1.

- Intro to Queantum Estimation theory (~90min ?)
- Appl. to Heisenberg's uncertainty relation (~30min ?)

Q estimation theory

- examples

State est (or tomography)

$$\underset{\text{unknown}}{\rho} = \rho(\underset{\text{unknown}}{\theta}) \longrightarrow \underset{\text{measurement}}{\mathbb{K}} \longrightarrow \underset{x}{\text{outcomes}} \longrightarrow \theta^{est}(x)$$

- measurement est

$$\underset{\text{known}}{\rho} \longrightarrow \mathbb{K}^{\leftarrow \text{unknown}} = \mathbb{K}(\underset{\text{unknown}}{\theta}) \longrightarrow \overset{\text{outcome}}{x} \longrightarrow \theta^{est}(x)$$

- Process est

$$\rho \to \underset{\text{unknown}}{\overset{\text{channel}}{\boxed{\mathcal{E}(\theta)}}} \longrightarrow \rho'(\underset{\text{unknown}}{\theta}) \longrightarrow \mathbb{K} \longrightarrow x \to \theta^{est}(x)$$

Q est theory = ① Q measurement

+ ② chassical estimation.

2.

Q measurement theory

$$\sum_x P(x) = 1$$

$$\Rightarrow \sum_{x,i} K_{x,i}^\dagger K_{x,i} = 1$$

$\mathbb{K} = \{K_{x,i}\}$: measurement op or kraus op.

$$p(x) = \sum_i \mathrm{Tr}[K_{x,i}\, \rho\, K_{x,i}^\dagger] = \mathrm{Tr}[E_x \rho]$$

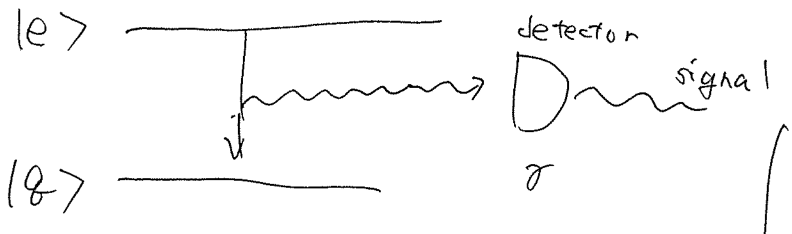
$$E_x := \sum_i K_{x,i}^\dagger K_{x,i} : \text{POVM}$$

$$\rho' = \sum_{x,i} M_{x,i}\, \rho\, M_{x,i}$$

e.g. Projective measurement $A = \sum_x a_x P_x$

$$\left(\begin{array}{l} P(x) = \mathrm{Tr}[P_x \rho] \\ \rho' = \sum_x P_x \rho P_x \end{array} \right.$$

2-d Hilbert space $|g\rangle, |e\rangle$



$x=0$ no signal

$x=1$ singnal

photonがでたら必ずこれ

$$\left(\begin{array}{l} K_{1,0} = \sqrt{\gamma}\, |g\rangle\langle g| \\ K_{0,0} = |g\rangle\langle g| \\ K_{0,1} = \sqrt{1-\gamma}\, |e\rangle\langle e| \end{array} \right)$$

measureできなかったら
計算すべきの問題
でーでなかった分の状態が
あるのでこれ2つある.

Classical est theory

e.g. ~~coin~~ Coin toss

$\Theta = [0, 1] \subseteq R$

$\mathcal{X} = \{0, 1\}$

$p(0, \theta) = \theta$

$p(1, \theta) = 1 - \theta$

$$\Theta^{est} : \mathcal{X} \to \Theta \subseteq R^m$$

(classical) ~~stasti~~ statistical model

$$\Theta \subseteq R^n$$

$$p(x; \theta), \quad \theta \in \Theta$$

$$\mathcal{X} = \{0, 1, \cdots n\}, \quad x \in \mathcal{X}$$

Gausian $\Theta = R \times R_+$

$\theta_1 = \mu$

$\theta_2 = \sigma^2$

$$P(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$\Theta^{est}(x) = (1, 1)$ ~~cost~~ const

↳ xによってない ⟹ meaningless!!

- restriction to estimator

unbiased est

$$\mathbb{E}[\theta^{est}(x)] = \theta$$

$$\parallel$$

$$\sum_x p(x; \theta)\, \theta^{est}(x)$$

$$\theta_1^{est}(x) = \mu^{est}(x) = \sum_i \frac{x_i}{n}$$

n: # of samples

$$\theta_2^{est}(x) = (\sigma^2)^{est} = \sum \frac{(x_i - \mu^{est}(x))}{n-1}$$

$\mathbb{E}(\mu^{est}(x)) = \mu$

· consistency

$\forall \varepsilon > 0 \quad \lim_{n \to \infty} \mathrm{Prob}(|\theta^{est}(x) - \theta| > \varepsilon) \to 0$

n ← # of sample

· Cramér – Rao ineq.

$$\lim_{n \to \infty} n \underbrace{\mathrm{Var}[\theta^{est}]}_{\text{Covariant matrix}} \geq J_\theta^{-1}$$

error var

J_θ ← matrix called Fisher information

$$J(\theta)_{ij} := \sum_x P(x;\theta) \left(\partial_i \log P(x;\theta)\right)\left(\partial_j \log P(x;\theta)\right)$$

$$\mathrm{Var}[\theta^{est}] = \mathbb{E}[(\theta^{est})^2] - \mathbb{E}[\theta^{est}]^2$$

Maximally Likelihood est, ($\cancel{\ }$ MLE)

$$\theta^{MLE}(x) := \operatorname*{arg\,max}_{\theta \in \Theta} \log P(x:\theta) \qquad P(x:\theta) = \prod_{i=1}^{n} P(x_i:\theta)$$

~~arg~~ max

$x = (x_1, x_2, \cdots x_n)$

$$= \operatorname*{arg\,max}_{\theta \in \Theta} \sum_x h_x \log P(x:\theta)$$

これはどうきまってる

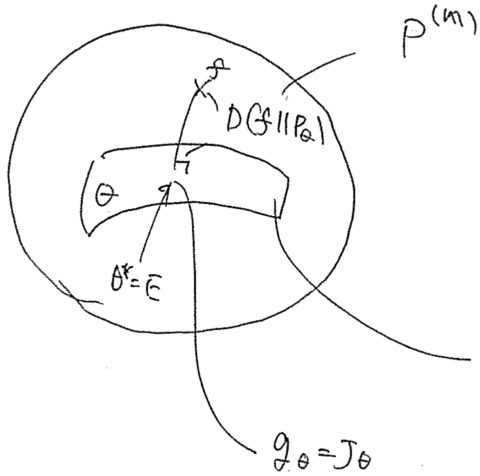
h_x ← ~~# of~~ # we obtain x

$$\hat{\Theta}^{MLE}(x) = \underset{\theta \in \Theta}{\arg\min}\, D(f \| P_\theta)$$

$f(x) = \frac{n_x}{n}$: frequency

relative entropy $D(p\|q) = \sum_x p(x)\log p(x) - p(x)\log q(x)$

· Information Geometrical aspect of MLE



$P^{(m)} = \{ p \mid p(x) > 0, \sum_{x \in X} p(x) = 1 \}$, $x = \{0, 1, \cdots m\}$

$M \subseteq P$: statistical model
$\quad \| $
$\{ P_\theta : \theta \in \Theta \}$

Fisher information = Fisher metric

· monotonicity of Fisher info

Markov map or Statistical map

$$q(y) = \sum_x \underbrace{K(y|x)}_{\text{conditional prob}} p(x)$$

e.g. p $x = \{1, \cdots 6\}$ dice

$\downarrow$

q : $y = \{\text{odd}, \text{even}\}$

$$k(y|x) = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

for all k

$$\underset{\underset{k(p)}{\shortparallel}}{J_q} \leq J_p$$

$$\begin{cases} k(y|x) \geq 0 \\ \sum_y k(y|x) = 1 \end{cases}$$

$(p^{(m)}, q^{(m)})$ $m = 1, 2$

If for all $k : p^{(n)} \to p^{(e)}$

$$g_{k(p)}^{(e)} \leq g_p^{(m)}$$

Then $g^{(m)} = cJ$ $c > 0$

$g^{(m)}$: 情報空間のmetric

J: fisher metric

~~relative~~

relation between $D(p||q)$ and Fisher info metric

$$\partial_i \partial_j D(P_\theta || P_{\theta_0})\Big|_{\theta=\theta_0} = \frac{1}{2} J_{ij}$$

$$D(P_{\theta_0} || P_{\theta_0 + \Delta\theta}) \simeq \frac{1}{2} \Delta\theta \cdot J_{\theta_0} \Delta\theta$$

f-divergence

$$D_f(p||q) := \sum_x p(x) f\left(\frac{q(x)}{p(x)}\right)$$

f: convex function

$$\begin{cases} D_f(p||q) \geq D_f(k(p)||k(q)) \\ D_f(p||p) = 0 \end{cases}$$

$$\partial_i \partial_j D_f(P_\theta || P_{\theta_0})\Big|_{\theta=\theta_0} = \frac{f''(1)}{2} J_{\theta_0, ij}$$

Quantum Estimation Theory

$$S(\mathcal{H}) = \{ \rho \in \mathcal{L}(\mathcal{H}) \mid \rho^\dagger = \rho,\ \mathrm{Tr}[\rho] = 1,\ \rho > 0 \}$$

$\ni p^{(n)}$

$\mathcal{M} \subseteq S(\mathcal{H})$

Quantum Stastical Manifold

e.g. $\rho(\theta) = \dfrac{I + \sum_{i=1}^{3} \theta_i \sigma_i}{2}$ $\quad \Theta \subseteq R^3$

$\ni$ Bloch spere

- $\rho(\theta) = \dfrac{e^{-\theta H}}{Z(\theta)}$: Gibbs state
- $\rho(\theta) = e^{-iH\theta} \rho_0 e^{iH\theta}$

$\to \Theta \subseteq R$

Perform measurement $p(x:\theta) = \mathrm{Tr}\left[\sum_i K_{x_i} \rho(\theta) K_{x_i}^\dagger\right]$

$\hookrightarrow x$

こっちはどう定義してる?

Cramer - Rao ineq $\quad \lim_{n\to\infty} n \mathrm{Var}[\Theta^{est}] \geq J^{-1} = J^{-1}(\mathbb{K})$

Quantum Cramér - Rao ineq $\quad$ for all $\mathbb{K}$

$$J(\mathbb{K}) \leq J^{Q} \qquad Q : \text{Fisher info}$$

Q Fisher information

general form (Petz)

f: op monotone function $(0, +\infty) \to (0, +\infty)$, $f(1) = 1$

$J^{a}_{i,j} = \langle \partial_i \rho, \left(f(L_\rho R_\rho^{-1}) R_\rho\right)^{-1} (\partial_j \rho) \rangle$ $\partial_i = \frac{\partial}{\partial \theta_i}$

$L_\rho(A) = \rho A$, $R_\rho(A) = A\rho$ $\langle A, B \rangle = \mathrm{Tr}[A^\dagger B]$

Hilbert Schmidt
inner-product

RLD Fisher info

Right logarithmic derivative

c.f. $J^{\text{classical}}_{ij} = \sum_x \frac{(\partial_i P(x))(\partial_j P(x))}{P(x)}$

$f(x) = 1$

↓

$J^{RLD}_{ij} = \langle \partial_i \rho, R_\rho^{-1}(\partial_j \rho) \rangle = \mathrm{Tr}\left[(\partial_j \rho)(\partial_i \rho)\rho^{-1}\right]$

Hermitian

• SLD Fisher info $f(x) = \frac{1+x}{2}$

Symmetric

L_i : operator satisfies $\partial_i \rho = \frac{1}{2}(L_i \rho + \rho L_i)$

$\partial_i \log \rho \longleftrightarrow L_i$

$J^{SLD}_{ij} = \mathrm{Tr}\left[\frac{1}{2}\{L_i, L_j\}\rho\right]$

symmetric real

CPTP map $\mathcal{E}$

$$S(\mathcal{H}) \to S(\mathcal{H}')$$

CP: completely positive

TP: trace preserving

monoticity for all $\mathcal{E}$, ρ, f

$$J_\rho^Q \geq J^Q_{\mathcal{E}(\rho)}$$

- max and min of Fisher info
  for all Q Fisher J^Q which is real symmetric

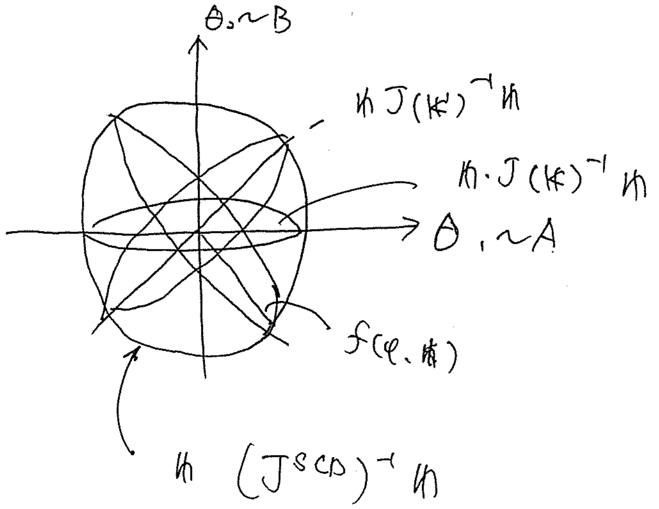
$$J^{SLD} \leq J^Q \leq \mathrm{Re}\,(J^{RLD})$$

Cramer-Rao inq

$$\lim_{n\to\infty} n \,\mathrm{Var}[\theta^{est}] \geq J^{-1} = J^{-1}(\mathbb{K}) \geq (J^{SLD})^{-1} \geq (J^{\theta})^{-1} \geq (\mathrm{Re} J^{RLD})^{-1}$$

Quantum Cramer-Rao ineq for all $\mathbb{K}$

$J(\mathbb{K}) \leq J^{Q}$ Q: Fisher info meaning of $J(\mathbb{K}) \leq J^{SLD}$ not attainabe



However

$\boldsymbol{n} (J^{SLD})^{-1} \boldsymbol{n} \leq \boldsymbol{n} J(\mathbb{K})^{-1} \boldsymbol{n}$

attrainable.

$$\boldsymbol{n} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$$

$$f(\varphi : \mathbb{K}) := \boldsymbol{n} \cdot J(\mathbb{K})^{-1} \boldsymbol{n}$$

$$D(\rho \| \sigma) = \mathrm{Tr}\left[\rho \ln \rho - \rho \log \sigma\right]$$

$$\partial_i \partial_j D(\rho_\theta \| \rho_{\theta_0}) \Big|_{\theta = \theta_0} = J^{BKM}$$

$$\downarrow$$

$$f(x) = \frac{1-x}{\log x}$$

Jのえらび方は 相関関数のorderのとり方によっている.

↳ある種、リロコをきめているようなもの?