Quantum Error Correction.

why?

- Important for building quantum computer
- An (extremely) efficient way to describe quantum many-body system.
  - → AdS/CFT, holographic principle (high energy)
  - → 〃 topological order (cond. mat) Beni & Daniel next week

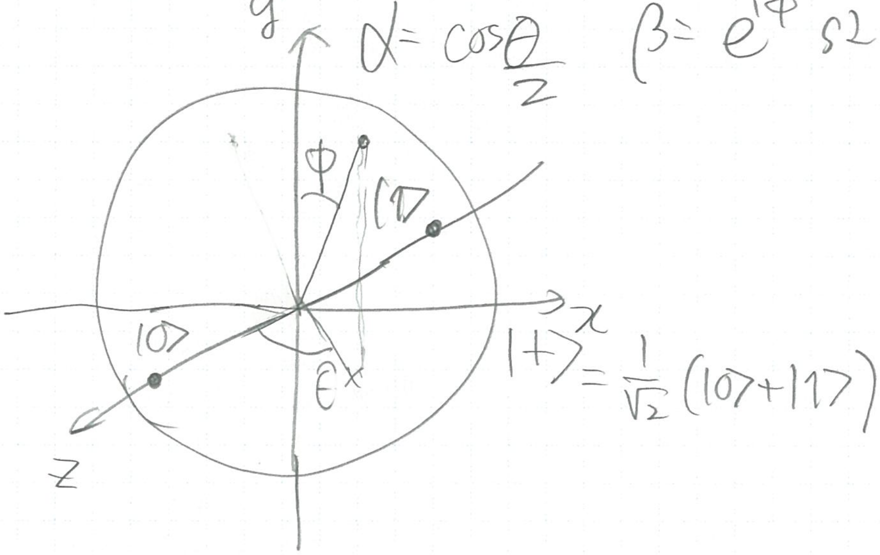
Outline.

- Introduction
  - quantum state & decoherence (noise)
- stabilizer QEC code
- Many-body phys & stabilizer code.
  - 2D toric code (topological order)

⑥ Introduction

○ qubit $\alpha|0\rangle+\beta|1\rangle$ $|\alpha|^2+|\beta|^2=1$

$\alpha=\cos\frac{\theta}{2}$ $\beta=e^{i\phi}\sin\frac{\theta}{2}$



$|+\rangle=\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$

○ Pauli operator

$\sigma_I=I=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $\sigma_x=X=\begin{pmatrix}0&1\\1&0\end{pmatrix}$, $\sigma_y=Y=\begin{pmatrix}0&-i\\i&0\end{pmatrix}$, $\sigma_z=Z=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$

$$A_i=I\otimes I\otimes I\otimes\cdots\underset{\text{ith qubit}}{A}\otimes I\otimes I\otimes I$$

○ pure & mixed state

ex) $\rho=|\Psi\rangle\langle\Psi|$

	pure	mixed
state	$|\Psi\rangle\in\mathcal{H}$	$\rho\in\mathcal{L}(\mathcal{H})$ s.t. $\rho^\dagger=\rho$, $\mathrm{Tr}[\rho]=1$, $\rho\geq0$
time evolution	U	$\mathcal{E}(\rho)=\sum_i K_i\rho K_i^\dagger$ linear map CPTP map
meas.	$\langle\Psi|P_i|\Psi\rangle=p_i$ (projective or POVM)	$p_i=\mathrm{Tr}[P_i\rho]$

○ Decoherence

system $\qquad\qquad$ environment

$|\psi\rangle \equiv \alpha|0\rangle + \beta|1\rangle \qquad \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$e^{-i\theta Z\otimes Z}$

$\downarrow$ interaction

$$|\Psi\rangle \equiv \frac{1}{\sqrt{2}}\left[e^{-i\theta}(\alpha|00\rangle + \beta|11\rangle) + e^{i\theta}(\alpha|01\rangle + \beta|10\rangle)\right]$$

$\downarrow$ ignore about environment

$$\rho_S = \mathrm{Tr}_E[|\Psi\rangle\langle\Psi|]$$

$$= |\alpha|^2|0\rangle\langle 0| + |\beta|^2|1\rangle\langle 1| + \cos\theta\,(\alpha^*\beta|1\rangle\langle 0| + \alpha\beta^*|0\rangle\langle 1|)$$

$\theta \to 0, \quad \rho_S = (\alpha|0\rangle + \beta|1\rangle)(\text{——})^\dagger$

pure state

$\theta \to \frac{\pi}{2} \quad \rho_S = |\alpha|^2|0\rangle\langle 0| + |\beta|^2|1\rangle\langle 1|$

classical mixture of $|0\rangle$ & $|1\rangle$ states.

for general θ

$$\mathcal{E}(\rho) = \left(\frac{1}{2} + \frac{\cos\theta}{2}\right)\rho + \left(\frac{1}{2} - \frac{\cos\theta}{2}\right)Z\rho Z$$

do nothing $\qquad$ apply Z.

Z error with prob. $\frac{1}{2} - \frac{\cos\theta}{2}$

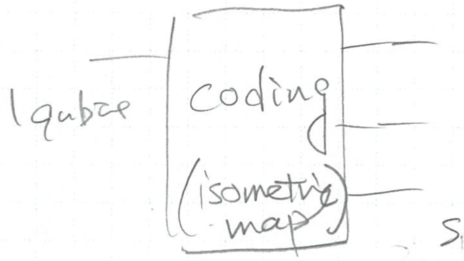
⊙ Stabilizer code

∘ 3 qubit code

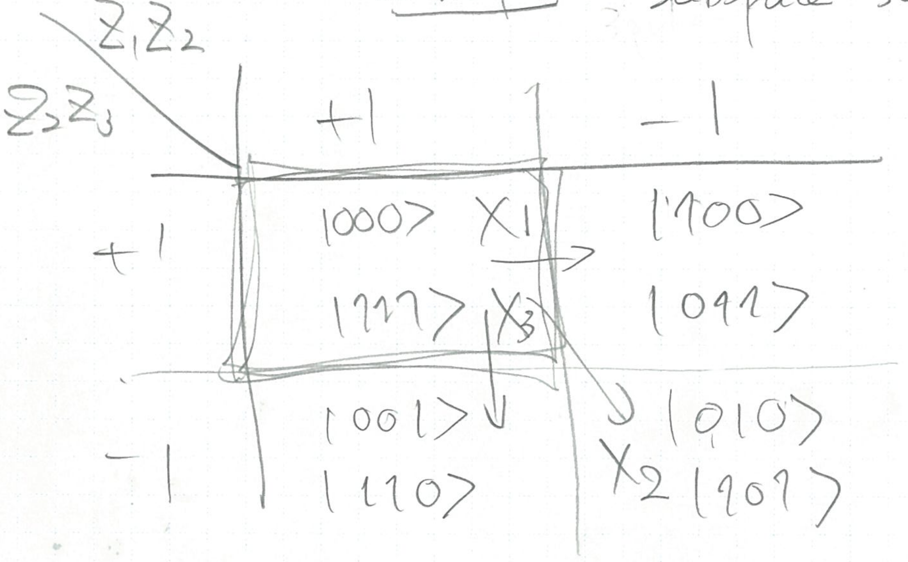
(assume errors are only X type)

$\alpha|0\rangle + \beta|1\rangle \longrightarrow \alpha|000\rangle + \beta|111\rangle$

2 dim subspace of 3 qubit space.



subspace sapanned by $\{|000\rangle, |111$



Z_2Z_3 \ Z_1Z_2	+1	−1
+1	$|000\rangle$ $|111\rangle$	$|100\rangle$ $|011\rangle$
−1	$|001\rangle$ $|110\rangle$	$|010\rangle$ $|101\rangle$

each error maps the code state into a diffierent orthogonal subspace

$\Rightarrow$ by measuring Z_1Z_2, Z_2Z_3

we know what type of error occurs.

$$P_{fail} = 3P^2(1-P) + (1-P)^3$$

if $P_{fail} < P$, we obtain gain from QEC.

○ 5qubit code.

can correct both X and Z errors.

$S_1 = X\,Z\,Z\,X\,I$

$S_2 = I\,X\,Z\,Z\,X$

$S_3 = X\,I\,X\,Z\,Z$

$S_4 = Z\,X\,I\,X\,Z$

} mutually commutable

→ stabilizer operator

constraint

$S_i|\Psi\rangle = |\Psi\rangle$

for all i.

$2^5/2^4 = 2$

degenerate subspace.

stabilizer subspace.

what is the operator that characterize degenerated subspace? ⇒ "logical operator"

$L_X = X\,X\,X\,X\,X$

$L_Z = Z\,Z\,Z\,Z\,Z$

} mutually anti-commute

Act as "logical" Pauli operators.

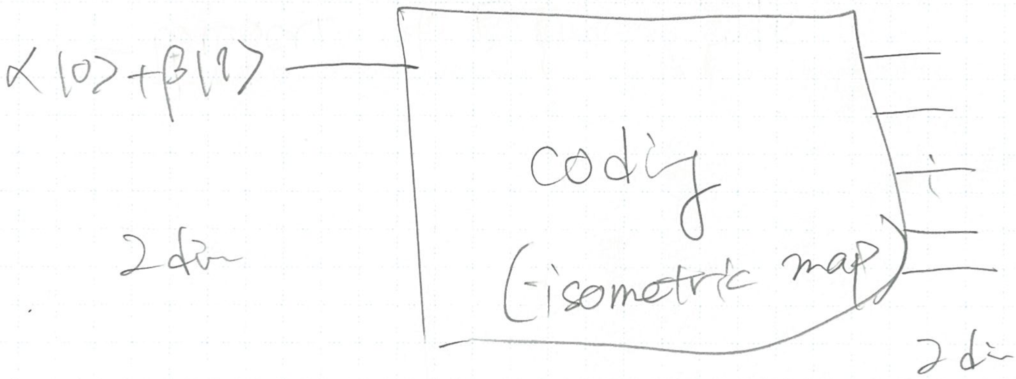
-- the choice is not unique!!

$L_X S_1 = -IYYIX$ } at least 3-body operator ⇒ code distanc 3

⋮

$A \sim B$ if $\exists S \in \mathcal{S}$ s.t. $A = SB$.

the action of logical operators is an equivalent class quotiented by stabilizer group



2 dim subspace
of 2^5 dim space

constrain $S_i|\Psi\rangle=|\Psi\rangle$

"raw" Pauli operator

$X, Z \overset{}{\underset{\text{the same quantum information}}{\longleftrightarrow}}$ "logical" Pauli operator

— a lot of physical representation

XXXXX ~ −IYYIX ~ ~~

A toy model for AdS / CFT correspondence.

[HaPPY '15]

∘ property of 5-qubit code.

ex) $X_1 \to$ anti-commutes with S_4

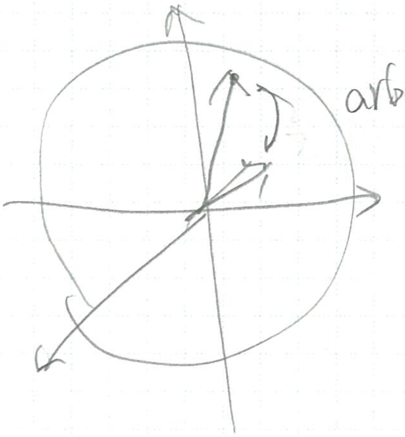
$X_3 \to$ 〃 with S_1, S_2

→ Each single qubit Pauli operator maps the code state into different orthogonal subspace.

# of single qubit Pauli error

$$\underset{\#\text{ of qubit}}{5}\times\underset{X,Y,Z}{3}+\underset{\text{no error}}{1}=16=\underset{\#\text{ of orthogonal space.}}{\underline{2^4}}$$

∘ X, Y, Z error tolerance is enough



$$\mathcal{R}\circ\mathcal{E}(|\Psi\rangle\langle\Psi|)$$

code state

arbitrary single qubit CPTP map

recovery operation

Assume that

$$R_k \overset{\in\{I,X,Y,Z\}}{A}|\Psi\rangle \propto |\Psi\rangle$$

$$\mathcal{R}(\rho)=\sum_k R_k\rho R_k^\dagger$$

then

$$\mathcal{R}\circ\mathcal{E}(|\Psi\rangle\langle\Psi|)=|\Psi\rangle\langle\Psi|$$

$$\mathcal{E}(\rho)=\sum_i E_i\rho E_i^\dagger$$

$$E_i=\sum_{A=I,X,Y,Z} c_A^{(i)} A$$

$$R_k E_i|\Psi\rangle \propto |\Psi\rangle$$

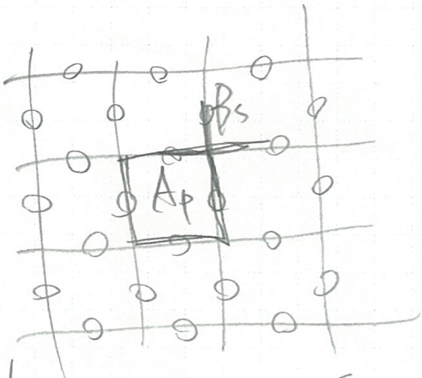
⇒ discrete X, Y, Z errors are enogh to model an arbitrary CPTP noise

○ Many body physics and stabilizer code.

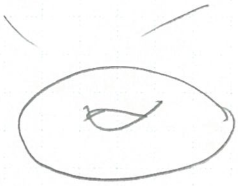
\ locality & translation invariance

⟷ topological stabilizer code

○ Kitaev's toric code.



N×N square lattice with p.b.c.



Stabilizer operators:

$$A_p = \prod_{i\in\partial p} Z_i$$

$$B_s = \prod_{j\in\delta s} X_j$$

$[A_p, B_s]=0$. mutually commutable

# of qubits	$\lvert E\rvert$	(edge)	$2N^2$
# of A_p	$\lvert F\rvert$	(face)	N^2
# of B_s	$\lvert V\rvert$	(vertex)	N^2

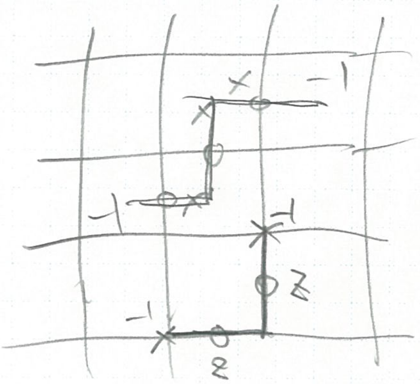
$$2^{|E|-(|F|+|V|\underbrace{-2}_{})} = 2^2 = 2^{2\underset{\text{genus}}{g}}$$

$\prod_p A_p = I$

$\prod_s B_s = I$ not independent!

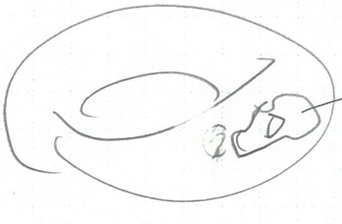
degeneracy depends only on topology.

◦ logical operator



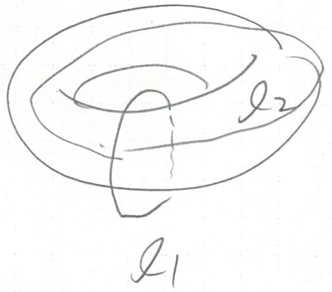
errors are detected (anti-commute) at the boundary of error chain

logical operator → boundaryless (cycle)



→ trivial cycle act trivially

$$\prod_{i\in\partial D} Z_i = \prod_{p\in D} A_p \leftarrow \text{stabilizer operator}$$



$$L_1^Z = \prod_{i\in l_1} Z_i$$

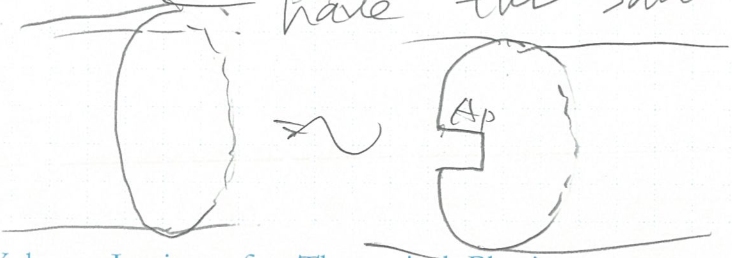
$$L_2^Z = \prod_{i\in l_2} Z_i$$

on dual lattice

$$L_1^X = \prod_{i\in \bar{l}_1} X_i$$

$$L_2^X = \prod_{i\in \bar{l}_2} X_i$$

the action of logical operators are equivalent if two cycles have the same topology



code distance: N

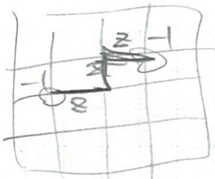
(in order to map a code state
into another code state
you have to address at least
N qubits)

○ Toric code Hamiltonian (topological order)

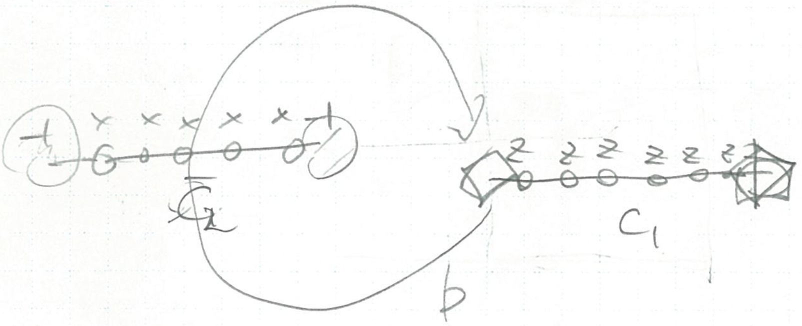
$$H_{toric} = -J \sum_p A_p - J \sum_s B_s$$

- ground state → code state

operator that
- characterize the deg. → logical operator



- boundary of error chains → excitations (anyonic)



$Z(c_1) X(\bar{c}_2) |\Psi_{g.s}\rangle$

$Z(b) Z(c_1) X(\bar{c}_2) |\Psi_{g.s}\rangle$

$= - Z(c_1) X(\bar{c}_2) Z(b) |\Psi_{g.s}\rangle$

↑ trivial cycle

$= - Z(c_1) X(\bar{c}_2) |\Psi_{g.s}\rangle$

– robustness aginst perturbation

$$\tilde{H} = H_{toric} + h^x \sum_i X_i + h^z \sum_j Z_j$$

LG theory

symmety breaky, local order parameter

ex) Ig $\cancel{\left|\begin{matrix}\uparrow\uparrow\uparrow\\ \uparrow\uparrow\uparrow\\ \uparrow\uparrow\uparrow\end{matrix}\right\rangle + \left|\begin{matrix}\downarrow\downarrow\downarrow\\ \downarrow\downarrow\downarrow\\ \downarrow\downarrow\downarrow\end{matrix}\right\rangle}$ unstable.

$$H_{Ig} = -J \sum_{\langle ij \rangle} Z_i Z_j + \boxed{h^z \sum_i Z}$$

↑ split ground state degeneracy

g.s.d. of H_{toric}

is robust up Nth order perturbation

kth order perturbation

$$\overset{\text{g.s.}\searrow}{P^{(i_1)}}\, V P^{(i_2)} V \cdots V \overset{\nearrow \text{g.s.}}{P^{(i_{k+1})}} \rightarrow L_x = X^{\otimes N}$$

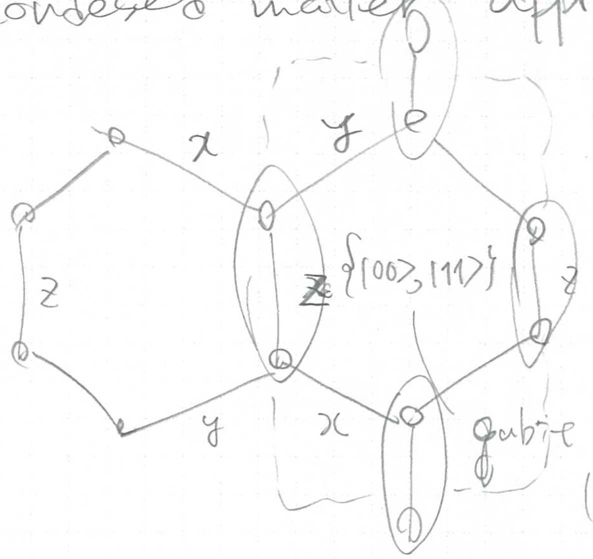
↑ perturbation Hamiltonian (1-body)

↑ $P^{(i)}$ projector onto ith eigen space

error correctability ⟷ robustness aginst perturbation

∘ experimental progress.

— condesed matter approach



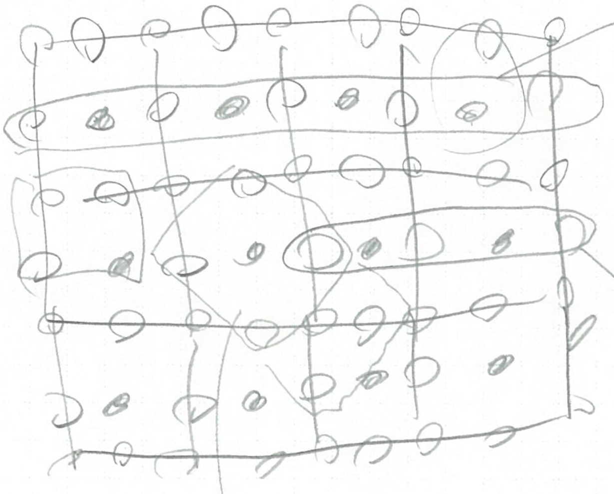
$$H_{\text{honey}} = -J_x \sum_{x\text{-link}} XX - J_z \sum_{z\text{-link}} ZZ - J_y \sum_{y\text{-link}} YY$$

$J_z >> J_x, J_y$

low energy model

→ toric code Haltn

— quantum ~~z~~information approach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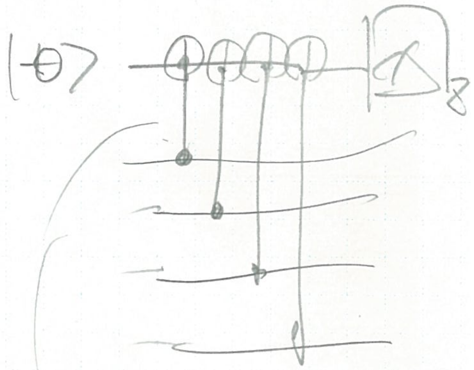

$$= |0\rangle\langle 0|_c \otimes I + |1\rangle\langle 1|_c \otimes X.$$