}{4}\right) Z_r Z_{r+1} + i\left(\tfrac{1+\sqrt{2}}{4}\right) X_r Y_{r+1} + i\left(\tfrac{1-\sqrt{2}}{4}\right) Y_r X_{r+1}$$

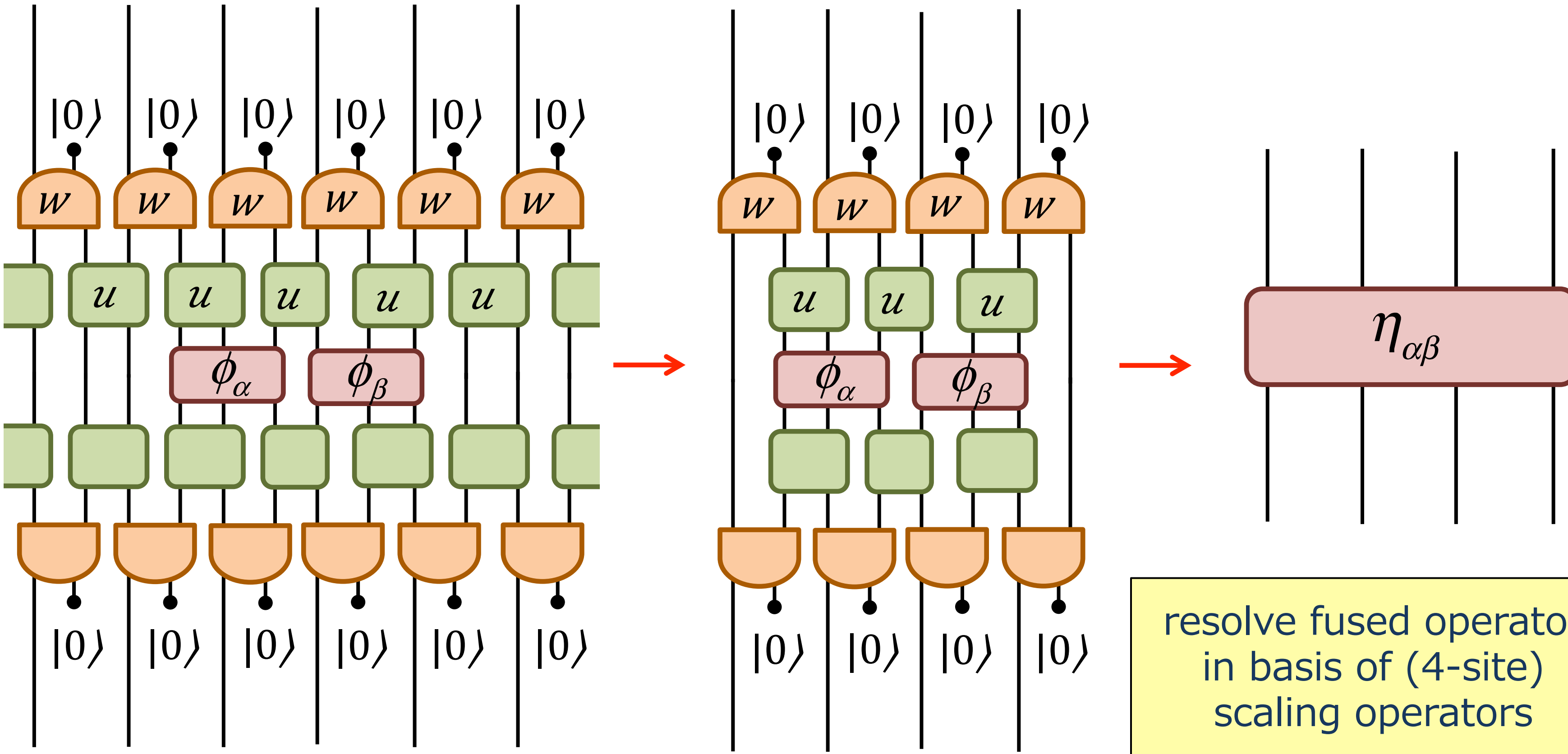
Disentanglers: $$u_{r,r+1} = \left(\tfrac{\sqrt{3}+2}{4}\right) I_r I_{r+1} + \left(\tfrac{\sqrt{3}-2}{4}\right) Z_r Z_{r+1} + \left(\tfrac{i}{4}\right) X_r Y_{r+1} + \left(\tfrac{i}{4}\right) Y_r X_{r+1}$$

	exact	**MERA** D4 wavelets
I	0	0
σ	0.125	0.140
ε	1	1
	1.125	1.136
	1.125	1.150
	2	2
	2	2
	2	2
	2	2
	2.125	2.113
	2.125	2.113
	2.125	2.131

	exact	**MERA** D4 wavelets
μ	0.125	0.144
ψ	0.5	0.5
$\bar{\psi}$	0.5	0.5
	1.125	1.100
	1.125	1.133
	1.5	1.5
	1.5	1.5
	2.125	2.085
	2.125	2.085
	2.125	2.127
	2.5	2.5
	2.5	2.5
	2.5	2.5
	2.5	2.5

Conformal data from MERA

How to extract **OPE coefficients** from MERA?
Consider fusion of two scaling operators...



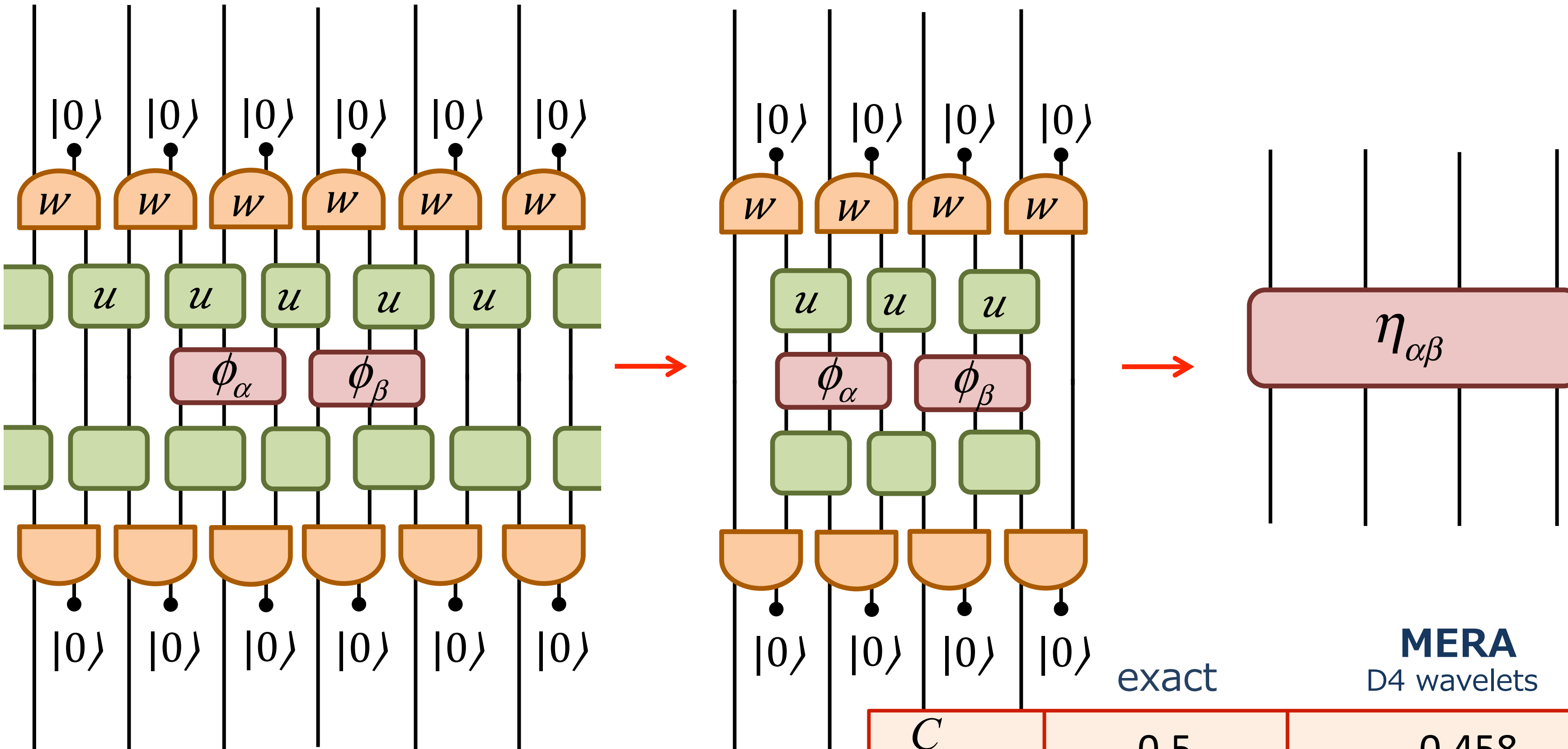
resolve fused operator in basis of (4-site) scaling operators

$$\eta_{\alpha\beta} = \sum_\gamma C_{\alpha\beta\gamma}\phi_\gamma$$

↑ OPE coefficients

Conformal data from MERA

How to extract **OPE coefficients** from MERA?
Consider fusion of two scaling operators...



	exact	MERA D4 wavelets
$C_{\varepsilon,\sigma,\sigma}$	0.5	0.458
$C_{\varepsilon,\mu,\mu}$	-0.5	-0.420
$C_{\psi,\mu,\sigma}$	0.5 - 0.5i	0.571 - 0.571i
$C_{\bar{\psi},\mu,\sigma}$	0.5 + 0.5i	0.571 + 0.571i
$C_{\varepsilon,\psi,\bar{\psi}}$	1i	1.23i
$C_{\varepsilon,\bar{\psi},\psi}$	-1i	-1.23i

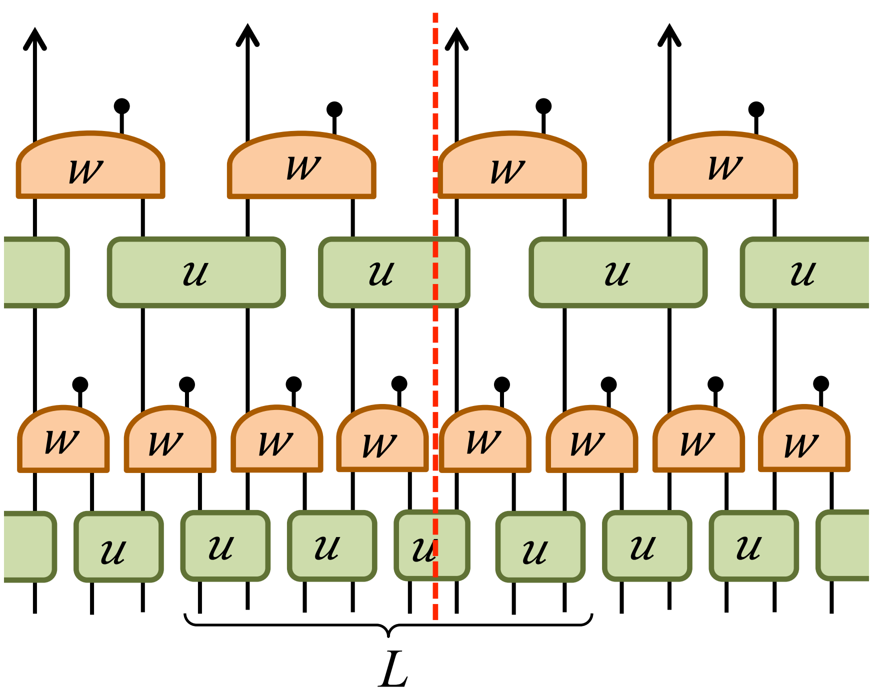
Conformal data from MERA

Central charge from MERA?
Many ways to do this (based on scaling of entanglement entropy)...

1. Compute entanglement entropy of different blocks length *L and use formula:*

$$S_L = \frac{c}{3}\log(L) + \text{const.}$$

2. Compute entanglement contribution (per scale) to the density matrix for half-infinite system



central charge

exact	**MERA** D4 wavelets
c = 0.5	0.495

MERA for critical Ising

Quantum critical Ising model

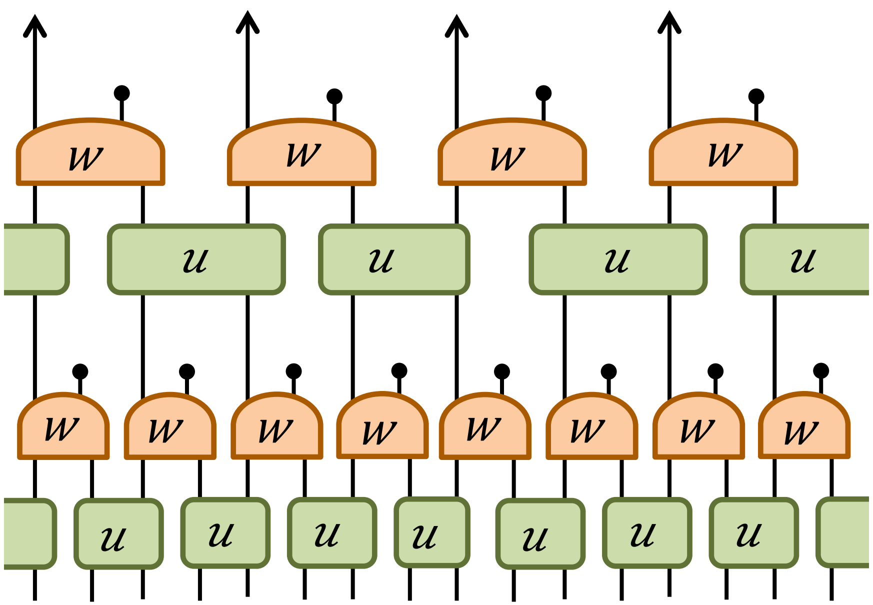
$$H_{\text{Ising}} = \sum_r \left(-X_r X_{r+1} + Z_r \right)$$

Expressed in Pauli matrices:

Isometries: $$w_{r,r+1} = \left(\tfrac{\sqrt{3}+\sqrt{2}}{4}\right) I_r I_{r+1} + \left(\tfrac{\sqrt{3}-\sqrt{2}}{4}\right) Z_r Z_{r+1} + i\left(\tfrac{1+\sqrt{2}}{4}\right) X_r Y_{r+1} + i\left(\tfrac{1-\sqrt{2}}{4}\right) Y_r X_{r+1}$$

Disentanglers: $$u_{r,r+1} = \left(\tfrac{\sqrt{3}+2}{4}\right) I_r I_{r+1} + \left(\tfrac{\sqrt{3}-2}{4}\right) Z_r Z_{r+1} + \left(\tfrac{i}{4}\right) X_r Y_{r+1} + \left(\tfrac{i}{4}\right) Y_r X_{r+1}$$

Wavelet based MERA does a remarkably good job of **encoding the Ising CFT!** (considering its simplicity...)

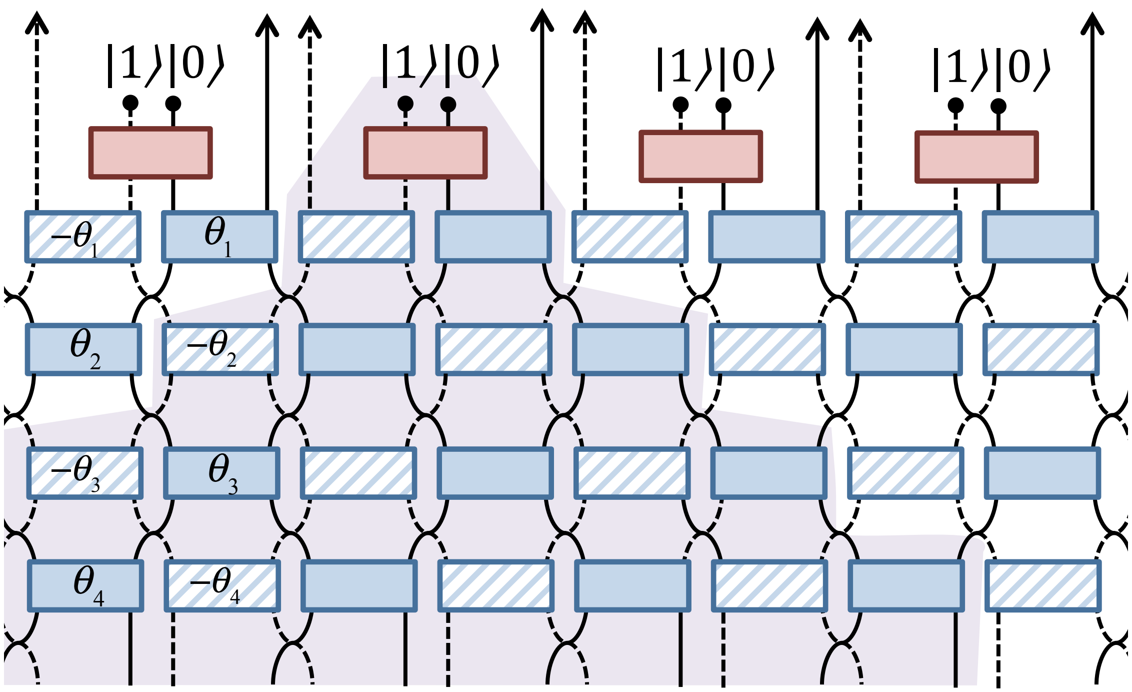


Higher order solutions

Is there a systematic way to generate better approximations to the ground state?

Yes! Use **higher order** wavelets (which correspond to circuits with a greater depth of unitary gates in each layer)

How does MERA with many layers of unitaries relate to standard (binary) MERA?



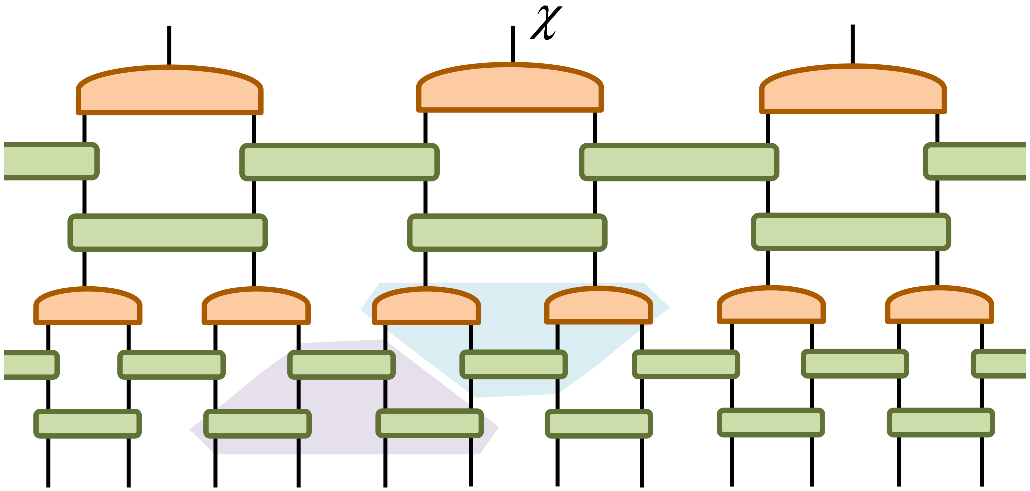
Four free parameters in the ansatz

$$[\theta_1, \theta_2, \theta_3, \theta_4]$$

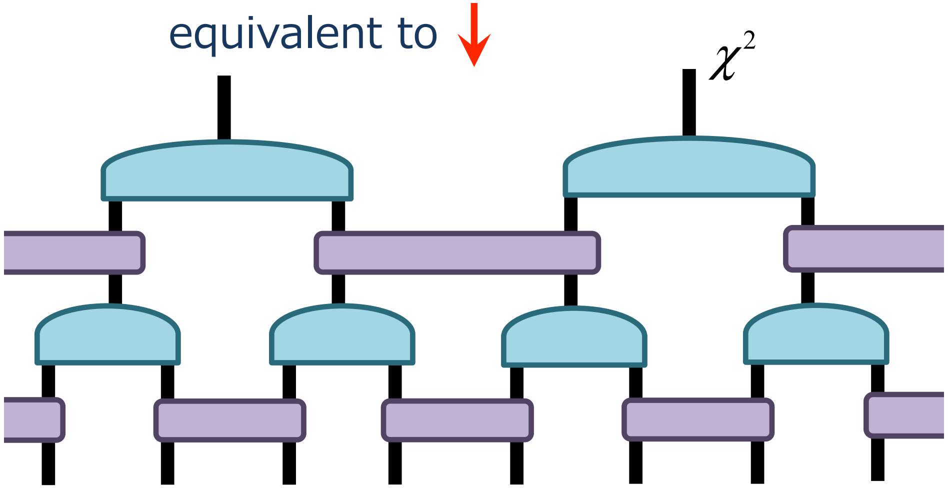
Wavelets have larger support (more compact in momentum space)

Higher order solutions

MERA with two layers of disentanglers:

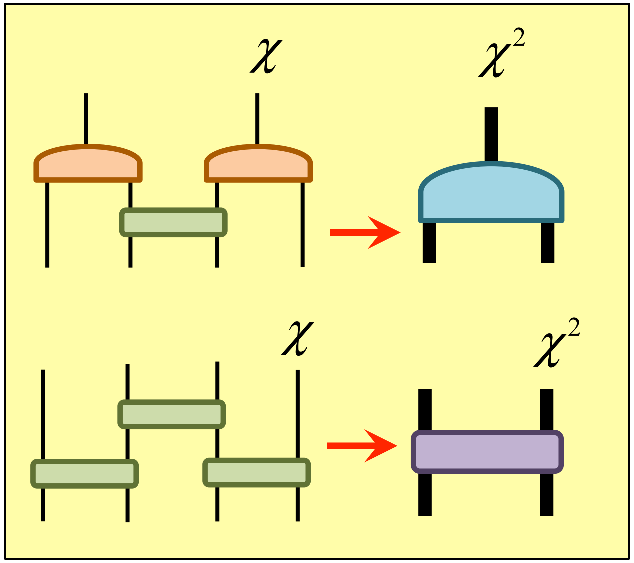


equivalent to



Binary MERA of larger bond dimension

group tensors:

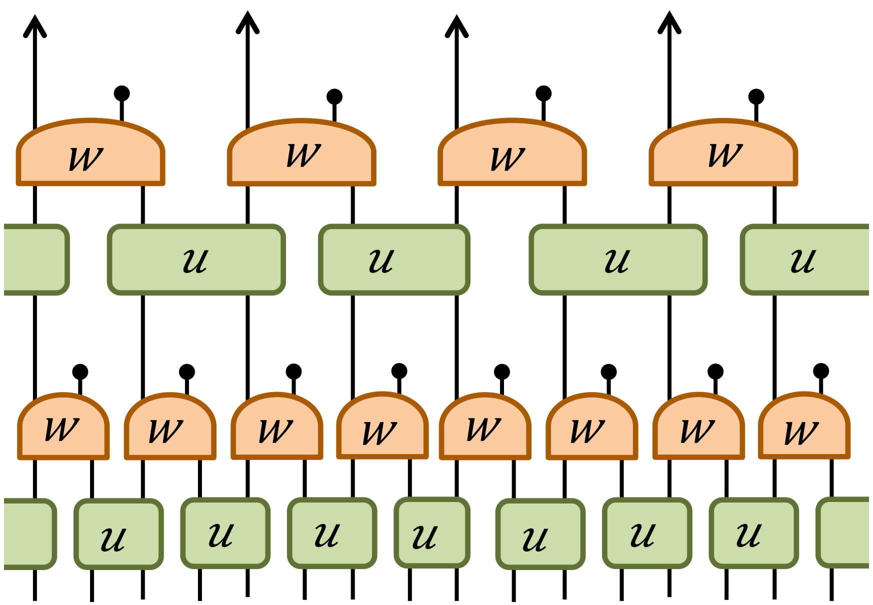


MERA for critical Ising

Quantum critical Ising model

$$H_{\text{Ising}} = \sum_r (-X_r X_{r+1} + Z_r)$$

- higher order wavelets = larger bond dimension MERA
- higher order wavelets offer systematic improvement in accuracy



ground energy

exact:	-1.27323...	
MERA:		rel. err.
2 parameter ($\chi=2$)	-1.24211	2.4%
3 parameter ($\chi=8$)	-1.26773	0.4%
5 parameter ($\chi=16$)	-1.27296	0.02%

How accurately can a MERA of finite bond dimension χ approximate the ground state of a CFT? Analytic bounds?

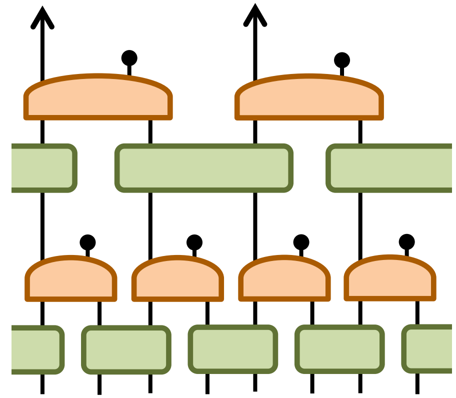
Summary

G.E., Steven. R. White, **Phys. Rev. Lett 116.** 140403 (April `16).

G.E., Steven. R. White, arXiv: 1605.07312 (May `16).

Real-space renormalization and wavelets have many conceptual similarities...
... but can one establish a precise connection?

↔



generalize from **ordinary** functions to
many-body **wavefunctions**

wavelets → ← **MERA**

restrict to **Gaussian MERA**

Applications: →
- better understanding of MERA
- construction of analytic examples of MERA (e.g. for Ising CFT)
- analytic error bounds for MERA?

Applications: ←
- design of better wavelets (e.g. for image compression)

Outline: Entanglement renormalization and Wavelets

G.E., Steven. R. White, **Phys. Rev. Lett 116.** 140403 (April `16).

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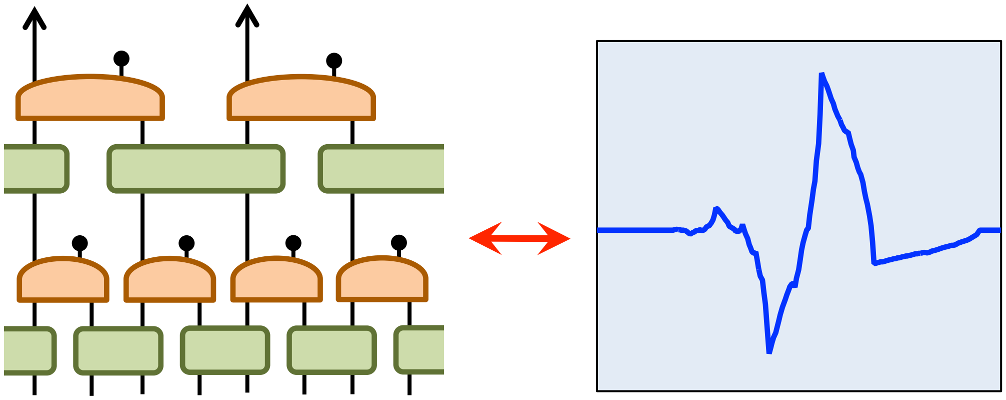
Overview

Wavelet solution to free fermion model

Representation of wavelets as unitary circuits and MERA

Benchmark calculations from wavelet based MERA

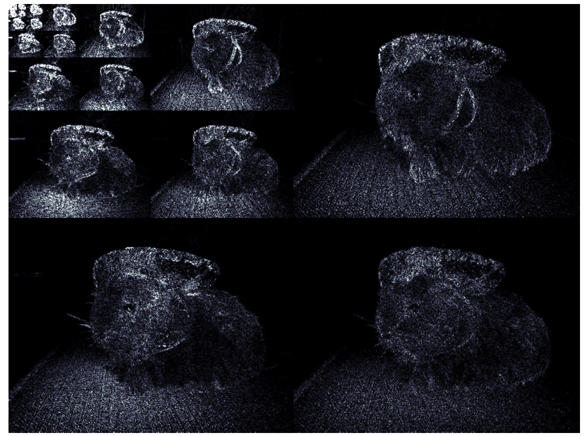
Further application of wavelet–MERA connection



Wavelets for image compression

image

→ transform to wavelet basis



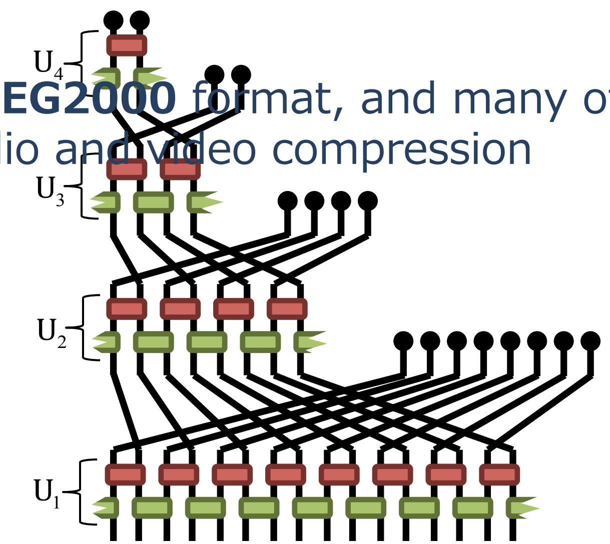
truncate (keep only largest 2% of coefficients)

→ inverse transform

compressed image

peak signal to noise:
PSNR: 37.0 dB

- This is the key part of **JPEG2000** format, and many other standards for image, audio and video compression

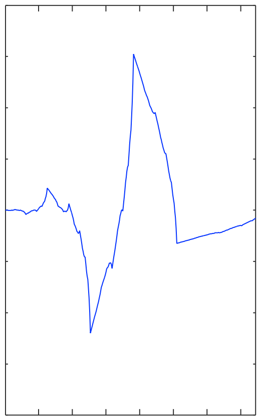


Wavelets for image compression

Can we do better?

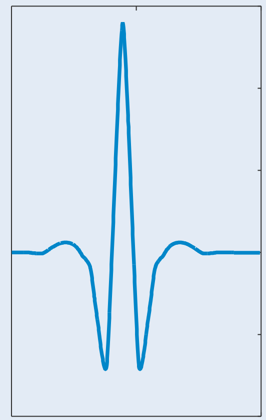
Representation and design of wavelets using unitary circuits, G.E., Steven. R. White, arXiv: 1605.07312 (May `16).

Daubechies

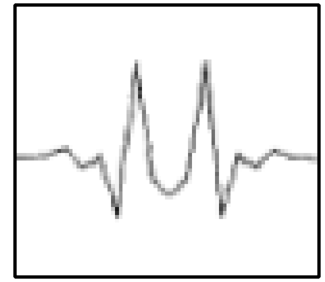


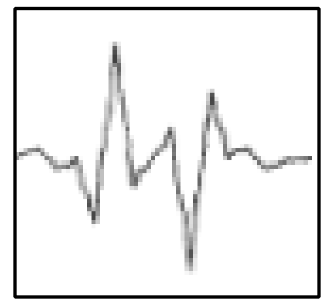
Coiflets

CDF wavelets



Scale-3 symmetric





desirable properties of wavelets	Daubechies	Coiflets	CDF wavelets	Scale-3 symmetric
orthogonality?	**yes**	**yes**	**near orthogonal**	**yes**
symmetric?	**no**	**near symmetric**	**yes**	**yes**
compression ratio?	**okay**	**good**	**good**	**bad**
			JPEG2000	

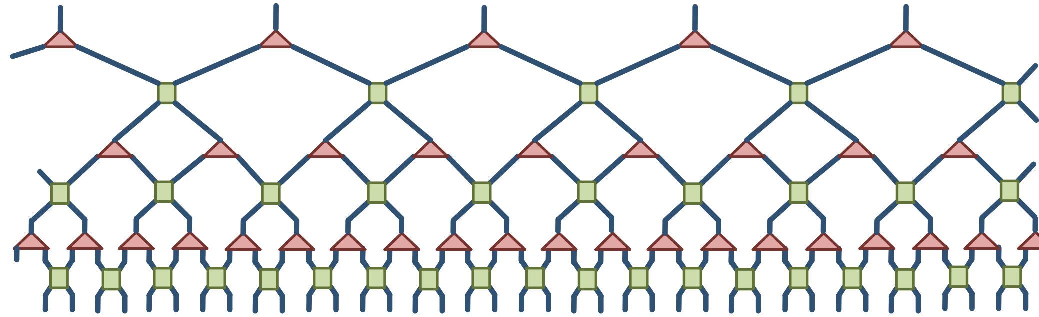
Application: wavelet design

Representation and design of wavelets using unitary circuits, G.E., Steven. R. White, arXiv: 1605.07312 (May `16).

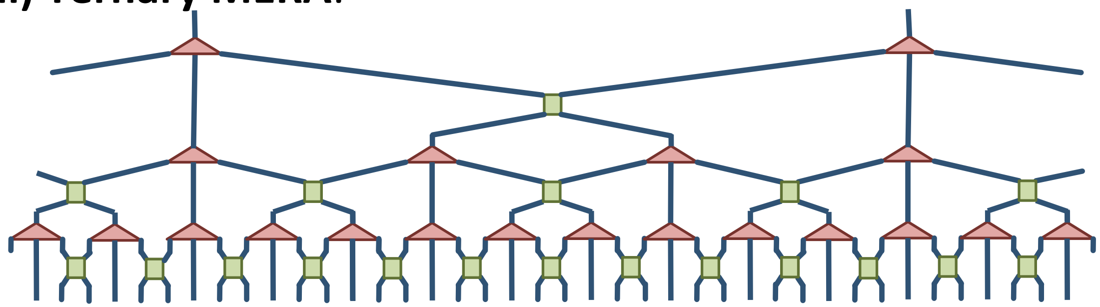
Things learned in the context of tensor networks / MERA:

- how to construct circuits with different forms and scaling factors
- incorporation of spatial and global internal symmetries
- optimization of networks!

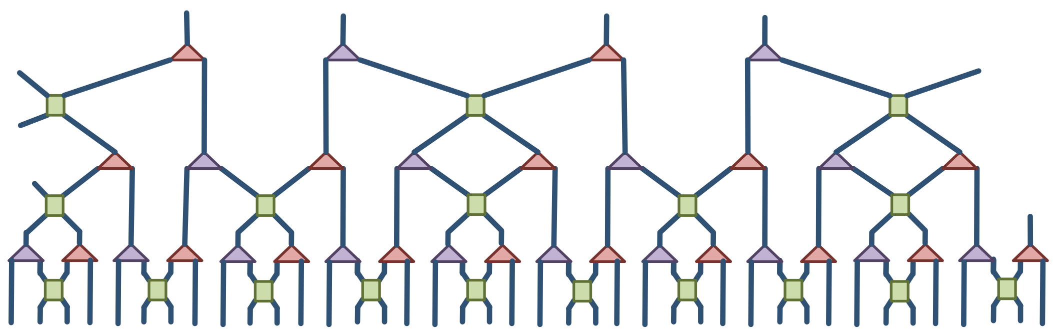
(i) Binary MERA



(ii) Ternary MERA:



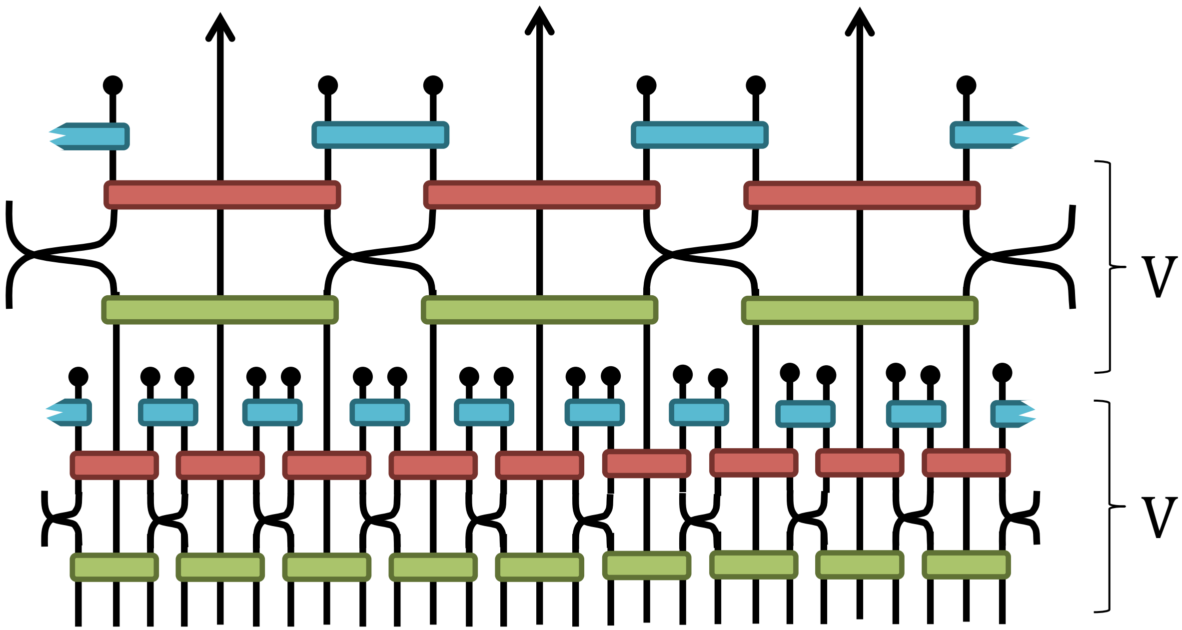
(iii) Modified Binary MERA:



Application: wavelet design

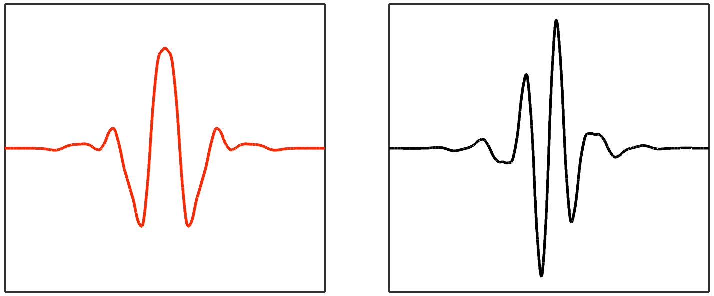
Representation and design of wavelets using unitary circuits,
G.E., Steven. R. White, arXiv: 1605.07312 (May `16).

Design a family of symmetric / antisymmetric wavelets based upon ternary unitary circuits:



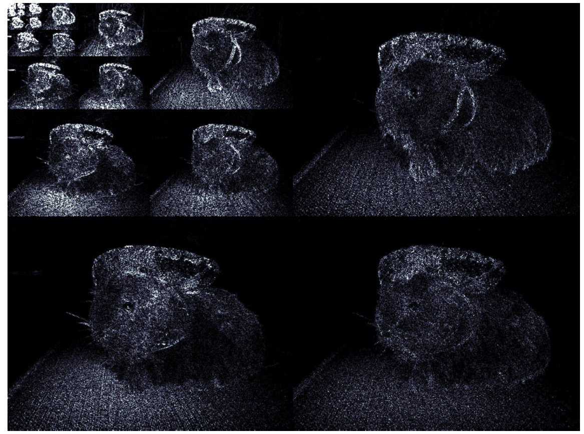
symmetric wavelet

antisymmetric wavelet



Wavelets for image compression

JPEG2000 wavelets



image

→ transform to wavelet basis

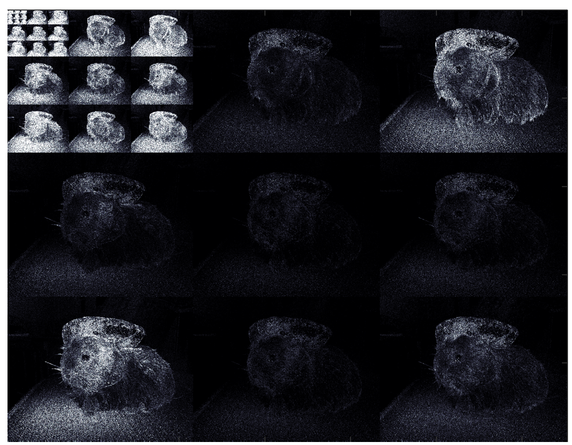
truncate (keep only largest 2% of coefficients)

→ inverse transform

compressed image

PSNR: 37.0 dB

new scale-3 wavelets



PSNR: 37.4 dB

Wavelets for image compression

Can we do better?

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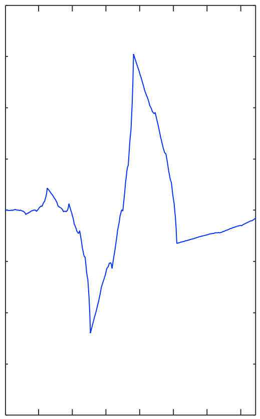
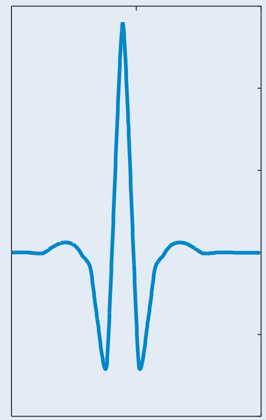
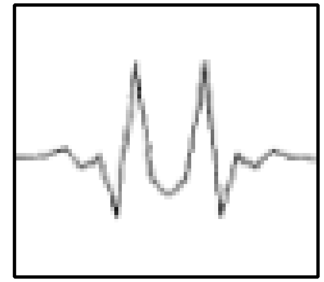
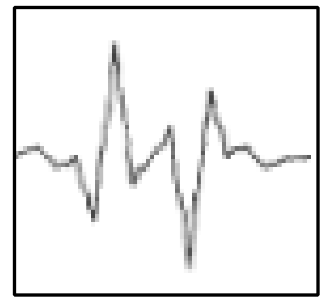
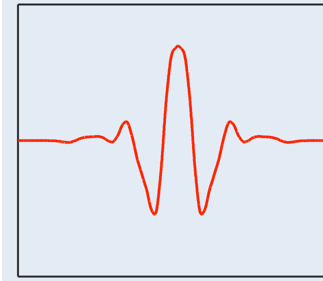
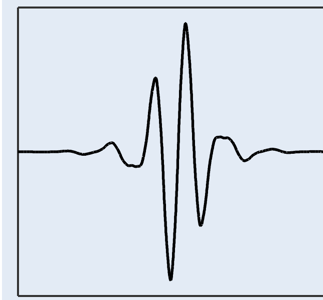
desirable properties of wavelets	Daubechies	Coiflets	CDF wavelets	Scale-3 symmetric	New scale-3!
				 	 
orthogonality?	yes	yes	near orthogonal	yes	yes
symmetric?	no	near symmetric	yes	yes	yes
compression ratio?	okay	good	good	bad	good
			JPEG2000		

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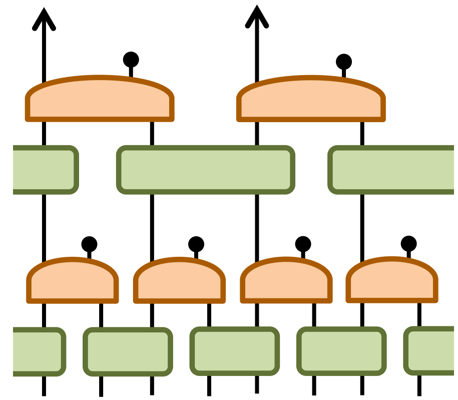

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