Schmidt decomposition

$H = H_1 \otimes H_2$ bipartition ; $\{|\Psi^{(i)}_m\rangle, m=1,..,d_i\}$ orthonormal basis of H_i $i=1,2$

$$|\Psi\rangle = \sum_m^{d_1}\sum_n^{d_2} A_{m,n} |\Psi^{(1)}_m\rangle \otimes |\Psi^{(2)}_n\rangle \in H$$

rectangular matrix

$\boxed{\text{SINGULAR VALUE decomposition}}$ $A = U D V^{+}$

U, V unitary matrices $\Rightarrow$ change of basis in H_i

$D = (\ddots | 0)$ or $\left(\frac{\ddots}{0}\right)$ rectangular and diagonal

$\lambda_i \geq 0$ diagonal elements of D

$i = 1, \dots, d_0 \equiv \min(d_1, d_2)$

$$|\Psi\rangle = \sum_{i=1}^{d_0} \lambda_i |\Phi^{(1)}_i\rangle \otimes |\Phi^{(2)}_i\rangle \qquad \text{SCHMIDT DECOMPOSITION}$$

$\{\lambda_j > 0,\ j = 1, .., r_0 \leq d_0\}$ Schmidt coefficients

$\hookrightarrow$ Schmidt rank

(*) $\langle\Psi|\Psi\rangle = 1 \Rightarrow \sum_{i=1}^{d_0} |\lambda_i|^2 = 1$ normalization of $|\Psi\rangle$

Entanglement properties are encoded in the λ_i's

(1) $\lambda_i = 0 \ \forall i \neq i_0 \Rightarrow |\Psi\rangle = |U_1\rangle \otimes |U_2\rangle$ is a PRODUCT STATE (not entangled)

otherwise $\Rightarrow |\Psi\rangle$ is ENTANGLED

(2) all λ_i have equal size $|\lambda_i| = \frac{1}{\sqrt{r_0}} \Rightarrow |\Psi\rangle$ is maximally entangled

pure state

$\rho = |\Psi\rangle\langle\Psi| \qquad H = H_1 \otimes H_2$

$$|\Psi\rangle = \sum_{\mu=1}^{d_0} \lambda_\mu \, |\Phi_\mu^{(1)}\rangle \otimes |\Phi_\mu^{(2)}\rangle \qquad \rho^{12}$$

$$\rho = |\Psi\rangle\langle\Psi| = \sum_{\mu,m} \lambda_\mu \bar{\lambda}_m \left(|\Phi_\mu^{(1)}\rangle \otimes |\Phi_\mu^{(2)}\rangle \right) \left(\langle \Phi_m^{(1)}| \otimes \langle \Phi_m^{(2)}| \right)$$

$$\Rightarrow \rho_i = \mathrm{tr}_{H_j}(\rho) = \sum_{k=1}^{d_j} \langle \Phi_k^{(j)} | \rho | \Phi_k^{(j)} \rangle = \sum_{k=1}^{d_0} |\lambda_k|^2 \, |\Phi_k^{(i)}\rangle\langle\Phi_k^{(i)}|$$

$j \neq i \; ; \; i = 1,2$

$\boxed{\text{REDUCED DENSITY MATRICES}}$

(1) ρ_1 and ρ_2 have the same non-zero eigenvalues

$$\{ |\lambda_k|^2 \equiv w_k > 0, \; k = 1, \dots, r_0 \} = \text{ENTANGLEMENT SPECTRUM}$$

$\hookrightarrow$ schmidt rank

(2) $$S_i \equiv -\mathrm{tr}_{H_i} (\rho_i \log \rho_i) = -\sum_{k=1}^{r_0} w_k \log w_k$$

$S_1 = S_2$ is the ENTANGLEMENT ENTROPY

(2.1) $|\Psi\rangle = |u_1\rangle \otimes |u_2\rangle$ product state $\Rightarrow S = 0$

(2.2) maximally entangled state $\Rightarrow S = \log(r_0)$

$w_k = 1/r_0$

r_0 is the effective number of states coupled in parts 1 and 2

(3) $|\Psi\rangle$ entangled $\Rightarrow$ ρ_i are MIXED STATES;
namely they cannot be written as projectors
$(\rho_i = |\chi_i\rangle\langle\chi_i|)$

Inequalities

density matrix

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \qquad \text{state } \rho$$

pure $\rho = |\psi\rangle\langle\psi|$ (projector)

mixed (cannot be described as a projector) $\rho = \sum_s p_s |\psi_s\rangle\langle\psi_s|$

$$\rho_i = \mathrm{tr}_{\mathcal{H}_j}(\rho) \qquad i,j = 1,2 \qquad j \neq i$$

$$S(\rho_i) = -\,\mathrm{tr}_{\mathcal{H}_i}(\rho_i \log \rho_i)$$

full system. $\rho = \rho_{12} = \rho_{1\cup 2}$

(1)

$$|S(\rho_1) - S(\rho_2)| \leq S(\rho) \leq S(\rho_1) + S(\rho_2)$$

Araki-Lieb inequality [CMP (1970)] sub-additivity (**)

$\mathcal{H}_2 = \phi$

$$\rho = |\psi\rangle\langle\psi| \Rightarrow S(\rho) = 0 \Rightarrow S(\rho_1) = S(\rho_2)$$

(2) $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$

see back for the mutual information

STRONG SUBADDITIVITY

$$\begin{cases} S(\rho_{12}) + S(\rho_{23}) \geq S(\rho_2) + S(\rho_{123}) & (*) \\ S(\rho_{12}) + S(\rho_{23}) \geq S(\rho_1) + S(\rho_3) \end{cases}$$

(·) these inequalities can be shown to be equivalent by PURIFICATION, namely enlarge $\mathcal{H}$ s.t. $\tilde{\mathcal{H}} = \mathcal{H} \otimes \mathcal{H}_4$

$\rho_{1234} = |\psi\rangle\langle\psi|$ pure

(·) characteristic property of the entanglement entropy

$$\rho_{123} = \mathrm{tr}_{\mathcal{H}_4} |\psi\rangle\langle\psi|$$

(·) "strong" because if $\mathcal{H}_2 = \phi$ (*) becomes (**)

Entanglement Entropy in QFT

Holzey, Larsen, Wilczek hep-th/9403108
Calabrese, Cardy hep-th/0405152

$x \sim 0$

$\mathrm{tr}_A \rho_A = 1 \qquad \log(x) = x + \dots$

$\Rightarrow \log\left(\mathrm{tr}_A \rho_A^m\right) = (m-1)\,\partial_m \mathrm{tr}\rho_A^m\big|_{m=1} + \dots$

$\hookrightarrow 1 + (m-1)\,\partial_m \mathrm{tr}\rho_A^m\big|_{m=1} + \dots$

(1) spatial bipartition $\rightsquigarrow$ "geometric entropy"
(2) replica trick

Renyi entropies $S_A^{(m)}$

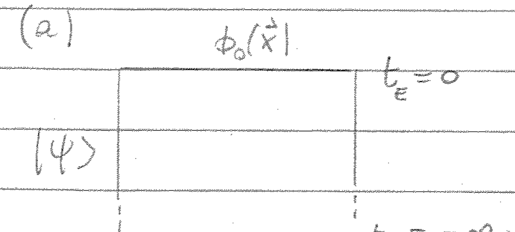
$$S_A = -\mathrm{tr}_A\left(\rho_A \log \rho_A\right) = -\partial_m\left(\mathrm{tr}\rho_A^m\right)\Big|_{m=1} = \lim_{m\to 1} \frac{1}{1-m} \log\left(\mathrm{tr}\rho_A^m\right)$$

$$-\partial_m\left(\mathrm{tr}\rho_A^m\right)\Big|_{m=1} = -\partial_m\left(\mathrm{tr}\, e^{m \log \rho_A}\right)\Big|_{m=1} = -\mathrm{tr}\left(\log\rho_A\, e^{m\log\rho_A}\right)\Big|_{m=1}$$

$\boxed{\mathrm{tr}\rho_A^m}$ computed through the replica trick (Renyi entropies = entanglement measures for pure states)

$(t_E, \vec{x})$ $\quad t_E$ Euclidean time $\quad \vec{x} \in \mathbb{R}^{d-1} \qquad \phi(t_E, \vec{x})$

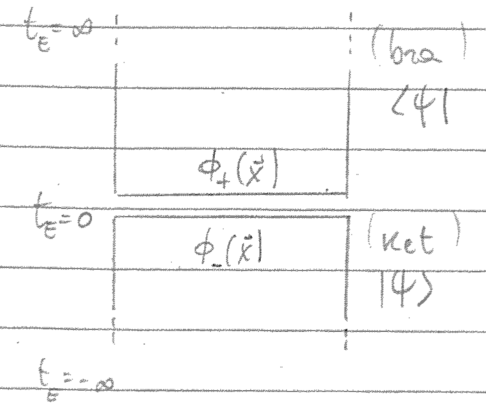
(a)



$$Z = \int_{t_E \in \mathbb{R}} D\phi\, e^{-S[\phi]} \qquad t_E \in \mathbb{R} \rightsquigarrow \vec{x} \in \mathbb{R}^{d-1} \text{ assumed}$$

$$|\Psi(\phi_0(\vec{x}))\rangle = \frac{1}{Z^{1/2}} \int_{t_E<0} D\phi\, \delta\left(\phi(0,\vec{x}) - \phi_0(\vec{x})\right) e^{-S[\phi]}$$

(b) density matrix



$$|\psi\rangle\langle\psi| = \frac{1}{Z} \int_{t_E<0} D\phi \int_{t_E>0} D\phi\; \delta\left(\phi(0^+,\vec{x}) - \phi_+(\vec{x})\right)\delta\left(\phi(0^-,\vec{x}) - \phi_-(\vec{x})\right) e^{-S[\phi]}$$

$$S[\phi] = \int_{t_E<0} \mathcal{L} + \int_{t_E>0} \mathcal{L}$$

$\phi_\pm(\vec{x})$ are "indices" of the matrix

$\phi_\pm$ $\rightarrow$ rows, $\rightarrow$ columns

Taking the trace:

(·) e.g.: matrix M_{ij} $\rightsquigarrow$ $\mathrm{tr}\, M = \sum_i M_{ii} = \sum_{ij} \delta_{ij} M_{ij}$ $\rightsquigarrow$ see the complement (*) [next sheet]

in particular $\mathrm{tr}\, |\psi\rangle\langle\psi|$ see example (*)

(·) $\mathrm{tr}_H(|\psi\rangle\langle\psi|) =$

$$= \frac{1}{Z}\int_{t_E=0} D\phi_+ D\phi_-\, \delta(\phi_+(\vec{x})-\phi_-(\vec{x})) \int_{t_E<0} D\phi \int_{t_E>0} D\phi\, \delta(\phi(0^+,\vec{x})-\phi_+(\vec{x}))\,\delta(\phi(0^-,\vec{x})-\phi_-(\vec{x}))\, e^{-S[\phi]} = 1$$

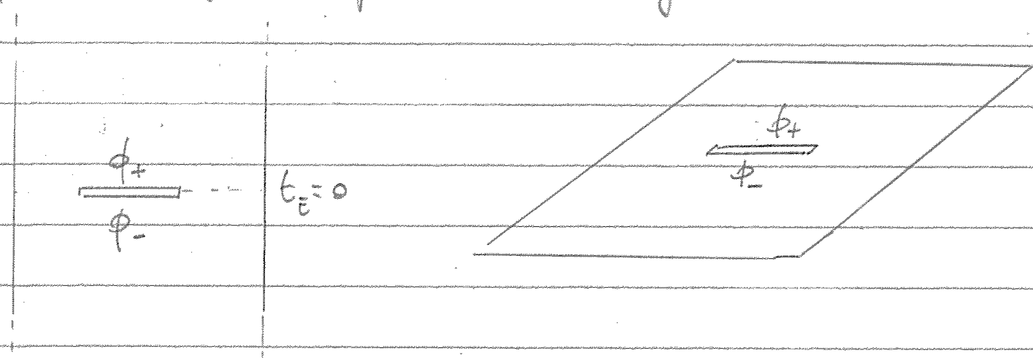
(·) Reduced density matrix

$$\rho_A = \mathrm{tr}_B\, |\psi\rangle\langle\psi| =$$

$$= \frac{1}{Z}\int_{t_E>0} D\phi \int_{t_E<0} D\phi \int_{\substack{t_E=0 \\ x\in B}} D\phi_+ D\phi_-\, \delta(\phi_+-\phi_-)\, e^{-\int_{t_E<0} L}\, e^{-\int_{t_E>0} L}$$

$$= \frac{1}{Z}\int_{t_E\in\mathbb{R}} D\phi \prod_{x\in A} \delta(\phi(0^+,\vec{x})-\phi_+(\vec{x}))\,\delta(\phi(0^-,\vec{x})-\phi_-(\vec{x}))\, e^{-S[\phi]}$$

path integral representation of the reduced density matrix



(*) finite T $\rightsquigarrow$ t_E compactified on S^1 of length $\beta = \frac{1}{T}$

(*) $T=0$, finite size $\rightsquigarrow$ x compactified on S^1 length L

(*) spatial boundaries

(complement)

$\circledast$

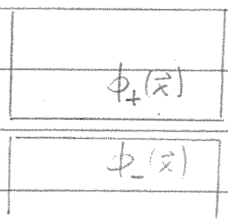
taking the trace

$$\underbrace{\langle\psi|\psi\rangle}_{\text{id}} = \sum_i \langle\psi|i\rangle\langle i|\psi\rangle = \sum_i \langle i|\psi\rangle\langle\psi|\overset{\text{id}}{\downarrow} i\rangle = \operatorname{tr}|\psi\rangle\langle\psi| = \underbrace{1}_{\text{normalization}}$$

$$= \sum_{ij} \langle i|\psi\rangle\langle\psi|j\rangle \underbrace{\langle j|i\rangle}_{\delta_{ij}}$$

$$\operatorname{tr}|\psi\rangle\langle\psi| = \underbrace{\sum_{ij}}\ \underbrace{\overbrace{\langle i|\psi\rangle\langle\psi|j\rangle}^{M_{ij}}}\ \underbrace{\delta_{ij}}$$

$$|\psi(\phi_-(\vec{x}))\rangle \qquad \langle\Psi(\phi_+(\vec{x}))|$$



$$\int_{\substack{t_E=0\\ x\in\mathbb{R}^{d-1}}} D\phi_+ D\phi_-\ \delta(\phi_+-\phi_-) \quad \text{trace all over } \vec{x}$$

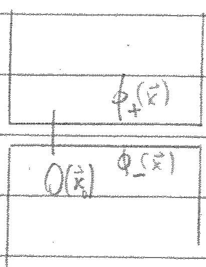
generalization:

$\circledast$

$$\langle\psi|\overset{O(\vec{x}_0)}{O}|\psi\rangle = \sum_i \langle\psi|O|i\rangle\langle i|\psi\rangle = \sum_i \overbrace{\langle i|\psi\rangle\langle\psi|}^{\rho}O|i\rangle = \operatorname{tr}(\rho O)$$

id id

$$= \sum_{i,j} \langle i|\psi\rangle\langle\psi|j\rangle\langle j|O|i\rangle$$



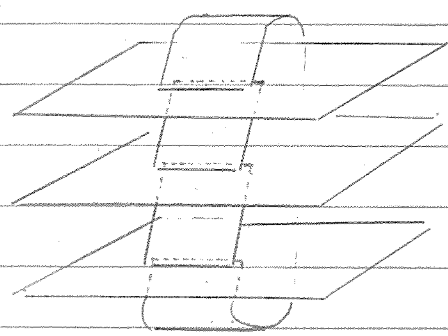
(a) Renyi entropies

$$\mathrm{tr}(M^n) = \sum_{i_1 \dots i_n} M_{i_1 i_2} M_{i_2 i_3} M_{i_3 i_4} \cdots M_{i_n i_1}$$

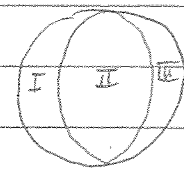
$$= \sum_{\substack{i_1 \dots i_n \\ j_1 \dots j_n}} M_{i_1 j_1} \delta_{j_1 i_2} M_{i_2 j_2} \delta_{j_2 i_3} M_{i_3 j_3} \cdots M_{i_n j_n} \delta_{j_n i_1}$$

1^{st} copy ↗ ↖ n-th copy

$M = \rho_A \Rightarrow$ $\mathcal{R}_n$ n-sheeted Riemann surface



topologically $\rightsquigarrow$



Sphere genus $= 0$

$$Z_n = \int_{\mathcal{C}} D\phi_1 \cdots D\phi_n \, e^{-\sum_{i=1}^{n} S[\phi_i]} \neq Z^n \text{ because of}$$

↳ boundary conditions: $\vec{x} \in A$ $\quad \phi_i(0^-, \vec{x}) = \phi_{i+1}(0^+, \vec{x})$

$$\boxed{\mathrm{tr}\, \rho_A^n = \frac{Z_n}{Z^n}}$$

(1) normalized to $\mathrm{tr}\, \rho_A = 1$

(2) cyclic symmetry $i \to i+1$ (global)

- ⊙ it holds $\forall$ QFT
- ⊙ generalizable to higher dimensions $\rightsquigarrow$ ∂A extended
- ⊙ $N \geq 2$ disjoint intervals.

Di Francesco yellow book
Ginsparg hep-th/9108028

Conformal group

$x^\mu \to x'^\mu$ change of coords $\Rightarrow$ $g_{\mu\nu}(x) \longrightarrow g'_{\mu\nu}(x') = \dfrac{\partial x^\alpha}{\partial x'^\mu}\dfrac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x)$

conformal group = subgroup of coordinate transformations such that

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \Omega^2(x)\, g_{\mu\nu}(x)$$

$\leadsto$ it contains. 1) Poincaré group $\left\{\begin{array}{l}\text{translations}\\ \text{rotations}\end{array}\right.$
2) dilatations
3) special conformal transformations.

1) $\begin{array}{l} x \to x' = x+a \\ x \to x' = \Lambda x \end{array}$ ↳ matrix (rotations)

2) $x \to x' = \lambda x$ ↳ $\lambda \in \mathbb{R}^+$

3) $x \to x' = \dfrac{x + b x^2}{1 + 2 b\cdot x + b^2 x^2}$ $\quad$ x^μ, b^μ

$\boxed{d=2}$
- finite group for $d>2$
- infinite group for $d=2$ $\qquad z \to w = f(z)$ $\quad \forall$ analytic function

global conformal group $\quad z \to \dfrac{az+b}{cz+d}$ $\quad \left.\begin{array}{l} a,b,c,d \in \mathbb{C} \\ ad-bc = 1 \end{array}\right\}$ 6 real parameters

generators : $\{L_{-1}, L_0, L_1\} \cup \{\bar{L}_{-1}, \bar{L}_0, \bar{L}_1\}$ $\qquad SL(2,\mathbb{C})/\mathbb{Z}_2$

$(a,b,c,d) \to -(a,b,c,d)$

$L_{-1}, \bar{L}_{-1}$ $\leadsto$ translations.
$L_0 + \bar{L}_0$ $\leadsto$ dilatation (translations of r)
$i(L_0 - \bar{L}_0)$ $\leadsto$ rotations (" " θ)
$L_1, \bar{L}_1$ $\leadsto$ special conformal transformations

$$\begin{cases} z = x + iy \\ \bar{z} = x - iy \end{cases}$$

$\mathbb{C}$ plane $\quad ds^2 = dx^2 + dy^2 = dz\, d\bar{z} \qquad g_{\mu\nu} = \delta_{\mu\nu}$

$$\begin{cases} \partial^\mu T_{\mu\nu} = 0 \\ T^\mu{}_\mu = 0 \end{cases} \qquad \begin{cases} T_{z\bar{z}} = T_{\bar{z}z} = 0 \\ \partial_{\bar{z}} T_{zz} = \partial_z T_{\bar{z}\bar{z}} = 0 \end{cases}$$

$$T_{zz} \equiv T(z) = \sum_{n\in\mathbb{Z}} \frac{1}{z^{n+2}} L_n \;;\quad T_{\bar{z}\bar{z}} = \bar{T}(\bar{z}) = \sum_{n\in\mathbb{Z}} \frac{1}{\bar{z}^{n+2}} \bar{L}_n$$

Virasoro algebra

$$\begin{cases} [L_n, L_m] = (n-m) L_{n+m} + \frac{c}{12}(n^2-1)n\, \delta_{n+m,0} \\ [L_n, \bar{L}_m] = 0 \\ [\bar{L}_n, \bar{L}_m] = (n-m)\bar{L}_{n+m} + \frac{\bar{c}}{12}(n^2-1)n\, \delta_{n+m,0} \end{cases}$$

primary fields: $\quad z \to w = w(z)$

$$\Phi(z,\bar{z}) \longrightarrow \Phi'(w,\bar{w}) = \left(\frac{dw}{dz}\right)^{-h} \left(\frac{d\bar{w}}{d\bar{z}}\right)^{-\bar{h}} \Phi(z,\bar{z})$$

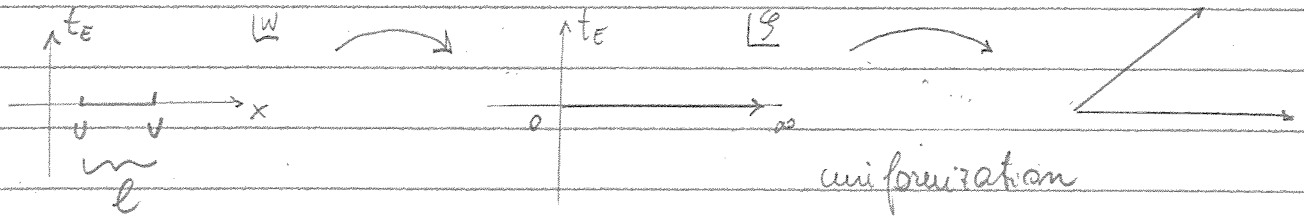
$(h,\bar{h})$ = conformal weights

$$\begin{cases} h+\bar{h} = \Delta & \text{scaling dim} \\ h-\bar{h} = s & \text{spin} \end{cases}$$

correlators: $\quad h_i = \bar{h}_i$

$$\Big\langle \prod_{i=1}^{n} \Phi_i(z_i,\bar{z}_i) \Big\rangle = \prod_{i=1}^{n} \left|\frac{dw}{dz}\right|_{z=z_i}^{2h_i} \Big\langle \prod_{i=1}^{n} \Phi(w_i,\bar{w}_i) \Big\rangle$$

2D CFT : Entanglement entropy of one interval



$$\zeta = \frac{w-u}{w-v} \qquad\qquad z = \zeta^{1/n}$$

$w \to z$ maps the n-sheeted Riemann surface R_n into the $\mathbb{C}$ plane z

two steps:

(1) Schwarzian derivative $\quad \{z, w\} \equiv \frac{1}{(z')^2}\left[z''' z' - \frac{3}{2}(z'')^2 \right]$

$$T(w) = \left(\frac{dz}{dw}\right)^2 T(z) + \frac{c}{12}\{z;w\}$$

$$\hookrightarrow \langle T(w)\rangle_{R_n} = \frac{c}{12}\{z, w\}$$

$$\langle T(z)\rangle_{\mathbb{C}} = 0$$

$$S_A^{(n)} = \frac{1}{1-n}\log \operatorname{tr}\rho_A^n =$$

$$= \frac{c}{6}\underbrace{\frac{1}{n-1}\left(n-\frac{1}{n}\right)}_{\left(1+\frac{1}{n}\right)}\log\frac{\ell}{a} + \frac{\log c_n}{1-n}$$

(2) Conformal Ward identity

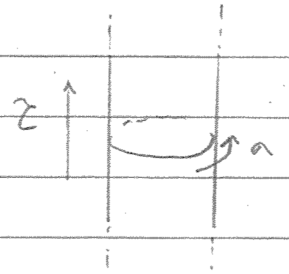
$$\rightsquigarrow \boxed{\operatorname{tr}\rho_A^n = \langle \mathcal{T}_n(u)\,\bar{\mathcal{T}}_n(v)\rangle_{\mathbb{C}} = \frac{c_n}{\left|\frac{u-v}{a}\right|^{2\Delta_n}}} \quad \boxed{\Delta_n = \frac{c}{12}\left(n-\frac{1}{n}\right)} \quad \boxed{c_1 = 1}$$

$$\rightsquigarrow -\partial_n\left(\operatorname{tr}\rho_A^n\right)\Big|_{n=1} = -c_1' - \underbrace{c_1}_{1}\,\partial_n\left(e^{-2\Delta_n\log(\ell/a)}\right)\Big|_{n=1} =$$

$$= \underbrace{\partial_n(2\Delta_n)}\Big|_{n=1}\log\left(\frac{\ell}{a}\right) - c_1' = \frac{c}{3}\log\left(\frac{\ell}{a}\right) - c_1'$$

easy replica limit $\quad \frac{c}{6}\left(1+\frac{1}{n^2}\right)$

(·) mapping the cylinder (∞) into the plane



$z = e^{\frac{2\pi}{L}(\tau + i a)}$

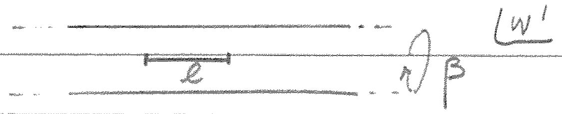
(·) transformation rule (primaries) $a \in [0, L]$

$$\langle \prod_{i=1} \Phi_i(z_i, \bar z_i) \rangle = \prod_i \left|\frac{dw}{dz}\right|_{z=z_i}^{2\Delta_i} \langle \prod_i \Phi(w_i, \bar w_i) \rangle$$

$\Delta_i = \bar\Delta_i$

each sheet coord

$w \to w' = \frac{\beta}{2\pi} \log w$



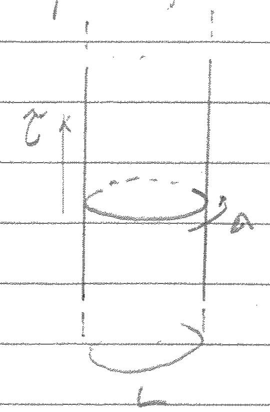
⊛ finite T, ∞ size $\Big\}$ $\Rightarrow$ branch cut $\parallel$ axis of the cylinder

$$S_A = \frac{c}{3} \log\left[\frac{\beta}{\pi a} \sinh\left(\frac{\pi \ell}{\beta}\right)\right] + \text{const}$$

$\rightsquigarrow$ $\ell \ll \beta$ $(T = 0)$ $\Rightarrow$ $S_A = \frac{c}{3} \log\left(\frac{\ell}{a}\right) + \text{const}$

$\rightsquigarrow$ $\ell \gg \beta$ $\Rightarrow$ $S_A = \frac{c}{3}\left(\frac{\pi \ell}{\beta}\right) + \dots$ Gibbs entropy (extensive) of an isolated system of length ℓ

⊛ finite size L, $T = 0$ $\Big\}$ branch cut $\perp$ axis of the cylinder



$$S_A = \frac{c}{3} \log\left[\frac{L}{\pi a} \sin\left(\frac{\pi \ell}{L}\right)\right] + \text{const}$$

$\rightsquigarrow$ $\ell \ll L$ $\Rightarrow$ $S_A = \frac{c}{3} \log\left(\frac{\ell}{a}\right) + \text{const}$

⊠ boundary CFT $\rightsquigarrow$ $\langle \mathcal{T}_n(i\ell) \rangle_{\mathbb{H}}$ $\rightsquigarrow$ $S_A = \frac{c}{6} \log\left(\frac{2\ell}{a}\right) + \text{const}$

[Calabrese - Cardy hep-th/0405152]

$$S_A = -\operatorname{tr}(\rho_A \log \rho_A) = \lim_{n\to 1^+} \underbrace{\left[\frac{1}{1-n}\log\left(\operatorname{tr}\rho_A^n\right)\right]}_{S_A^{(n)}} = -\partial_n \operatorname{tr}\rho_A^n\Big|_{n=1}$$

$\uparrow$ $\operatorname{tr}_{\mathcal{H}_A}$

$S_A^{(n)}$ Renyi entropies $n \geq 2$

$$\operatorname{tr}\rho_A^n = \frac{Z_n}{Z_1^n}$$

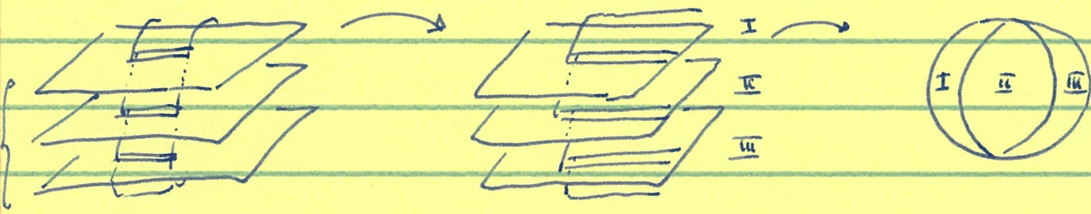
2-dim $\Rightarrow$ A = interval length ℓ $\Rightarrow$ R_n is a <u>sphere</u>

$$\operatorname{tr}\rho_A^n = \langle \mathcal{T}_n(u)\,\bar{\mathcal{T}}_n(v)\rangle$$
$$= \frac{Z_n}{Z_1^n} \rightsquigarrow \text{partition function on } R_n$$

$\hookrightarrow$ n-sheeted Riemann surf.

CFT

n copies



$$\operatorname{tr}\rho_A^n = \frac{c_n}{\underbrace{|u-v|}_{\hookrightarrow\, \ell/a}{}^{2\Delta_n}} \qquad \Delta_n = \frac{c}{12}\left(n - \frac{1}{n}\right) \qquad c_1 = 1$$

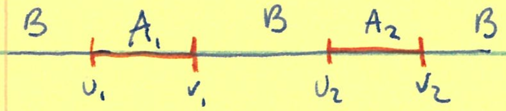
$\begin{cases} \infty \text{ line (size)} \\ T = 0 \end{cases}$

$\rightsquigarrow$ $\begin{cases} \infty \text{ size} \\ T > 0 \end{cases}$ $\quad \ell \to \dfrac{\beta}{\pi}\sinh\left(\dfrac{\ell\pi}{\beta}\right)$

$\rightsquigarrow$ $\begin{cases} L < \infty \\ T = 0 \end{cases}$ $\quad \ell \to \dfrac{L}{\pi}\sin\left(\dfrac{\pi\ell}{L}\right)$

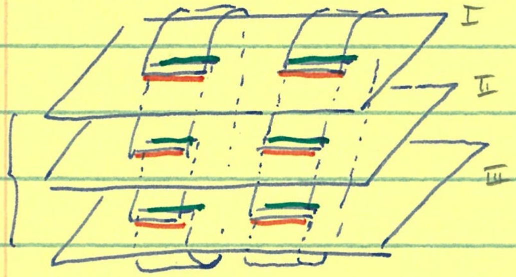
1 interval $\Rightarrow$ $S_A^{(n)}$ contains only the <u>central charge</u>

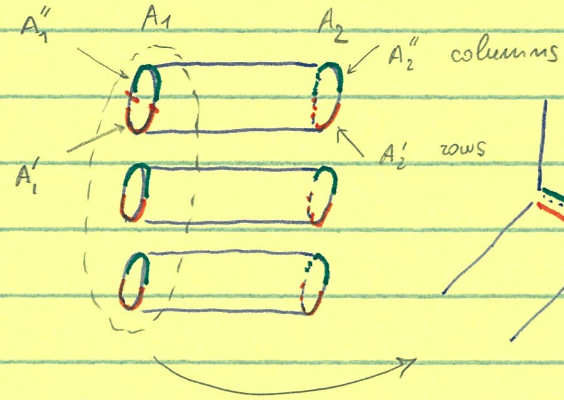
2-dim: <u>two disjoint intervals</u> $A = A_1 \cup A_2$ $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

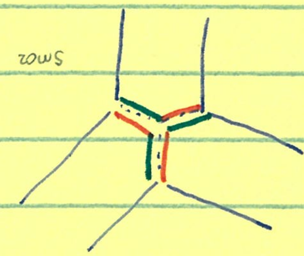


$S_A = ?$ bipartite entropy

n copies

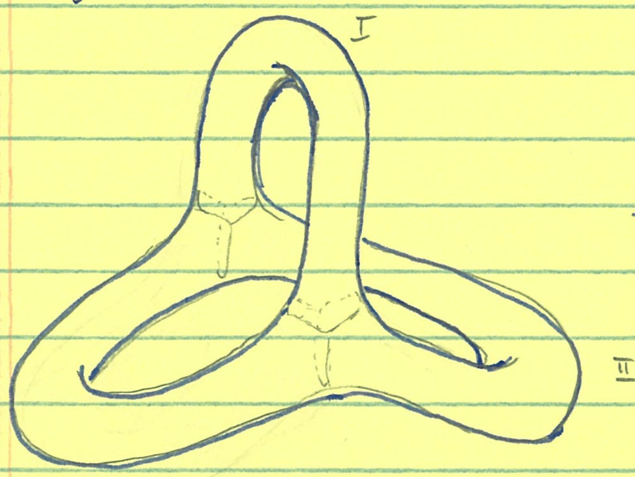






$n = 2 \rightsquigarrow$ torus

$n = 3$

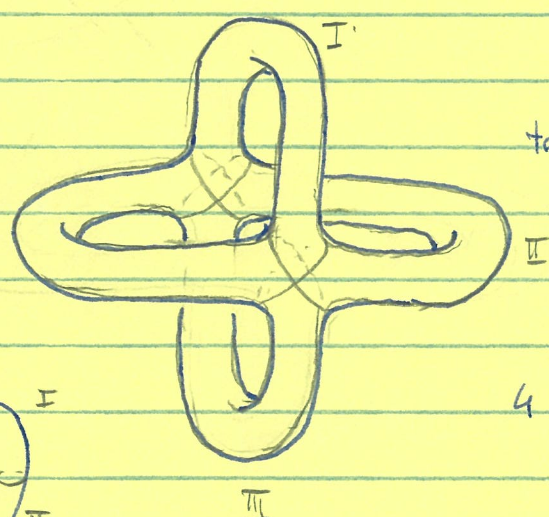


topologically $\rightsquigarrow$ $g = 2$

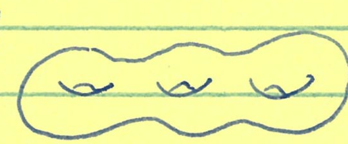
replica construction

$\Rightarrow$ 3 equal handles

$n = 4$



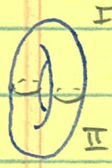
topologically $\rightsquigarrow$ $g = 3$



4 equal handles

$n = 2$

$\Rightarrow$ torus



n copies $\Rightarrow$ $g = n - 1$

($N = 2$ intervals)

N intervals

n copies $\Rightarrow$ $g = (N-1)(n-1)$

$\mathrm{tr}\,\rho_A^n = \dfrac{Z_n}{Z_1^n}$ ⇝ n-sheeted Riemann surface R_n

R_n genus $g = n-1$ } genus g ⇒ moduli space
3g − 3 complex parameters
($g \geq 2$)

$\mathrm{tr}\,\rho_A^n = \langle \mathcal{T}_n(u_1)\,\bar{\mathcal{T}}_n(v_1)\,\mathcal{T}_n(u_2)\,\bar{\mathcal{T}}_n(v_2)\rangle =$ four point function in CFT
⇝ 1 harmonic ratio

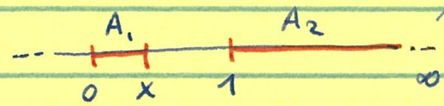
CFT — 4-point function of primaries with Δ_n — CFT

$$= c_n^2 \left(\frac{|u_1-u_2|\,|v_1-v_2|}{|u_1-v_1|\,|u_2-v_2|\,|u_1-v_2|\,|u_2-v_1|} \right)^{2\Delta_n} \mathcal{F}_n(x)$$

$\mathcal{F}_n(x)$: all CFT data
} model dependent

$$x = \frac{(u_1-v_1)(u_2-v_2)}{(u_1-u_2)(v_1-v_2)} \in (0,1)$$

$$\Delta_n = \frac{c}{12}\left(n - \frac{1}{n}\right)$$



(*) only one real parameter $x \in (0,1)$ ⇒ subclass of Riemann surfaces

examples:

⊙ non compact free boson ($c=1$)

$$\mathcal{F}_n(x) \propto \frac{1}{\prod_{k=1}^{n-1} F_{k/n}(x)\, F_{k/n}(1-x)} \; ; \quad F_{k/n}(x) \equiv {}_2F_1\!\left(\tfrac{k}{n}, 1-\tfrac{k}{n}; 1; x\right)$$

→ def of Θ behind

⊙ compact free boson
$\eta \propto R^2$
target space

$$\mathcal{F}_n(x) = \frac{\Theta(\tau/\eta)\,\Theta(\eta\tau)}{\Theta(\tau)^2}$$

τ: period matrix of R_n ($g\times g$)
symmetric
$\mathrm{Im}\,\tau > 0$

$(y,z) \in \mathbb{C}^2$
$y^n = (z-u_1)(z-u_2)\left[(z-v_1)(z-v_2)\right]^{n-1}$

$$\tau_{rs}(x) = i\,\frac{2}{n} \sum_{k=1}^{n-1} \sin\!\left(\pi \frac{k}{n}\right) \frac{F_{k/n}(1-x)}{F_{k/n}(x)} \cos\!\left[2\pi \frac{k}{n}(r-s)\right] \qquad r,s = 1,\dots,n-1$$

⊙ Ising model ($c = 1/2$)

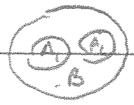
$$\mathcal{F}_n(x) = \frac{1}{2^{n-1}} \sum_{\varepsilon,\delta} \left| \frac{\Theta\left[{\varepsilon \atop \delta}\right](0|\tau)}{\Theta(0|\tau)} \right|$$

⊙ Dirac fermion ($c=1$)

$$\mathcal{F}_n(x) = \frac{1}{2^{n-1}} \sum_{\varepsilon,\delta} \left| \dots \right|^2$$

ENTANGLEMENT NEGATIVITY

why? entanglement mixed states



entanglement between A_1 and A_2 $\Rightarrow$ no mutual info

$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ bipartite system $\qquad \rho$ density matrix

$- \; |e_i^{(1)}\rangle |e_j^{(2)}\rangle$

$$\rho^{T_2} \rightsquigarrow \langle e_i^{(1)} e_j^{(2)} | \rho^{T_2} | e_k^{(1)} e_l^{(2)} \rangle \equiv \langle e_i^{(1)} e_k^{(2)} | \rho | e_j^{(1)} e_l^{(2)} \rangle$$

PARTIAL TRANSPOSE

⊙ ρ^{T_2} has also negative eigenvalues $\qquad$ NEGATIVITY $\mathcal{N}$

$$\|\rho^{T_2}\|_1 = \mathrm{tr}|\rho^{T_2}| = \sum_i |\lambda_i| = \sum_{\lambda_i>0} \lambda_i - \sum_{\lambda_i<0} \lambda_i \overset{\Uparrow}{=} 1 - 2\sum_{\lambda_i<0} \lambda_i = 1 + 2\sum_{\lambda_i<0} |\lambda_i|$$

: normalization $\mathrm{tr}\,\rho^{T_2} = \sum_i \lambda_i = \sum_{\lambda_i>0} \lambda_i + \sum_{\lambda_i<0} \lambda_i = 1$

$$\boxed{\mathcal{E} \equiv \log \|\rho^{T_2}\|_1}$$ LOGARITHMIC NEGATIVITY

⊙ $\mathcal{N}$ is an entanglement monotone
$\mathcal{E}_1 = \mathcal{E}_2$ for any state ($S_1 \neq S_2$ finite T) $\Big\}$ $\mathcal{E}$ measure of entanglement for mixed states.

⊙ replica approach
$$\begin{cases} \mathrm{tr}(\rho^{T_2})^{n_e} = \sum \lambda_i^{n_e} = \sum_{\lambda_i>0} |\lambda_i|^{n_e} + \sum_{\lambda_i<0} |\lambda_i|^{n_e} \\ \mathrm{tr}(\rho^{T_2})^{n_o} = \sum_i \lambda_i^{n_o} = \sum_{\lambda_i>0} |\lambda_i|^{n_e} - \sum_{\lambda_i<0} |\lambda_i|^{n_o} \end{cases}$$

$$\mathcal{E} = \log \sum_i |\lambda_i| = \lim_{n_e \to 1} \log \mathrm{tr}(\rho^{T_2})^{n_e}$$ make 1 an even number

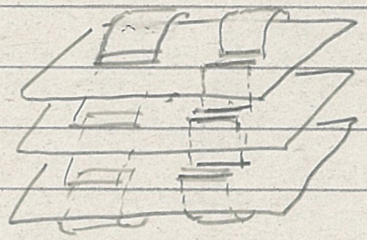
$$\lim_{n_e \to 1} \mathrm{tr}(\rho^{T_2})^{n_e}$$

ENTANGLEMENT NEGATIVITY QFT

$$\mathcal{E} = \lim_{n_e \to 1} \log \mathrm{tr}\left(\rho_A^{T_2}\right)^{n_e}$$

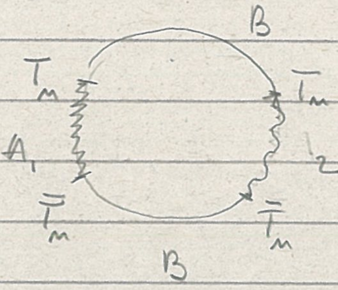
$\rho_A^{T_2}$ partial transpose

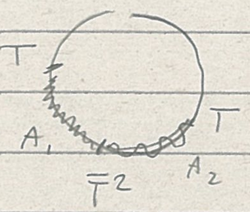
$$\mathrm{tr}\left(\rho_A^{T_2}\right)^{n} =$$



$$= \langle \mathcal{T}_n(u_1)\, \bar{\mathcal{T}}_n(v_1)\, \bar{\mathcal{T}}_n(u_2)\, \mathcal{T}_n(v_2) \rangle$$

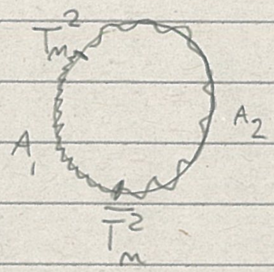
$A = A_1 \cup A_2$





adjacent intervals

$B \to \emptyset$



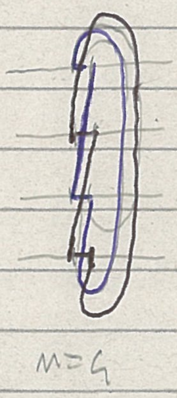
pure state (bipartite entanglement)

PURE STATES : single interval

⊙

$$\mathrm{tr}\left(\rho_A^{T_2}\right)^{n} = \langle \mathcal{T}_n^2(u)\, \bar{\mathcal{T}}_n^2(v) \rangle \overset{\text{CFT}}{=} \frac{c_n^{(2)}}{|u-v|^{2\Delta_n^{(2)}}} \qquad \Delta_n^{(2)} = ?$$

$\mathcal{T}_n^2 \rightsquigarrow$ jump of two sheets turning around — $\mathcal{T}_n^2$ $\Rightarrow$ parity effect



$n=4$

$\Downarrow$

factorization for even n

$$\mathrm{tr}(\rho_A^{T_2})^n = \langle \mathcal{T}_n^2(u)\,\bar{\mathcal{T}}_n^2(v)\rangle = \begin{cases} \left(\langle \mathcal{T}_{n/2}(u)\,\bar{\mathcal{T}}_{n/2}(v)\rangle\right)^2 & n = n_{even} \\ \langle \mathcal{T}_n(u)\,\bar{\mathcal{T}}_n(v)\rangle & n = n_{odd} \end{cases}$$

$$\Rightarrow \quad \Delta_n^{(2)} = \begin{cases} 2\Delta_{n/2} = \frac{c}{6}\left(\frac{n}{2} - \frac{2}{n}\right) & n = n_{even} \\ \Delta_n = \frac{c}{12}\left(n - \frac{1}{n}\right) & n = n_{odd} \end{cases}$$

$\odot$ this is a general result $\forall$ quantum system (see Schmidt decomposition)

$$\text{pure state} \qquad \mathrm{tr}(\rho^{T_2})^n = \begin{cases} \mathrm{tr}_A\,\rho_A^n & n \text{ odd} \\ \left(\mathrm{tr}\,\rho_A^{n/2}\right)^2 & n \text{ even} \end{cases}$$

$$\Rightarrow \quad \mathcal{E} = \lim_{n_e \to 1} \log \mathrm{tr}\,\rho_A^{n_e} = 2\log \mathrm{tr}\,\rho_A^{1/2} = \overbrace{\underbrace{\frac{1}{1-n}\log \mathrm{tr}\,\rho_A^n}_{S^{(n)}}\Big|_{n=1/2}}^{S_A^{(1/2)}}$$

$$\text{CFT} \quad \Rightarrow \quad \mathcal{E} = \frac{c}{2}\log\frac{\ell}{a} + \text{const}$$

<u>two adjacent intervals</u>

$-\ell_1 \qquad 0 \qquad \ell_2$

$$\mathrm{tr}(\rho_A^{T_2})^n = \langle \mathcal{T}_n(-\ell_1)\,\bar{\mathcal{T}}_n^2(0)\,\mathcal{T}_n(\ell_2)\rangle = \dots$$

3-point function of a CFT behind

$$\mathcal{E} = \frac{c}{4}\log\left(\frac{\ell_1\ell_2}{\ell_1+\ell_2}\right) + \text{const}$$

two disjoint intervals

$$\mathrm{tr}(\rho_A^{T_2})^n = \tilde{c}_n^2 \left(\frac{|u_1-u_2||v_1-v_2|}{\prod_{i,j}|u_i-v_j|} \right)^{2\Delta_n} \mathcal{G}_n(x)$$

$$\boxed{\mathcal{G}_n(x) = (1-x)^{\frac{c}{3}(n-1/n)} \mathcal{F}_n\left(\frac{x}{x-1}\right)}$$

$$\mathcal{E}(x) = \lim_{n_e \to 1} \log \mathrm{tr}(\rho_A^{T_2})^{n_e} = \lim_{n_e \to 1} \log \mathcal{G}_{n_e}(x) =$$

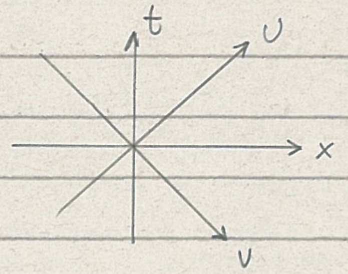
$$= \lim_{n_e \to 1} \log \mathcal{F}_{n_e}\left(\frac{x}{x-1}\right)$$

very difficult

$$x \in (0,1) \Rightarrow \frac{x}{x-1} \in (-\infty, 0)$$

c-theorem through Strong Sub-additivity

Casini - Huerta hep-th/0405111



$$\begin{cases} U = x + t \\ V = x - t \end{cases} \qquad dU dV = -dt^2 + dx^2$$

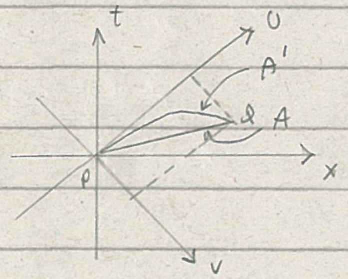
$$\begin{cases} P_i = (t_i, x_i) = (U_i, V_i) \\ P_j = (t_j, x_j) = (U_j, V_j) \end{cases} \qquad \overline{P_i P_j} \text{ spacelike}$$

$$\begin{aligned} U_{ij} V_{ij} = (U_i - U_j)(V_i - V_j) &= [(x_i + t_i) - (x_j + t_j)][(x_i - t_i) - (x_j - t_j)] \\ &= [(x_i - x_j) + (t_i - t_j)][(x_i - x_j) - (t_i - t_j)] \\ &= -(t_i - t_j)^2 + (x_i - x_j)^2 > 0 \end{aligned}$$

↳ spacelike

$\overline{P_i P_j} = r^2 = U_{ij} V_{ij} > 0$

(1)



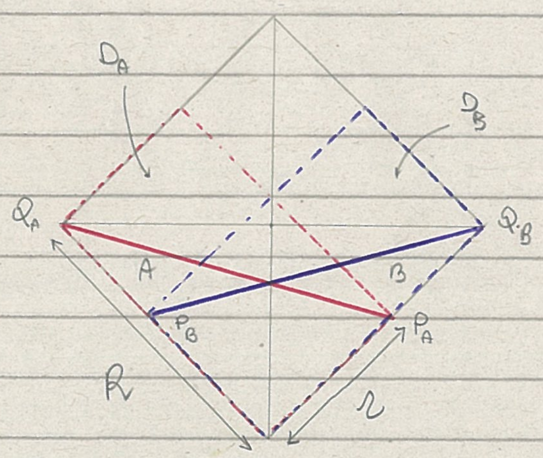
spacelike

A and A' same domain of dependence $D_A = D_{A'}$

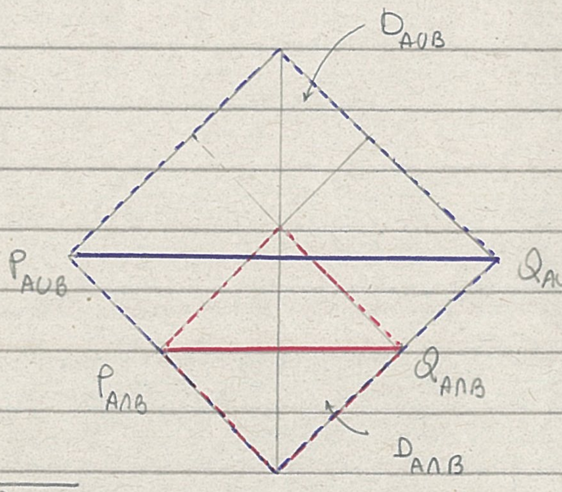
$\Rightarrow S_A = S_{A'} = s(r)$

↳ $\overline{PQ}$

(2) pick A and B two equal boosted spacelike intervals



$\overline{P_A Q_A} = \overline{P_B Q_B} = \sqrt{rR}$



$$\begin{cases} \overline{P_{A\cap B} Q_{A\cap B}} = r \\ \overline{P_{A\cup B} Q_{A\cup B}} = R \end{cases}$$

(3) Strong sub-additivity

$$S_A + S_B \geq S_{A\cup B} + S_{A\cap B}$$

$$2\, S(\sqrt{Rr}) \geq S(R) + S(r) \qquad (*)$$

$R = r + \varepsilon$ with $\varepsilon > 0$ and $\varepsilon \to 0^+$

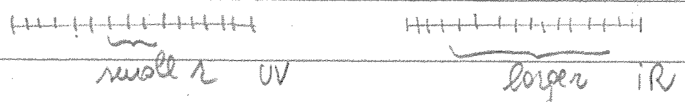
Expanding $(*)$ for $\varepsilon \to 0$ the $O(1)$ and $O(\varepsilon)$ terms cancel

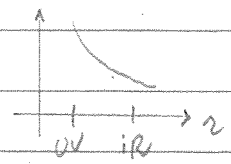
$O(\varepsilon^2) \rightsquigarrow \quad r S'' + S' \leq 0$

$\boxed{C(r) \equiv r S'(r)} \qquad \boxed{C'(r) \leq 0}$

$C(r)$ is (1) dimensionless and (2) decreasing

- At the critical point $S(r) = \frac{c}{3} \log \frac{r}{a} + \text{const} \Rightarrow C(r) = \frac{c}{3}$
- increasing r we go from UV to IR





$\rightsquigarrow \quad \boxed{c_{UV} \geq c_{IR}}$ Zamolodchikov (1986) c-theorem

(4) 2+1 dims. $S(r) = \gamma \frac{2\pi R}{\epsilon} - F \qquad F_{UV} \geq F_{IR}$ (F theorem)

[Casini - Huerta 1202.5650]

(5) 2D $\langle T^\mu{}_\mu \rangle = \frac{c}{24\pi} R$

4D $\langle T^\mu{}_\mu \rangle = a E_4 - c W^2$

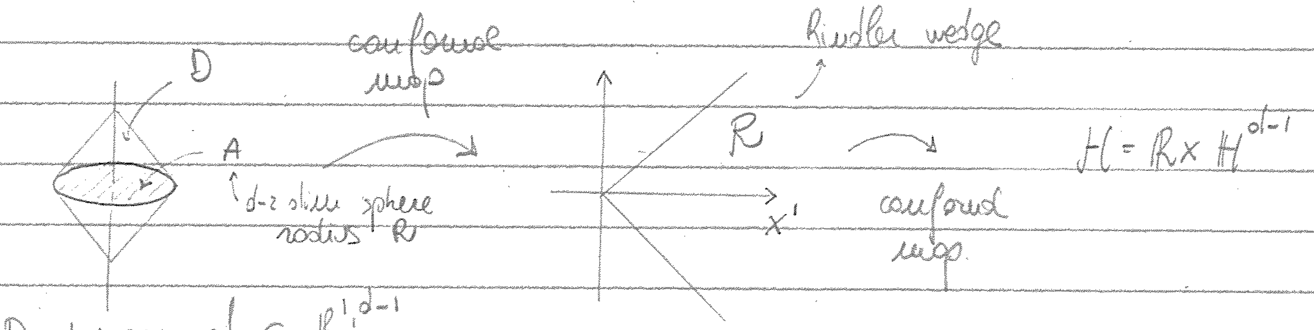
1107.3987

[Cardy 1988] [Komargodski - Schwimmer]

$$\begin{cases} E_4 = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \\ W^2 = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 2 R_{\mu\nu} R^{\mu\nu} + \frac{1}{3} R^2 \end{cases} \qquad a_{UV} \geq a_{IR}$$

$\hookrightarrow W_{\mu\nu\rho\sigma} W^{\mu\nu\rho\sigma}$ Weyl squared

Spherical region in CFT_d [Casini - Huerta - Myers] 1102.0440



D domain of dependence $\subset \mathbb{R}^{1,d-1}$

$T_0 = \frac{1}{2\pi R}$

$\rho_D = U^{-1} \rho_{th}(T_0)\, U$

ρ_D

$\left\{ \begin{array}{l} \cdot \text{ Bisognano - Wichmann theorem} \\ \cdot \text{ Unruh effect} \end{array} \right.$ $\rightarrow$ $\forall$ QFT modular Hamiltonian = boost generator along x^1

$\mathcal{R}$ = Rindler wedge = causal development of half space $x^1 > 0$

review of the map in Euclidean signature $\mathbb{R}^d \longrightarrow S^1 \times \mathbb{H}^{d-1}$

$$ds^2_{\mathbb{R}^d} = dt_E^2 + dr^2 + r^2 d\Omega^2_{d-2} = dw\, d\bar{w} + \left(\frac{w+\bar{w}}{2}\right)^2 d\Omega^2_{d-2} \qquad w = r + i t_E$$

entangling surface $(t_E, r) = (0, R)$ or $w = R$

$$e^{-a} = \frac{R-w}{R+w} \qquad a = u + i\frac{\tau_E}{R} \in \mathbb{C} \Rightarrow \text{entangling surface } u \to +\infty$$

$$ds^2_{\mathbb{R}^d} = \Omega^{-2} R^2 \left[da\, d\bar{a} + \sinh^2 \underbrace{\left(\frac{a+\bar{a}}{2}\right)}_{u} d\Omega^2_{d-2} \right]; \quad \Omega = \frac{2R^2}{|R^2 - w^2|} = |1 + \cosh a|$$

Weyl rescaling

R curvature of $\mathbb{H}^{d-1}$

$$ds^2_{S^1 \times \mathbb{H}^{d-1}} = \Omega^2 ds^2_{\mathbb{R}^d} = d\tau_E^2 + R^2 \left[du^2 + \sinh^2 u\, d\Omega^2_{d-2} \right]$$

$\hookrightarrow$ lenght of S^1 = period of $\tau_E = 2\pi R$ $\Rightarrow$ thermal CFT $T_0 = \frac{1}{2\pi R}$ on $S^1 \times \mathbb{H}^{d-1}$

$$\rho_A \sim \rho_D = U^{-1} \rho_H U \qquad \rho_{th} = \underbrace{\frac{e^{-\beta H}}{\mathrm{tr}(e^{-\beta H})}}_{Z(T)} \qquad \mathrm{tr}\,\rho_{th} = 1$$

sphere in $\mathbb{M}^d|_{t=\mathrm{const}}$

$\mathbb{R} \times \mathbb{H}^{d-1}$ thermal $T_0 \equiv 1/2\pi R$

$$\Rightarrow \rho_A^m = U^{-1} \frac{e^{-\frac{m}{T_0}H}}{Z(T_0)^m} U \Rightarrow \mathrm{tr}\,\rho_A^m = \frac{\mathrm{tr}\, e^{-\frac{m}{T_0}H}}{Z(T_0)^m} = \frac{Z(T_0/m)}{Z(T_0)^m}$$

$$\Rightarrow S_A^m = \frac{1}{1-m} \log \mathrm{tr}\,\rho_A^m = \frac{1}{1-m}\left[\log Z(T_0/m) - m \log Z(T_0)\right]$$

$$\overset{!}{=} \frac{1}{1-m}\left[-\frac{m}{T_0} F(T_0/m) + \frac{m}{T_0} F(T_0)\right]$$

(*) def FREE ENERGY

$$Z(T) = e^{-\beta F(T)}$$

$$\overset{!}{=} \frac{m}{1-m}\frac{1}{T_0}\left[F(T_0) - F(T_0/m)\right]$$

(**) from thermodynamics

$$S_{th} = -\frac{\partial F}{\partial T}$$

$$F(T) = \int_T^{T_*} S_{th}(\tilde{T})\, d\tilde{T}$$

$$\overset{!}{=} \frac{m}{1-m}\frac{1}{T_0}\left[\int_{T_0}^{T_*} S_{th}(\tilde{T})\,d\tilde{T} - \int_{T_0/m}^{T_*} S_{th}(\tilde{T})\,d\tilde{T}\right]$$

$$= \frac{m}{1-m}\frac{1}{T_0}\int_{T_0}^{T_0/m} S_{th}(\tilde{T})\,d\tilde{T}$$

$$= \frac{m}{m-1}\frac{1}{T_0}\int_{T_0/m}^{T_0} S_{th}(\tilde{T})\,d\tilde{T}$$

eq. (1.10) of 1110.1084
Hung-Myers-Smolkin-Yale

thermal entropy of the CFT on $\mathbb{R} \times \mathbb{H}^{d-1}$

$$\Rightarrow \lim_{m\to 1} S_A^{(m)} = S_{th}(T_0)$$

(see back of this sheet)