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Noncommutative homotopy algebras associated with open strings

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Abstract

We discuss general properties of A_∞-algebras and their applications to the theory of open strings. The properties of cyclicity for A_∞-algebras are examined in detail. We prove the decomposition theorem, which is a stronger version of the minimal model theorem, for A_∞-algebras and cyclic A_∞-algebras and discuss various consequences of it. In particular it is applied to classical open string field theories and it is shown that all classical open string field theories on a fixed conformal background are cyclic A_∞-isomorphic to each other. The same results hold for classical closed string field theories, whose algebraic structure is governed by cyclic L_∞-algebras.

Contents

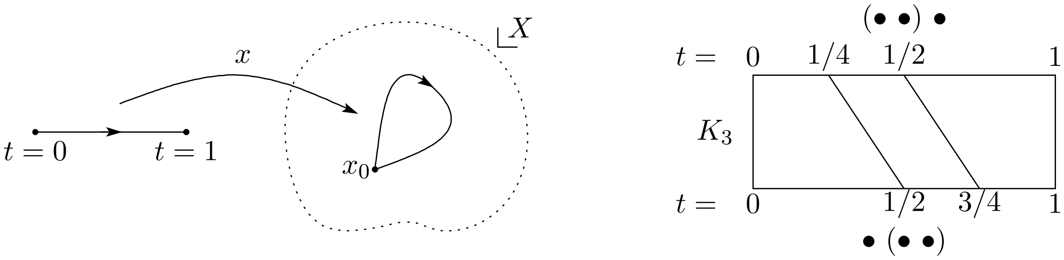
1 Introduction and Summary

This paper is the extended version of [53]. We shall discuss general properties of homotopy algebras and their application to string theory. Homotopy algebras and string theory are related to each other. General properties of homotopy algebra govern general properties of field theory of string, whereas, string theory or field theory gives some insight into the theory of homotopy algebras. The direct connection between them is realized in terms of formal supermanifolds. Homotopy algebras are described (by taking their duals) in terms of formal supermanifolds, of which their coordinates are just the fields of field theories. We concentrate on the theory related to tree-level open strings, whose relevant homotopy algebraic structures are A_∞-algebras [107]. A_∞-algebras appearing in open string theory have an additional structure, the cyclicity. We call them cyclic A_∞-algebras and examine their properties in detail. We also give a statement of formal noncommutative (odd) symplectic supergeometry and examine its local properties. It serves as a realization of A_∞-algebras equipped with cyclicity. The minimal model theorem [50] plays a key role in studying homotopy algebraic properties of A_∞-algebras. We prove a stronger version of the minimal model theorem, which we call the decomposition theorem, for A_∞-algebras and cyclic A_∞-algebras. For A_∞-algebras, a similar result is obtained independently in [72]. Various consequences of the decomposition theorem are then discussed. In particular it is applied to the classification of classical open string field theories.

In this section, we shall provide some background and main ideas of the present work. In subsection 1.1, we shall first recall some background history of A_∞-algebras. The construction of string field theory and the relevance of homotopy algebraic structures to them are reviewed in subsection 1.2. In subsection 1.3, we present some of our notations related to formal supermanifolds, which play a central role in this paper. Subsection 1.4 consists of additional comments for the noncommutativity of formal supermanifolds and their connection to physics of open strings. Subsection 1.5 is devoted to showing the idea of the construction of formal noncommutative symplectic supergeometry inspired from open strings. Subsection 1.2, subsection 1.3 and subsection 1.5 include our basic concept and tools leading to some of the main results of this paper. The contents and the results of this paper are summarized in subsection 1.6. Since in later sections we assume no knowledge presented in this section, the readers can skip this section and begin with section 2.

1.1 A_∞-spaces and A_∞-algebras

An A_∞-*space* was introduced by J. Stasheff as a tool in the study of $H(opf)$-*spaces* [106, 108]. Roughly speaking, H-spaces are group-like topological spaces. A typical example is a *based loop space*. Let $Y = \Omega X$ be the space of based loops in X. For a based point $x_0 \in X$, an element of Y is a map $x : [0,1] \to X$ where $x(0) = x(1) = x_0$ (Figure 1 (a)). We have a product as a



(a) (b)

Figure 1: (a). An element in Y. (b). A homotopy between $m_2(m_2(\bullet,\bullet))$ and $m_2(\bullet, m_2(\bullet,\bullet))$, where $\bullet$ symbolizes an element in Y.

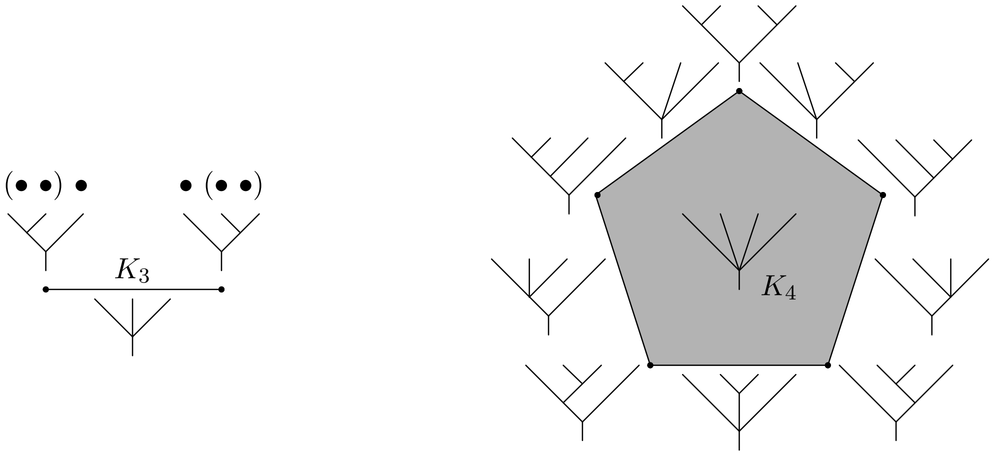
group-like space

$$m_2 : Y \times Y \to Y \ .$$

It is given by connecting two loops as $m_2(x,x')(t) = x(2t)$ for $0 \le t \le 1/2$ and $m_2(x,x')(t) = x'(2(t-1/2))$ for $1/2 \le t \le 1$. m_2 is not associative but clearly there exists a homotopy described by an interval K_3 (Figure 1 (b))

$$m_3 : K_3 \times Y \times Y \times Y \longrightarrow Y \ .$$

When we represent the product by a trivalent planar tree, the relation above is characterized pictorially as in Figure 2(a). Next, when considering possible operations of $(Y)^{\times 4} \to Y$ by m_2,



(a) (b)

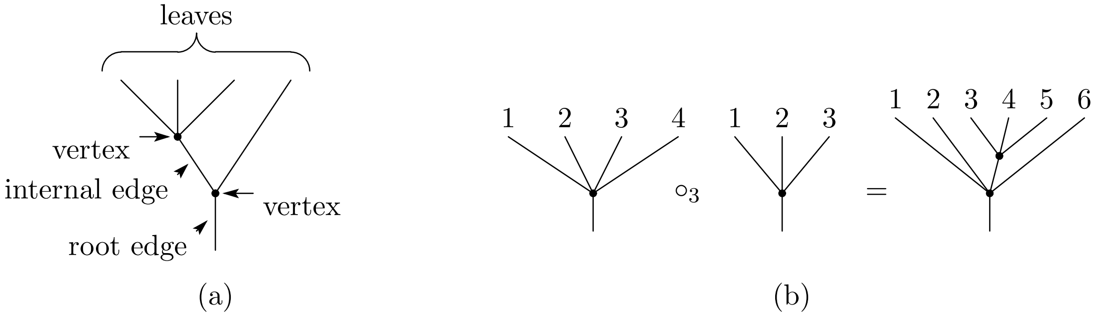
Figure 2: (a). An interval as the associahedron K_3. (b). A pentagon as the associahedron K_4.

we have five vertices corresponding to tree graphs which consists of trivalent trees. Then one gets Figure 2(b) corresponding to the 'homotopy pentagon relation'. One can see that each edge corresponds to K_3 and K_4 bounded by these edges is a pentagon. The corresponding homotopy $m_4 : K_4 \times (Y)^{\times 4} \to Y$ is then defined. Repeating this procedure then produces higher homotopies

$$m_n : K_n \times (Y)^{\times n} \longrightarrow Y \ .$$

For $n \geq 2$, K_n is a polytope of dimension $(n-2)$; K_2 is a point, K_3 is a interval, K_4 is a pentagon as above, and so on. As indicated in Figure 2(a) or (b), K_n is associated with an n-corolla, where an n-corolla is an n-tree without internal edges and an n-tree is a planar rooted tree with n leaves. For a planar rooted k-tree, l-tree and an integer $1 \leq i \leq k$, one can consider the *grafting* of l-tree to k-tree along leaf i, given by identifying the root edge of the l-tree with the i-th leaf of k-tree (see Figure 3 (b)). Associated to the grafting, one can consider the following inclusion map

$$K_k \circ_i K_l \hookrightarrow K_{k+l-1} \ .$$



(a) (b)

Figure 3: (a). Notation for planar rooted tree. The above one is a 4-tree. (b). An example of grafting, grafting of a 3-corolla to a 4-corolla along leaf 3.

Then, by construction, the $\{K_n\}_{n\geq 2}$ have the following recursion relation

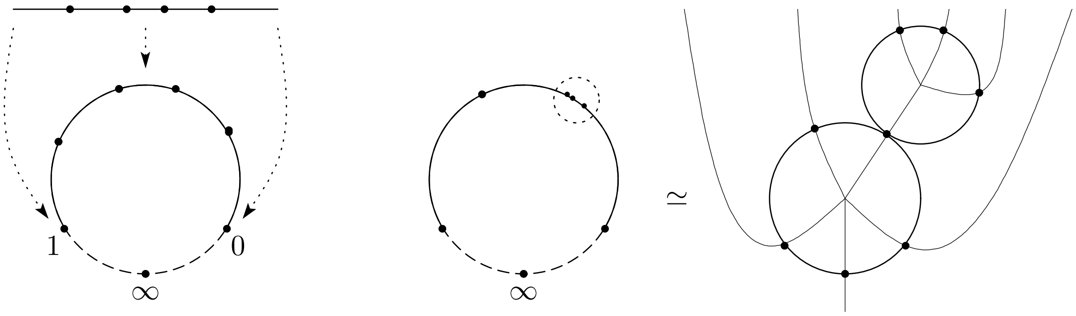
$$\partial K_n = \sum_{\substack{k+l=n+1 \\ k,l\geq 2}} \sum_{i=1}^{k} K_k \circ_i K_l \tag{1.1}$$

for the codimension one boundary of K_n. One can confirm eq.(1.1) in the case of $n = 4$, where the summation in the right hand side produces five terms; the terms for $k = 2,\ i = 1, 2$ and $k = 3,\ i = 1, 2, 3$. They corresponds to the five edges of the pentagon in Figure 2 (b). The trees associated to the edges are just the ones associated to $K_k \circ_i K_l$. There also exist other relations for lower components (codimension greater than one boundaries) of K_n.

Generally, a topological space Y equipped with higher homotopies $\{m_n\}_{n\geq 2}$ as above is called an A_∞-space [106] (for a brief review see [111], an origin of this concept is M. Sugawara's work [114]). It is applied to the study of loop spaces [1, 14, 81]. Conversely, it is known that any topological space Y that admits the structure of an A_∞-space and whose connected components form a group is homotopy equivalent to a loop space [1]. It also appears in the construction of a classical open string field theory as will be mentioned in the next subsection.

The set of associahedra $\{K_n\}_{n\geq 2}$ is one of the most typical example of topological operads. Though we avoid presenting the complicated definition, a $(\cdots)$ operad [81] is a set of $(\cdots)$-objects that correspond to corollas and are equipped with natural structures associated with trees and their grafting. (In the case here, $(\cdots) =$ 'topological'.) The set $\{K_n\}_{n\geq 2}$ is associated to a *non-symmetric* operad for which the corresponding trees are *planar.* [1] It is known that for any topological operad, the singular chain complex forms a differential graded (dg) operad. Since the associahedra are presented as cell complexes and the composition of trees is cellular, the cellular chains form a dg operad. Then the algebra over the dg operad is an A_∞-algebra [107], see below.

The theory of operads and trees are closely related to compactification of configuration spaces. It is known that K_n is obtained as the real compactification of $(n-2)$ distinct points in an interval (cf. the *little interval operad*; see [80], p94). The configuration space can further be related to the real compactification of the moduli space $\mathcal{M}_{n+1}$ of a disk with $(n+1)$ points on the boundary as indicated in Figure 4. The compactified moduli space $\mathcal{M}_{n+1}$ is defined as the configuration space of $(n+1)$-punctures on S^1 divided by conformal transformations. In the case when the Riemann surface is the disk, the conformal transformations are elements of $SL(2,\mathbb{R})$. The degree of freedom can be killed by fixing three points on the boundary. As usual we denote the three points by 0, 1 and ∞. By choosing ∞ as the 'root edge', the interval is naturally identified with the arc between 0 and 1. The pattern of the degeneration of points on the boundary is just the same as the right hand side of eq.(1.1), which has $2+3+\cdots+(n-1)$ terms corresponding to the boundary components. Other interesting examples of topological operads and their connection to compactifications can be found for instance in [31, 110].



(a) (b)

Figure 4: (a). The identification of the interval with $(n-2)$ points on it with the boundary of the disk with $(n+1)$ points on the boundary. (b). Compactification of moduli space $\mathcal{M}_7$. The figure above represents a boundary component of $\mathcal{M}_7$. It just corresponds to the grafting of trees in Figure 3 (b).

Let $\mathcal{H}$ be a $\mathbb{Z}$-graded vector space and $\mathfrak{m} := \{m_n : (\mathcal{H})^{\otimes n} \to \mathcal{H}\}_{n\geq 1}$ a collection of multilinear maps. The pair $(\mathcal{H},\mathfrak{m})$ is then an A_∞-algebra iff $\mathfrak{m}$ satisfies the following relations (see also Definition 2.5)

$$m_1 m_n + \sum_{i=1}^{n} m_n(\mathbf{1}^{\otimes i-1}\otimes m_1 \otimes \mathbf{1}^{\otimes n-i}) = -\sum_{\substack{k+l=n+1\\ k\geq 2,l\geq 2}} \sum_{j=1}^{k} m_k(\mathbf{1}^{\otimes j-1}\otimes \mathfrak{m}_l \otimes \mathbf{1}^{\otimes k-j}) \qquad (1.2)$$

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[1] Here non-symmetric corresponds to noncocommutative in coalgebra description of A_∞-algebras in section 2.

on $\mathcal{H}^{\otimes n}$ for each $n \geq 1$. The equation for $n = 1$ is just $(m_1)^2 = 0$, which implies that $(\mathcal{H}, m_1)$ forms a complex. The equation for $n = 2$ is then the Leibniz rule for the action of derivation m_1 on m_2. For $n = 3$ eq.(1.2) describes the associativity of m_2 up to homotopy. Comparing this with an A_∞-space, one can see that a topological space Y corresponds to a graded vector space $\mathcal{H}$ with the m_i for $i \geq 2$ on the two sides corresponding to each other, where the action of ∂ on K_n corresponds to the action of m_1 on m_n in the left hand side of eq.(1.2). Namely, the correspondence is in some sense similar to the one between singular homology and deRham cohomology. This paper deals with this algebra side, some 'deRham rings up to homotopy'.

Such algebraic treatments of homotopy theory were developed in rational homotopy theory by Quillen [89] and Sullivan [115, 34]. In particular [115] deals with differential forms on a manifold M, which form a differential graded algebra (dga). It is then shown that the dga of differential forms on M has the information of the rational homotopy type of M. Note that a dga is an A_∞-algebra $(\mathcal{H}, \mathfrak{m})$ with $m_3 = m_4 = \cdots = 0$. In particular, in this situation, the graded vector space $\mathcal{H}$ is the space of differential forms on M, m_1 is the exterior derivative and m_2 is the wedge product. For A_∞-algebras, there is a notion of homotopy. Two homotopy equivalent A_∞-algebras are transformed to each other by an A_∞-quasi-isomorphism, where quasi-isomorphisms are morphisms which preserves the cohomology with respect to m_1 (Definition 2.8). Then it is known that, for a given A_∞-algebra $(\mathcal{H}, \mathfrak{m})$, there exists an A_∞-structure on $H(\mathcal{H})$, the cohomology of the complex $(\mathcal{H}, m_1)$, which is A_∞-quasi-isomorphic to the original A_∞-algebra $(\mathcal{H}, \mathfrak{m})$ [50] ([49] for the case $(\mathcal{H}, \mathfrak{m})$ is a dga). This fact is called the *minimal model theorem.* The way of constructing minimal models of A_∞-algebras, in particular dgas, has been developed in the framework of *homological perturbation theory* as an important subject in algebraic topology (for example [35, 40, 36, 37, 38, 45], and see also [46] for the dg Lie algebra case). The minimal model theorem implies, for the case of dga of differential forms, that one can recover the rational homotopy type of M by considering the A_∞-structure on the deRham cohomology instead of the original deRham complex. In this case, the higher operations $\{m_n\}_{n\geq 3}$ are related to the higher Massey-Yoneda products. One may also consider the (complex of) modules over M and *Ext* between them. Correspondingly, there are the notion of A_∞-modules over M and an A_∞-category on M (see [59, 72]). It is then known that the stories stated above hold in a similar way as for A_∞-algebras.

Such notions are applied to mathematical physics in many ways. One of the application is the *homological mirror symmetry conjecture* [63] which states some equivalences between an A_∞-category [22] on Calabi-Yau manifolds M (A-model side in physical terms) and the category of coherent sheaves on the mirror dual manifold $\hat{M}$ (B-model side). This conjecture implies that both sides, that is, not only the A but also the B-model sides have some A_∞-structures. It is known that in some restricted situations both A and B model are described by topological Chern-Simons field theories [125] and one can obtain so-called D-brane superpotentials from the topological Chern-Simons field theories [73, 118, 74]. This is nothing but the minimal model theorem, where a topological Chern-Simons field theory has a structure of dga and a D-brane superpotential is regarded as the collection of higher Massey-Yoneda products [125].

Furthermore, not only the Chern-Simons field theory above but any field theory has a homotopy algebraic structure generally only if it satisfies a classical Batalin-Vilkovisky (BV-) master equation (see subsection 1.3). The typical examples are classical string field theories explained in the next subsection.

1.2 A_∞-structures and classical open string field theory

String field theory is defined on a fixed conformal background of space-time (target space) M to which world sheet of strings (Riemann surfaces) are mapped, where a conformal background is a background (metric, etc.) of M so that the action of a string on M has conformal symmetry (see [95]). There exists several classes of string field theories corresponding to the classes of Riemann surfaces. The most general one is open-closed string field theory [129], which associates to the most general class of Riemann surfaces; Riemann surfaces with boundaries, genera and punctures. It includes various 'sub-string field theories'; classical open string field theories - associated to disks (one boundary and no genus) with punctures only on the boundary, classical closed string field theories - associated to spheres (no boundary and no genus) with punctures, quantum closed string field theories - associated to Riemann surfaces with punctures (and genera) and without boundary, and so on. Genus and multi-boundaries relate to loops of closed strings and open strings, respectively. We use the term 'classical' (resp. quantum) for theory without such loop (resp. with such loops). There exists an abstract standard way for constructing these string field theories [77, 128]. We shall review it briefly in the case of classical open string field theories below. The essence is the same for the other SFTs.

The open string Hilbert space $\mathcal{H}$ is a $\mathbb{Z}$-graded vector space. The conformal field theory technique gives us a basis system $\{\mathbf{e}_i\}$ of open string states (in terms of the oscillators in the mode expansions), where the grading of these basis is related to the ghost number of string states ([128, 85]). For each state $\mathbf{e}_i$, consider a field ϕ^i (in the sense of field theory) whose degree is minus the degree of $\mathbf{e}_i$ so that the degree of $\Phi := \mathbf{e}_i\phi^i$ is set to be zero. Φ is called a *string field.* [2] Moreover we have a degree one coboundary operator $Q : \mathcal{H} \to \mathcal{H}$ and a degree minus one antisymmetric bilinear form $\omega(\ ,\ ) : \mathcal{H}\otimes\mathcal{H} \to \mathbb{C}$ that are also defined canonically on the conformal background. Q and ω are called the BRST-operator [58] and the BPZ-inner product [13], respectively. They in fact define a degree-zero graded-symmetric bilinear form $\mathcal{V}_2 := \omega(\mathbf{1}\otimes Q) : \mathcal{H}\otimes\mathcal{H} \to \mathbb{C}$, which defines the kinetic term (quadratic term with respect to field $\{\phi\}$) of the action of a classical open string field theory. The action is of the following form,

$$S(\Phi) = \frac{1}{2}\omega(\Phi, Q\Phi) + \sum_{k\geq 3}\frac{1}{k}\mathcal{V}_k(\Phi,\cdots,\Phi)\ ,$$

where $\mathcal{V}_k : \mathcal{H}^{\otimes k} \to \mathbb{C}$ is a degree zero cyclic multilinear map. We call $\{\mathcal{V}_k\}_{k\geq 3}$ the *vertex maps.* The term 'cyclic' indicates that $\mathcal{V}_k$ satisfies $\mathcal{V}_k(\mathbf{e}_{i_1},\cdots,\mathbf{e}_{i_k}) = (-1)^{\mathbf{e}_{i_1}(\mathbf{e}_{i_2}+\cdots+\mathbf{e}_{i_k})}\mathcal{V}_k(\mathbf{e}_{i_2},\cdots,\mathbf{e}_{i_k},\mathbf{e}_{i_1})$ for any $\mathbf{e}_i \in \mathcal{H}$. It holds for $k \geq 2$, where the case $k = 2$ is equivalent to the fact that $\mathcal{V}_2$ is graded-symmetric stated above. All the (multi-)linear maps introduced here are extended naturally to the polynomials of fields ϕ^i. Thus, the action $S(\Phi)$ is a degree zero polynomial function that has the cyclicity. To construct a string field theory is then to construct vertex maps $\{\mathcal{V}_k\}_{k\geq 3}$ satisfying certain conditions explained below. In order to do it, some conformal field theory

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[2] For the relation to the usual notations in physics see also [53] subsection 5.2.

technique provides us with the following set-up [121, 4, 77, 128]. Let us consider a disk D with cyclic ordered n holomorphic half disks on the boundary S^1 for $n \geq 3$. Namely, we have n holomorphic maps f_i, $i = 1, \cdots, n$, from a half disk $\{z \in \mathbb{C} | \mathrm{Im}(z) \geq 0, |z| \leq 1\}$ in an upper half plane $H_+ = \{z \in \mathbb{C} | \mathrm{Im}(z) \geq 0\}$ to the disk D which are injective and map the boundary $\mathrm{Im}(z) = 0$ of the half disk to intervals on the boundary of the disk D with preserving the orientations. Thus, f_i maps the origin o of the half disk to a point (puncture) on the boundary of D, and these n holomorphic maps are defined so that $f_1(o), \cdots, f_n(o)$ are counterclockwise cyclic ordered and the images of the half disks by any two holomorphic maps do not overlap with each other. In particular, in order to fix the $SL(2,\mathbb{R})$ automorphisms of the disk D, we fix three points $f_1(o) =: 0$, $f_{n-1}(o) =: 1$ and $f_n(o) =: \infty$. We denote the space of such disks with cyclic ordered n holomorphic half disks by $\widetilde{\mathcal{M}}_n$. It forms an infinite dimensional space. For a disk $\Sigma_n \in \widetilde{\mathcal{M}}_n$, the image of the arc defined by $|z| = 1$, $|\mathrm{Im}(z)| \geq 0$ by each holomorphic map f_i is regarded as an open string. The disk Σ_n thus describes the interaction of such n open strings, as in Figure 5 (a), with the initial condition of each open string being specified by the image of the origin $f_i(o)$. An open string state space $\mathcal{H}$ is associated to each origin o of the half disk. In particular, since $Q : \mathcal{H} \to \mathcal{H}$ is a differential, $(\mathcal{H}, Q)$ forms a complex called the *BRST-complex*. The kernel (resp. cokernel) of Q is called the *on-shell* (resp. *off-shell*) state space, and the cohomology $H(\mathcal{H})$ with respect to Q, the *BRST cohomology*, is called the *physical state* space. [3] For each $\Sigma_n \in \widetilde{\mathcal{M}}_n$, the corresponding *correlation function (expectation value)* of conformal field theory gives a map $\Sigma_n : \mathcal{H}^{\otimes n} \to \mathbb{C}$ (as above we denote the map also by Σ_n). Moreover, one can consider the tangent space $T\widetilde{\mathcal{M}}_n$ and the space of differential k-forms $\tilde{\Omega}^k_{diff}(\widetilde{\mathcal{M}}_n)$ on $\widetilde{\mathcal{M}}_n$ for each $k \geq 0$ [121, 4, 128]. In particular, associated to the infinitesimal deformations of Σ_n, one can define a map $\tilde{\Omega}^k_n : \mathcal{H}^{\otimes n} \to \tilde{\Omega}^k_{diff}(\widetilde{\mathcal{M}}_n)$ for each k.

Let $\mathcal{M}_n$, $n \geq 3$, be a suitable compactification of the moduli space of disks with n punctures on the boundary. The dimension of $\mathcal{M}_n$ is $(n-3)$. There is a projection $\pi : \widetilde{\mathcal{M}}_n \to \mathcal{M}_n$ obtained by forgetting the holomorphic maps f_i, $i = 1, \cdots, n$, except the image of the origin $f_i(o)$. Namely, for $\Sigma_n \in \widetilde{\mathcal{M}}_n$, $\pi(\Sigma_n)$ is the disk with n punctures specified by $f_1(o), \cdots, f_n(o)$.

Let us consider a map (section) $\sigma : \mathcal{M}_n \to \widetilde{\mathcal{M}}_n$ such that $\pi \circ \sigma$ is identity. When restricting every $\mathbf{e}_i \in \mathcal{H}$ to on-shell, the following map

$$\tilde{\mathcal{V}}_n(\mathbf{e}_{i_1}, \cdots, \mathbf{e}_{i_n}) := \int_{\mathcal{M}_n} \sigma^* \left(\tilde{\Omega}^{n-3}_n(\mathbf{e}_{i_1}, \cdots, \mathbf{e}_{i_n}) \right) \in \mathbb{C} \qquad (1.3)$$

is in fact independent of the choice of σ, where the degree of the differential form $(n-3)$ is the dimension of $\mathcal{M}_n$. These are nothing but the tree (on-shell) *(scattering) amplitudes* of open strings. Since the n insertions (=punctures) are on the boundary of the disk, $\mathcal{A}_n$ is a cyclic map. Moreover it is known that the on-shell correlation function vanishes if one of the external states is Q-exact. Thus the collection of open string scattering amplitudes can be defined on the physical state space $H(\mathcal{H})$.

In this situation, the vertex maps $\{\mathcal{V}_n\}_{n \geq 3}$ should be constructed so that the perturbation theory reproduces the open string scattering amplitudes (1.3). In perturbation theory the on-

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[3] Here we assume for simplicity that the basis $\mathbf{e}_i$ are taken so that the subbasis of $\{\mathbf{e}_i\}$ can span the on-shell state space or the physical state space.

shell scattering amplitudes are calculated by Feynman graphs. The usual construction of string field theory is then to decompose each $\mathcal{M}_n$ into cells so that cells correspond one to one with Feynman graphs. The vertex map $\mathcal{V}_n$ is determined by the pair $(\mathcal{M}_n^0, \sigma)$, where $\mathcal{M}_n^0 \in \mathcal{M}_n$ is a cell of $\mathcal{M}_n$ and $\sigma : \mathcal{M}_n^0 \to \widetilde{\mathcal{M}}_n$ is a map such that $\pi\sigma$ is equal to the identity. We give such a pair $(\mathcal{M}_n^0, \sigma)$ so that $\sigma\mathcal{M}_n^0 \subset \widetilde{\mathcal{M}}_n$ has an $SL(2,\mathbb{R})$ automorphism on the disk D which transforms f_i to f_{i+1} for $1 \le i \le n-1$ and f_n to f_1. Then $\mathcal{V}_n$ is given as

$$\mathcal{V}_n(\Phi, \cdots, \Phi) := \int_{\mathcal{M}_n^0} \sigma^* \left(\tilde{\Omega}_n^{n-3}(\Phi, \cdots, \Phi) \right) \ . \tag{1.4}$$

By construction, $\mathcal{V}_n(\mathbf{e}_{i_1}, \cdots, \mathbf{e}_{i_n})$ is cyclic. [4]

Let us consider an on-shell tree n-point open string scattering amplitude. The corresponding Feynman graphs are tree planar graphs, each of which consists of cyclic vertices, internal edges, and external edges. Each internal edge has two distinct vertices. Each external edge, called a leaf, has a vertex at one end and the other end is free (see Figure 5 (b)). Clearly, by ignoring the distinction between the root edge and the leaves of a rooted planar tree and regarding the root edge also as a leaf, one gets a planar tree graph. Thus, we have a natural surjection $\check{r} : G_{n-1} \to G_n^{cyc}$, where G_{n-1} is the set of rooted planar $(n-1)$-trees and G_n^{cyc} is the set of planar graphs with n leaves. Let $G_n^{cyc,I}$ be the set of planar trees with n leaves and I internal edges. Each element $\Gamma_n^{cyc,I} \in G_n^{cyc,I}$ then has $I+1$ vertices. We assign v_m, $m = 1, 2, \cdots, I+1$ to the vertices and let e_m be the number of incident (both internal and external) edges. The following identity then holds

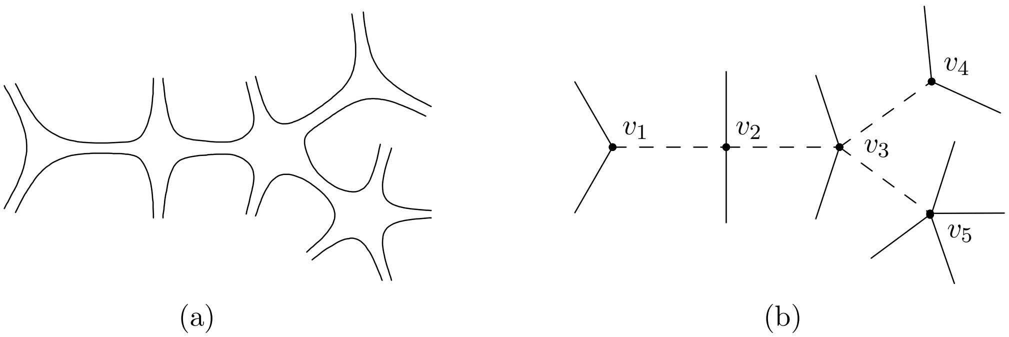

Figure 5: (a). A Riemann surface (disk) that describes an open string interaction. The corresponding Feynman diagram is the planar graph in Figure (b). The dashed lines denote the internal edges that correspond to propagators in the string field theory. Here the vertices are labeled by $v_1, \cdots, v_5$. The numbers of legs for the vertices are $e_1 = 3$, $e_2 = 4$, $e_3 = 5$, $e_4 = 3$, $e_5 = 5$. The number of the internal edges equal $I = 4$. The graph has twelve external edges, and eq.(1.5) holds because $12 = 3 + 4 + 5 + 3 + 5 - 2 \cdot 4$.

$$n + 2I = \sum_{m=1}^{I+1} e_m \ . \tag{1.5}$$

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[4] In section 3 in [53] the index for the differential form k in $\tilde{\Omega}_n^k$ is omitted. The correspondence of the notation between [53] and this paper is then given by $\Omega_n = \sigma^* \tilde{\Omega}_n$.

For $\Gamma_n^{cyc,I}$, one can consider a number $\tilde{\mathcal{V}}_{\Gamma_n^{cyc,I}}(\mathbf{e}_{i_1},\cdots,\mathbf{e}_{i_n})\in\mathbb{C}$. Essentially it is given by attaching $\mathcal{V}_{e_m}$ to each vertex v_m and to each internal edge a so-called propagator (Definition 6.4, denoted by $\mathcal{V}_L^+\in\mathcal{H}\otimes\mathcal{H}$) and $\mathbf{e}_{i_1},\cdots,\mathbf{e}_{i_n}$ to leaves cyclically (see Definition 6.9). The tree n-point open string scattering amplitude (1.3) is then reproduced by

$$\tilde{\mathcal{V}}(\mathbf{e}_{i_1},\cdots,\mathbf{e}_{i_n}):=\sum_{\Gamma_n^{cyc}\in G_n^{cyc}}\frac{1}{\sharp\mathrm{Aut}(\Gamma_n^{cyc})}\tilde{\mathcal{V}}_{\Gamma_n^{cyc}}(\mathbf{e}_{i_1},\cdots,\mathbf{e}_{i_n})\ , \tag{1.6}$$

where each $\mathbf{e}_i$ is on-shell, and $\mathrm{Aut}(\Gamma_n^{cyc})$ indicates the number of the automorphisms acting on Γ_n^{cyc}. The fraction $\frac{1}{\sharp\mathrm{Aut}(\Gamma_n^{cyc})}$ is called the *symmetric factor* of the Feynman graph. (We shall discuss these Feynman graphs in detail in subsection 6.3.)

On the other hand, the propagator $\mathcal{V}_L^+$ is represented by an integral over $[0,\infty]$. [5] Namely, the propagator or the internal edge has modulus $\tau\in[0,\infty]$ and, in a Riemann surface picture, a strip with fixed width and length τ is associated to it. Assume that $\{\mathcal{V}_k\}_{k\geq 3}$ are constructed so that the associated Riemann surfaces $\{\sigma:\mathcal{M}_k^0\to\widetilde{\mathcal{M}}_k\}_{k\geq 3}$ can be joined with the strip (propagator) by sewing Riemann surfaces. Then, each graph $\Gamma_n^{cyc,I}\in G_n^{cyc,I}$ is associated with a subspace of $\widetilde{\mathcal{M}}_n$, which we denote by $\widetilde{\mathcal{M}}_{\Gamma_n^{cyc,I}}\subset\widetilde{\mathcal{M}}_n$. The important point is that the compatibility with respect to the sewing of Riemann surface is known [77] (for classical open string theory, more explicitly in [93]), which implies, for instance,

$$\tilde{\mathcal{V}}_{\Gamma_n^{cyc,I}}(\mathbf{e}_{i_1},\cdots,\mathbf{e}_{i_n})=\int_{\widetilde{\mathcal{M}}_{\Gamma_n^{cyc,I}}}\tilde{\Omega}_n^{n-3}(\mathbf{e}_{i_1},\cdots,\mathbf{e}_{i_n})\ .$$

Here note that the dimension of $\widetilde{\mathcal{M}}_{\Gamma_n^{cyc,I}}$ is actually $n-3$ and independent of I. The fact can be confirmed by eq.(1.5) as $(e_1-3)+\cdots+(e_{I+1}-3)+I=(k+2I-3(I+1))+I=k-3$.

Suppose that the vertex maps $\{\mathcal{V}_k\}_{k\geq 3}$ in eq.(1.4) are constructed consistently up to $k=n-1$ and then concentrate on the n-point amplitude (1.6). The Feynman graph without propagator($I=0$) consists only of the vertex $\mathcal{V}_n$, which is not determined yet. For each $\Gamma_n^{cyc,I}$, $I>0$, we assume the projection of $\widetilde{\mathcal{M}}_{\Gamma_n^{cyc,I}}$ gives an inclusion,

$$\pi(\widetilde{\mathcal{M}}_{\Gamma_n^{cyc,I}})\subset\mathcal{M}_n\ , \tag{1.7}$$

and for any two distinct elements of G_n^{cyc} the images never have a common subspace except their boundaries. Let us denote

$$\mathcal{M}_k^I:=\bigcup_{\Gamma_n^{cyc,I}\in G_n^{cyc,I}}\pi(\widetilde{\mathcal{M}}_{\Gamma_n^{cyc,I}})\ . \tag{1.8}$$

Thus, $\mathcal{M}_n^0$ is determined as

$$\mathcal{M}_n=\mathcal{M}_n^0\cup\mathcal{M}_n^1\cup\mathcal{M}_n^2\cup\cdots\cup\mathcal{M}_n^{n-3}\ , \tag{1.9}$$

where the common subspace $\mathcal{M}^I\cap\mathcal{M}^{I'}$, $I\neq I'$, has codimension greater than one. Furthermore, define $\sigma:\mathcal{M}_n^0\to\widetilde{\mathcal{M}}_n$ so that $\sigma(\mathcal{M}_n^0)$ and $\bigcup_{I\geq 1}\bigcup_{\Gamma_n^{cyc,I}\in G_n^{cyc,I}}(\widetilde{\mathcal{M}}_{\Gamma_n^{cyc,I}})$ form a *continuous* section of the bundle $\widetilde{\mathcal{M}}_n\to\mathcal{M}_n$. Consequently one obtains $\mathcal{V}_n$ by eq.(1.4). One can see that the action

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[5] This corresponds to the length parameter of the open string evolution.

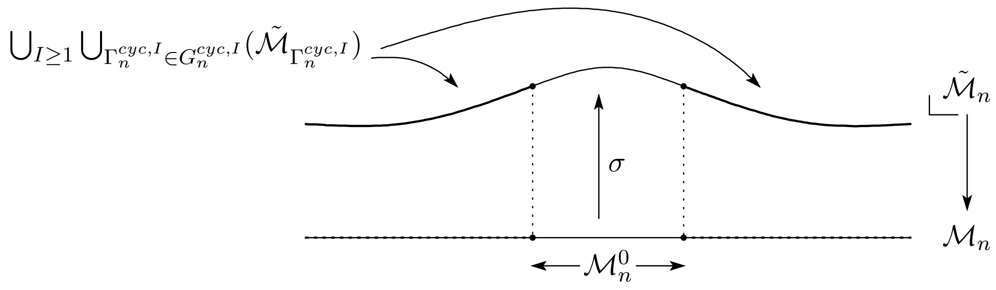

Figure 6: The determination of the pair $(\mathcal{M}_n^0, \sigma)$.

is obtained by repeating this procedure. By construction it is clear that the action reproduces the tree open string scattering amplitudes by perturbation theory.

The action constructed as above actually satisfies the classical BV-master equation (1.13). First, consider the infinitesimal variation of the decomposition of Riemann surfaces, or more precisely, take the boundary ∂ of eq.(1.9). Since in eq.(1.7) we assumed π is an inclusion, then π commute with ∂. If one takes the boundary of eq.(1.8), the boundary operator ∂ acts on each $\widetilde{\mathcal{M}}_{\Gamma_n^{cyc,I}}$ in the right hand side. Here $\widetilde{\mathcal{M}}_{\Gamma_n^{cyc,I}}$ is a topological space $\sigma(\mathcal{M}_{e_1}^0)\times\cdots\times\sigma(\mathcal{M}_{e_{I+1}}^0)\times[0,\infty]^I$ equipped with the information of the planar tree graph $\Gamma_n^{cyc,I}$. Then ∂ acts by the Leibniz rule on a vertex space $\mathcal{M}_{e_m}^0$ or a propagator $[0,\infty]$. Here the boundary of the propagator is $\{0\}-\{\infty\}$. It is natural to require that in these construction the sum of all the contributions $\{\infty\}$ corresponds to the boundary of $\mathcal{M}_n$. Then, acting by ∂ on eq.(1.9) yields

$$0=\partial(\mathcal{M}_n^0)+\sum_{\substack{k_1+k_2=n+2\\ k_1,k_2\geq 3}}\frac{1}{2}\left(\begin{array}{c}\partial(\mathcal{M}_{k_1}^0)\text{---}(\mathcal{M}_{k_2}^0)\\ +(\mathcal{M}_{k_1}^0)\text{---}\partial(\mathcal{M}_{k_2}^0)\\ +(\mathcal{M}_{k_1}^0)\leftarrow\!\!\ominus\!\!\rightarrow(\mathcal{M}_{k_2}^0)\end{array}\right)+\sum_{\substack{k_1+k_2+k_3=n+4\\ k_1,k_2,k_3\geq 3}}(\cdots)+\cdots \tag{1.10}$$

for $n\geq 3$. We should explain some of the notations used above. First, we identify the image of the composition of two maps $\sigma:\mathcal{M}_{k_i}^0\to\widetilde{\mathcal{M}}_{k_i}$ and $\pi:\widetilde{\mathcal{M}}_{k_i}\subset\widetilde{\mathcal{M}}_n\to\mathcal{M}_n$ with $\mathcal{M}_{k_i}^0$ itself and wrote $\mathcal{M}_{k_i}^0$. Note that each $\mathcal{M}_{k_i}^0$ is associated with a vertex. We then denoted by — a topological space $[0,\infty]$ with the operation of connecting two vertices with the propagator. Alternatively, '$\leftarrow\!\!\ominus\!\!\rightarrow$' indicates the operation of grafting two vertices with '$\{0\}$', the contracted propagator. The equation (1.10) is, in fact, equivalent to

$$0=\partial(\mathcal{M}_n^0)+\sum_{\substack{k_1+k_2=n+2\\ k_1,k_2\geq 3}}\frac{1}{2}(\mathcal{M}_{k_1}^0)\leftarrow\!\!\ominus\!\!\rightarrow(\mathcal{M}_{k_2}^0)\ . \tag{1.11}$$

The right hand side of the identity above is the sum of the first term and one of the second term in the right hand side of the identity(1.10). The equivalence holds because the other parts of eq.(1.10) cancel by induction. For example, $\partial(\mathcal{M}_{k_1}^0)\text{---}(\mathcal{M}_{k_2}^0)$ in the second term cancels one of the third term $(\cdots)$ of the form $\sum_{\substack{k+l=k_1+2\\ k,l\geq 3}}\left(\frac{1}{2}(\mathcal{M}_k^0)\leftarrow\!\!\ominus\!\!\rightarrow(\mathcal{M}_l^0)\right)\text{---}(\mathcal{M}_{k_2}^0)$. The recursion equation (1.11) is called the *string factorization equation* [105].

The string factorization equation (1.11) is actually equivalent to the BV-master equation. In fact, this identity (1.11) is an identity between $(n-4)$-dimensional moduli space and graphically an identity between planar tree graphs with n leaves. Thus, let us integrate $\sigma^*\left(\tilde{\Omega}_n^{n-4}(\Phi,\cdots,\Phi)\right)$ over the identity (1.11). The conformal field theory technique leads to the following result (**cf.** [128]):

$$0=\delta_1\left(\frac{1}{n}\mathcal{V}_n(\Phi,\cdots,\Phi)\right)+\frac{1}{2}\sum_{\substack{k_1+k_2=n+2\\k_1,k_2\geq 3}}\left(\frac{1}{k}\mathcal{V}_k(\Phi,\cdots,\Phi),\frac{1}{l}\mathcal{V}_l(\Phi,\cdots,\Phi)\right)\ . \tag{1.12}$$

In the equation above, $(\ ,\ ):=\frac{\overleftarrow{\partial}}{\partial\phi^i}\omega^{ij}\frac{\overrightarrow{\partial}}{\partial\phi^j}$ and ω^{ij} is the inverse of $\omega_{ij}:=\omega(\mathbf{e}_i,\mathbf{e}_j)$. This in fact defines an odd Poisson bracket and is called *the BV-bracket.* Also, δ_1 is defined by $\delta_1:=(\ ,\frac{1}{2}\omega(\Phi,Q\Phi))$. It satisfies $(\delta_1)^2=0$ corresponding to the nilpotency $Q^2=0$ (or $\partial^2=0$). Summing up eq.(1.12) for $n\geq 3$ and multiplying by two then lead to the *classical BV-master equation*

$$(S(\Phi),S(\Phi))=0\ . \tag{1.13}$$

To summarize, to construct a string field theory is to construct $\{\sigma(\mathcal{M}_k^0)\}_{k\geq 3}$ so that they are compatible with the decomposition of the moduli spaces. The construction of $\{\sigma(\mathcal{M}_k^0)\}_{k\geq 3}$ is independent of the conformal background we choose. Whereas, $\tilde{\Omega}_\bullet^\bullet$ is determined canonically by the conformal background and taking a representation of $\{\sigma(\mathcal{M}_k^0)\}_{k\geq 3}$ by $\{\tilde{\Omega}_k^{k-3}\}_{k\geq 3}$ produces a string field theory action on the conformal background. Mathematically, $\{\mathcal{M}_n^0\}_{n\geq 3}$ forms an operad and, by taking its representation, one obtains an algebra $\mathcal{H}$ over the operad, where $\mathcal{H}$ is called an operad algebra [80].

As seen in the next subsection, an action which has cyclic vertices and satisfies the classical BV-master equation as above has an A_∞-structure. The A_∞-algebra in addition possesses an odd symplectic inner product and cyclicity. Such an algebra is called a cyclic A_∞-algebra (see Definition 2.11). The appearance of an A_∞-structure can already be seen from eq.(1.12). This identity is in fact a different but an equivalent expression of the A_∞-condition (1.2) under the situation cyclic symmetry exists. The structure of an A_∞-space can also be found in an explicit construction of the classical open string field theory in [85, 53], where $\mathcal{M}_{n+1}^0$ is just the associahedra K_n and string factorization equation (1.11) is just the cyclic version of eq.(1.1) [54]. The corresponding operad is called the *A_∞-operad* [80]. Similar stories hold for other classical string field theories. The underlying operad structure in classical closed string field theory is the L_∞-operad [60, 109]. For classical open-closed string case, see [55] and also a related earlier work [122].

The minimal model theorem appears naturally also in string theory. In [66] for any A_∞-algebra an explicit construction of the minimal model is given. The construction is just given by Feynman graphs. For classical open string field theories, these are just the Feynman graphs appearing above. This implies that the collection of the scattering amplitudes of open string theory forms a minimal cyclic A_∞-algebra. This statement is essentially already known. In [127] it is shown that the tree closed string theory has the structure of the L_∞-algebra (and it is extended to the quantum case in [121]). Thus, the minimal model theorem implies on a fixed conformal background all classical open string field theories are A_∞-quasi-isomorphic to each other [53]. Namely, the difference in the choice of the decomposition of moduli spaces leads to homotopy equivalence of A_∞-algebras, and the minimal model is obtained by homotopic deformation $\mathcal{M}_k^0 \to \mathcal{M}_k$. [6] Moreover one can show that these are not only quasi-isomorphic but A_∞-isomorphic (Theorem 6.18) due to Theorem 5.15.

Physically, string field theories have been investigated as a candidate for string theory which describes nonperturbative effects. This purpose requires the off-shell extension of string theory as above. A typical off-shell physics phenomenon is tachyon condensation [99]. Recently, string field theory has been applied in such a direction successfully [103, 86] (see also [67]). Though we assumed the existence of σ which required many consistency conditions as above, actually there exists many Lorentz-covariant string field theories (SFTs) [7] ; the covariant open or closed SFT with light cone type-like vertices(HIKKO's SFT) [42], a very simple open SFT which consists of only a three-point vertex (Witten's open SFT or cubic SFT) [124], nonpolynomial classical closed SFT constructed by 'restricted polyhedron' [68, 69], and so on. Witten's SFT is treated in the context of BV-formalism[116, 15](see [117]). HIKKO's closed SFT is also extended to quantum SFT by employing the quantum BV-master equation [41]. The quantum master equation is moreover applied to construct quantum closed SFT with symmetric vertices by Zwiebach [128]. Though this theory has infinite sort of vertices of higher punctures and higher genus, it has a very beautiful algebraic structure. For instance for the classical part, the set of the tree vertices has the structure of an L_∞-algebra. Open-closed SFT is also considered in this direction [129]. HIKKO-type open-closed SFT is given in [70, 5]. Recently a one parameter family of classical open string field theories, which possess A_∞-structures, has been constructed explicitly [85, 53, 54] by deforming the Witten's cubic SFT [124].

1.3 Dual description; formal noncommutative supermanifolds

For A_∞-algebras, we use mainly three descriptions; the coalgebra language, its dual language, and superfield description. Coalgebras are used to define A_∞-algebras precisely and simply. On the other hand, the dual description is geometric and intuitive as explained below. The superfield description, used in the previous subsection, is their mixed version. It is directly equivalent to the dual ones but the superfield description uses the notation used in coalgebra. This description is convenient to simplify indices. The operad structure is implicit in various arguments in this paper, but we shall not indicate it explicitly.

As in the previous subsection, given an A_∞-algebra $(\mathcal{H}, \mathfrak{m})$, denote by $\{\mathbf{e}_i\}$ a basis of $\mathcal{H}$ and $\{\phi^i\}$ the dual coordinates. Reflecting the non(co)commutativity of $\mathcal{H}$, the dual fields are treated as noncommutative as explained in section 3. We call $\Phi = \mathbf{e}_i\phi^i$ the superfield, which is the string field in string field theory. Let us describe the A_∞-structure in coordinates as

$$m_k(\mathbf{e}_{i_1}, \cdots, \mathbf{e}_{i_k}) = \mathbf{e}_j c^j_{i_1\cdots i_k}\ .$$

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[6] This also implies that moduli spaces of open string correlation functions $\{\mathcal{M}_k\}$, obtained by a suitable compactification, has the structure of an A_∞-space. Though in a slightly different context, an A_∞-space structure in open string theory is discussed in [21].

[7] String field theories of the type explained here are called (Lorentz) covariant string field theories in contrast to light cone type string field theories developed earlier.

For the collection $c^j_{i_1\cdots i_k}\in\mathbb{C}$ for $k\geq 1$, one can define the following degree one vector field, called the *homological vector field*, on a formal noncommutative supermanifold

$$\delta=\sum_{k=1}^{\infty}\frac{\overleftarrow{\partial}}{\partial\phi^j}c^j_{i_1\cdots i_k}\phi^{i_k}\cdots\phi^{i_1}\ . \tag{1.14}$$

We often use the Einstein convention of summing over repeated indices as above. Note that the A_∞-condition is then rewritten as $(\delta)^2=0$. We call this δ an A_∞-odd vector field.

On the other hand, let us consider a degree zero cyclic function of the form

$$S=\frac{1}{2}\mathcal{V}_{i_1i_2}\phi^{i_2}\phi^{i_1}+\sum_{k\geq 3}\frac{1}{k}\mathcal{V}_{i_1\cdots i_k}\phi^{i_k}\cdots\phi^{i_1}$$

where $\mathcal{V}_{i_1\cdots i_k}\in\mathbb{C}$ for $k\geq 2$. For a given odd nondegenerate constant Poisson bracket $(\ ,\ ):=\frac{\overleftarrow{\partial}}{\partial\phi^i}\omega^{ij}\frac{\overrightarrow{\partial}}{\partial\phi^j}$, the Hamiltonian vector field of the Hamiltonian S,

$$\delta=(\ ,S)\ ,$$

is nilpotent iff $(S,S)=0$. This δ is nothing but an A_∞-odd vector field (1.14), where the A_∞-structure is written as

$$c^j_{i_1\cdots i_k}=(-1)^{\mathbf{e}_m}\omega^{jm}\mathcal{V}_{mi_1\cdots i_k}\ .$$

Although one can obtain an A_∞-algebra in such a way, it has an additional structure; the A_∞-structure is cyclic with respect to the odd Poisson structure. Thus, we denote the corresponding algebra by $(\mathcal{H},\omega,\mathfrak{m})$ or $(\mathcal{H},\omega,S)$ and call it a cyclic A_∞-algebra (see Definition 2.11). Moreover one may notice that the condition $(S,S)=0$ is nothing but the classical BV-master equation (1.13) in the BV-formalism. Then one can see that any cyclic field theory equipped with a classical BV-structure, including classical open string field theories in the previous subsection, has a cyclic A_∞-structure (Theorem 6.1).

1.4 Noncommutativity, open strings, and D-branes

In the explanation above, we set $\{\phi^i\}$ to be formally-*noncommutative* coordinates. Mathematically, it is because, otherwise some informations of A_∞-algebras are lost in the dual supermanifold description. In the case of field theories equipped with classical BV-structure discussed in section 6, we identify the dual coordinates with the fields of field theory. The noncommutativity of fields then implies physically the presence of Chan-Paton factor in open string theory. Namely, the non(co)commutativity of $\mathcal{H}$ allows the freedom of the choice of the Chan-Paton factor, where the fields $\{\phi^i\}$ are described typically by $N\times N$ matrices with entries $\mathbb{C}$. In other words, we have a representation of the theory in terms of $N\times N$ matrices. [8] Note that in the theory of open strings there exist D-branes and open strings must end on the D-branes. The size of the matrices N then means there exist N (parallel) D-branes. The typical gauge structure group is $U(N)$, though the structure group depends on the (super)symmetry which

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[8] More precisely when we define the cyclic structure on the action we treat the real part and imaginary part of $\mathbb{C}$ separately (see subsection 6.1).

the theory has. Note that, although we can represent classical open string field theory with $N \times N$ matrices for any N, by definition the vertices $\mathcal{V}_{i_1 \cdots i_k} \in \mathbb{C}$ are independent of N. However, just as we fix a representation by $N \times N$ matrices, the theory reduces to the one equipped with cyclic L_∞-structure.

For instance, let us represent the noncommutative fields by $N \times N$ matrices as

$$\phi^i = \begin{pmatrix} \phi^i_{11} & \cdots & \phi^i_{1N} \\ \vdots & \ddots & \vdots \\ \phi^i_{N1} & \cdots & \phi^i_{NN} \end{pmatrix} .$$

The noncommutative product of ϕ^i's are the usual multiplication of the matrices. Then the A_∞-odd vector field, as in eq.(1.14), is written in terms of the component fields $\phi^i_{pq}, 1 \le p, q \le N$ which are graded *commutative*. Correspondingly, the coefficients $c^j_{i_1 \cdots i_k} \in \mathbb{C}$ of the A_∞-odd vector field are graded symmetrized and the results turns out to define an L_∞-structure (see [75] for L_∞-algebras from symmetrizations of A_∞-algebras without passing through the dual supermanifold description). Another choice of the structure groups leads to another L_∞-algebra as the results of the graded symmetrizations of the component fields. In particular, if the size of the matrices are one ($N = 1$), the coefficients $c^j_{i_1 \cdots i_k} \in \mathbb{C}$ are completely symmetrized as the dual supermanifold description of a result in [75].

Namely, when one fixes a structure group, one loses a part of the informations which the open string theory has. A more familiar example is a gauge theory. The action, before being treated in the BV-formalism, is of the form

$$S(A) = \int F_{\mu\nu} F^{\mu\nu} , \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] .$$

If the structure group is $U(N)$, each A_μ is an antiHermitian matrix. However in the case of $U(1)$ gauge theory the commutator $[A_\mu, A_\nu]$ vanishes. Namely, the structure constants except for the kinetic term (quadratic term of the action) are lost. For this reason it is reasonable to set the fields fully noncommutative. Then we discuss universal structures of open strings independent of the choice of Chan-Paton factor.

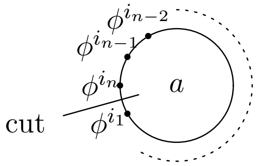
The statements above imply that many properties which hold for A_∞-algebras do also hold for L_∞-algebras. That is, at least as far as homotopy algebraic properties are concerned, classical closed string theory can be understood from that of open string theory. For this reason we shall discuss only on A_∞-side in this paper.

1.5 Formal noncommutative symplectic supergeometry

In order to discuss the algebraic properties of cyclic A_∞-algebras on a formal noncommutative supermanifold, we need some notions of noncommutative symplectic supergeometry, where a symplectic structure plays the role of a nondegenerate inner product defining the cyclicity of an A_∞-algebra (see subsection 2.3). Such a notion has appeared in [61, 62], where a constant symplectic structure is introduced. We shall extend it to a nonconstant one in the way inspired from the physics of open strings, and examine various mathematical properties of them such as the Darboux theorem (Theorem 4.15) in section 4. Note that another nonconstant extension is discussed in [30] based on Connes's noncommutative differential geometry [20]. Also, a different nonconstant extension of the inner product, called the homotopy inner product, is proposed in [119].

When one considers a Poisson algebra on a formal noncommutative supermanifold, one first needs functions on it. We define them so that they can describe linear combinations of open string disk correlation functions, which are cyclic with respect to the open string insertions (punctures) on the boundary S^1 of the disk. Pictorially, such a function is displayed as

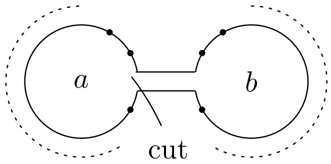
$$a_{i_1\cdots i_n}\phi^{i_n}\cdots\phi^{i_1} =$$



. (1.15)

In order to translate such cyclic objects to purely algebraic terms, one needs to cut the boundary of the disk S^1 as above. The cyclicity is then encoded in the coefficient, that is, $a_{i_1\cdots i_n}\in\mathbb{C}$ in the left hand side of (1.15) is graded symmetric with respect to the cyclic permutations of $i_1\cdots i_n$. When one considers a constant symplectic structure on a formal noncommutative supermanifold, the corresponding constant odd Poisson bracket is naturally defined so that the bracket of two open string disks becomes an open string disk. It is then natural to write the odd Poisson bracket as the following double lines

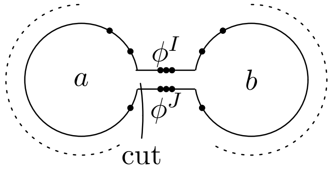
$$(A,B) =$$



. (1.16)

The choice of the place of the cut fixes the ambiguity of the sign ± for (A,B). The double line notation admits a natural extension to a nonconstant odd Poisson structure as follows

$$(A,B) =$$



, (1.17)

where I denotes a multiindex and so $\phi^I=\phi^{i_k}\cdots\phi^{i_1}$ if $I=[i_k\cdots i_1]$. The corresponding equation is

$$(A,B)=\sum_{ij,IJ}\pm\omega^{ij}_{JI}\left(\frac{A\overleftarrow{\partial}}{\partial\phi^i}\phi^I\frac{\overrightarrow{\partial}B}{\partial\phi^j}\phi^J\right)_c\ ,$$

(see eq.(4.2) in Definition 4.5), where $_c$ denotes the cyclic symmetrization and $\omega^{ij}_{JI}\in\mathbb{C}$ has an appropriate constraint so that the bracket satisfies $(B,A)=-(-1)^{AB}(A,B)$ and so on. One can define a notion of differential forms on formal noncommutative supermanifolds and the class of the odd Poisson brackets, which satisfy the Jacobi identity, can naturally be induced from closed two-forms (symplectic forms) on formal noncommutative supermanifolds (see section 4).

In string theory, the nonconstant symplectic structure here is relevant to background independent string field theory (recently preferably called a boundary string field theory) [126] (see also [43, 53]). Consequently, our definition as above seems to be natural also mathematically.

1.6 Plan of this paper

Section 2 is devoted mostly to fixing our conventions for A_∞-algebras. The precise definition of cyclic A_∞-algebras is also included. In subsection 2.1, we recall the notion of coalgebras. A_∞-algebras are then defined in terms of coalgebras (the bar construction) in subsection 2.2. The cyclic A_∞-algebras are presented in subsection 2.3. Some basic facts around Maurer-Cartan equations for A_∞-algebras are mentioned briefly in subsection 2.4.

In section 3, A_∞-algebras are realized geometrically in the dual picture. In subsection 3.1, the dual is defined explicitly through an inner product, and its graphical realization is also presented. The dual picture is used in many papers, but there are few where the explicit relation is presented. All the tools presented in subsection 2.2 are reinterpreted in terms of formal noncommutative supermanifolds in subsection 3.2. We shall then define 'superfield' to simplify conventions in the dual picture, and mention some mixed description that interpolates between the coalgebra side and its dual side in subsection 3.3.

In section 4, we shall explore local properties of symplectic structures on the formal noncommutative supermanifolds, which are relevant to the dual picture of cyclic A_∞-algebras. The notion of formal noncommutative symplectic geometry appears for instance in [61, 62]. However, nonconstant symplectic structures are not explicitly written. We first define such covariant symplectic structures inspired by open strings. Namely, we consider cyclic formal functions. In subsection 4.1, we shall observe some basic properties of constant symplectic structures, which serve as the starting point of more general cases. We then define covariant odd symplectic structures in subsection 4.2, where we show a key lemma (Lemma 4.8), the Poincaré lemma on formal noncommutative supermanifolds. Using the lemma, we examine the properties of symplectic diffeomorphisms in subsection 4.3, and show the Darboux theorem on the formal noncommutative supermanifolds (Theorem 4.15) in subsection 4.4. The study of the formal noncommutative symplectic supergeometry is directly related to the notion of cyclic A_∞-algebras. We look back over cyclic A_∞-algebras from these dual pictures in subsection 4.5.

The purpose of section 5 is to understand clearer the minimal model theorem [50], one of the key theorem in homotopy algebras. For the construction of minimal models of A_∞-structures, in particular on the homology of a differential graded algebra (dga), various versions of homological perturbation theory (HPT) have been developed, for instance, by [35, 40, 36, 37, 38, 45]. Alternatively, as mentioned in [65], there exists another stronger version of the minimal model theorem. It enables us to understand clearly the homotopical structures of homotopy algebras. We call it the decomposition theorem and prove it explicitly (Theorem 5.4) in subsection 5.1. The decomposition theorem for cyclic A_∞-algebras is then shown in subsection 5.2. The decomposition theorem guarantees the existence of an inverse A_∞-quasi-isomorphism of an A_∞-quasi-isomorphism (Theorem 5.17) as stated in [65]. We shall explain it in subsection 5.3. Though the minimal model theorem follows from the decomposition theorem, the proof relies on inductive arguments and the explicit form of a minimal model is unclear. On the other hand, it is known that for any A_∞-algebra a minimal model can be given explicitly by using some Feynman diagrams [66] (see also [38, 45, 82] and [46] for L_∞ case). We demonstrate in subsection 5.4 that the Feynman diagrams arise naturally from the issue of finding the solutions of the Maurer-Cartan equation for an A_∞-algebra [53]. The cyclic A_∞ version of the explicit minimal model is discussed in subsection 5.5, which is directly related to section 6.

In section 6, these homotopy algebraic structures are applied to field theories equipped with classical BV-structures. The appearance of cyclic A_∞-structures in field theories is explained in subsection 6.1. To consider the perturbative expansion in the BV-formalism, we shall review briefly the notion of gauge fixing and propagators in our language and examine some properties of propagators in subsection 6.2. Subsection 6.3 then shows that the tree on-shell correlation functions of a classical BV-field theory define just the minimal cyclic A_∞-algebra defined in subsection 5.5 (Corollary 6.14, cf.[53]). Moreover, in subsection 6.4, the arguments in section 5 are applied to classical open string field theories, and it is shown that all classical string field theories on a fixed conformal background are cyclic A_∞-isomorphic to each other (Theorem 6.18). Cyclic A_∞-isomorphic means physically equivalent.

Finally, in section 7, we shall come back to some basic problems in A_∞-algebras. In subsection 7.1, we shall define homotopy between A_∞-morphisms and discuss various homotopy invariant algebraic structures of A_∞-algebras. In subsection 7.2, the notion of gauge equivalence and then the moduli space of A_∞-algebras are defined. The properties of the moduli spaces are then examined.

Throughout this paper, we employ the dual picture, the formal noncommutative supermanifolds, in various places. To describe the dual of coalgebras by dual coordinates has some subtlety when the graded vector space is infinite dimensional. For instance, field theory is just such a case. However, since field theory is a theory of fields, it is well-defined as far as assuming that field theory itself is well-defined. Moreover, the dual language is used in this paper only for intuitive and geometric understanding. Hence almost all of the arguments on the dual can be rearranged in coalgebra language and hold even in the model where it is subtle to take a canonical basis system. One of the issues we do not discuss is some convergences. For instance A_∞-morphisms or the solutions of the Maurer-Cartan equations, which are formally preserved under the A_∞-morphisms, are defined by polynomials of *infinite* powers. Of course many of the arguments in this paper make sense as formal power series. For instance, in the application to field theories, each coefficient of the Maurer-Cartan equations defines an on-shell S-matrix element. However, it is also interesting to examine whether the solutions of Maurer-Cartan equations converge. This problem of convergence depends on the model equipped with an A_∞-structure. Thus looking for some 'good' models might be a good issue. Alternatively, one can also argue these on an appropriate subspace due to, for instance, the momentum conservation of the vertices in the case of field theory. Therefore, some well-defined solutions of the equations of motions may be obtained in the subspace.

2 A_∞-algebras

In this section, we shall summarize some basic facts about A_∞-algebras ((strong) homotopy associative algebras). These facts are applicable in a similar way to L_∞-algebras ((strong) homotopy Lie algebras). We restrict our arguments to A_∞-algebras over a field k of characteristic zero. For more simplicity we set $k = \mathbb{C}$.

A_∞- (and L_∞-) algebras are defined in different ways. One way is the operads. An A_∞-algebra is obtained by an algebra over a non-symmetric dg operad (see [80]). Another one is the bar construction and then A_∞-algebras are defined as coalgebras with some additional structures. The bar construction is useful to define A_∞-algebras in a simple manner and we take this definition in the present paper. For an intuitive or geometric realization of A_∞-algebras, the dual picture of coalgebras is suitable. It is the subject of the next section.

First we shall recall the notion of coalgebras in section 2.1. A_∞-algebras and A_∞-morphisms are then defined in terms of coalgebras in subsection 2.2. In subsection 2.3 we shall give a definition of A_∞-algebras with cyclic symmetry. For an A_∞-algebra, its Maurer-Cartan equation plays some important roles, which are explained briefly in subsection 2.4.

2.1 Coalgebras, coderivations, and cohomomorphisms

An element of an A_∞-algebra belong to a $\mathbb{Z}$-graded vector space $\mathcal{H}$. In the bar construction, the free tensor coalgebra of $\mathcal{H}$ is treated as a coalgebra. We first provide the notions of coalgebras.

Definition 2.1 (Coalgebra, Coassociativity) Let C be a (generally infinite dimensional) graded vector space. When a *coproduct* $\triangle : C \longrightarrow C \otimes C$ is defined on C and it is *coassociative*, *i.e.*

$$(\triangle \otimes \mathbf{1})\triangle = (\mathbf{1} \otimes \triangle)\triangle$$

then C is called a *coalgebra*.

Definition 2.2 (Coderivation) A linear operator $\mathfrak{m} : C \to C$ raising the degree of C by one is called *coderivation* when

$$\triangle\mathfrak{m} = (\mathfrak{m} \otimes \mathbf{1})\triangle + (\mathbf{1} \otimes \mathfrak{m})\triangle$$

is satisfied. Here, for $x, y \in C$, the sign is defined as $(\mathbf{1} \otimes \mathfrak{m})(x \otimes y) = (-1)^x(x \otimes \mathfrak{m}(y))$ where the x on (-1) denotes the degree of x.

Definition 2.3 (Cohomomorphism) Given two coalgebras C and C', a *cohomomorphism (coalgebra homomorphism)* $\mathcal{F}$ from C to C' is a map of degree zero satisfying the condition

$$\triangle\mathcal{F} = (\mathcal{F} \otimes \mathcal{F})\triangle\ . \tag{2.1}$$

Remark 2.4 Coassociativity of $\triangle$, the conditions of coderivations and cohomomorphisms imply that the following diagrams commute:

$$\begin{tikzcd} C \arrow[r, "\triangle"] \arrow[d, "\triangle"'] & C\otimes C \arrow[d, "\triangle\otimes\mathbf{1}"'] \\ C\otimes C \arrow[r, "\mathbf{1}\otimes\triangle"] & C\otimes C\otimes C \end{tikzcd} \ , \quad \begin{tikzcd} C \arrow[r, "\mathfrak{m}"] \arrow[d, "\triangle"'] & C \arrow[d, "\triangle"'] \\ C\otimes C \arrow[r, "\mathbf{1}\otimes\mathfrak{m}+\mathfrak{m}\otimes\mathbf{1}"] & C\otimes C \end{tikzcd} \ , \quad \begin{tikzcd} C \arrow[r, "\mathcal{F}"] \arrow[d, "\triangle"'] & C' \arrow[d, "\triangle"'] \\ C\otimes C \arrow[r, "\mathcal{F}\otimes\mathcal{F}"] & C'\otimes C' \end{tikzcd} \ .$$

If the orientation of these map are reversed and the coproduct is replaced by a product, then the coassociativity, the coderivation, and the cohomomorphism take place to associativity, a derivation, and a homomorphism of the corresponding algebra, respectively.

Reversing the orientation of the maps corresponds to taking the dual of the coalgebra. The precise meaning of the dual in the present paper is given in subsection 3.1.

Let $\mathcal{H}$ be a $\mathbb{Z}$-graded vector space. Namely, $\mathcal{H}=\oplus_{k\in\mathbb{Z}}\mathcal{H}^k$ where $\mathcal{H}^k$ is a vector space of degree k. Consider the free tensor coalgebra of $\mathcal{H}$

$$C(\mathcal{H})=\oplus_{n\geq 0}\mathcal{H}^{\otimes n}$$

as a coalgebra. Note that $\mathcal{H}^{\otimes 0}=\mathbb{C}$, which includes a counit $\mathbf{1}$. [9]

Then the coassociative coproduct $\triangle : C(\mathcal{H})\to C(\mathcal{H})\otimes C(\mathcal{H})$ is uniquely determined. For $\mathbf{e}_1\cdots\mathbf{e}_n\in\mathcal{H}^{\otimes n}$ it is given by

$$\triangle(\mathbf{e}_1\cdots\mathbf{e}_n)=\sum_{k=0}^{n}(\mathbf{e}_1\cdots\mathbf{e}_k)\otimes(\mathbf{e}_{k+1}\cdots\mathbf{e}_n)\ , \tag{2.2}$$

where the term for $k=0$ is $\mathbf{1}\otimes(\mathbf{e}_1\cdots\mathbf{e}_n)$ and the term for $k=n$ is $(\mathbf{e}_1\cdots\mathbf{e}_n)\otimes\mathbf{1}$. The form of the coderivation corresponding to this coproduct is also given as follows. Let $\{m_k\}_{k\geq 0}$ be a set of multilinear maps of degree one

$$\begin{array}{cccc} m_k\ : & \mathcal{H}^{\otimes k} & \longrightarrow & \mathcal{H} \\ & \mathbf{e}_1\otimes\cdots\otimes\mathbf{e}_k & \mapsto & m_k(\mathbf{e}_1,\cdots,\mathbf{e}_k) \end{array} \ . \tag{2.3}$$

$m_0:\mathbb{C}\to\mathcal{H}$ is defined so that $m_0(\mathbf{1})$ has degree one. The operation on $C(\mathcal{H})$ is given as

$$\mathfrak{m}_k(\mathbf{e}_1\cdots\mathbf{e}_n)=\sum_{p=1}^{n-k}(-1)^{\mathbf{e}_1+\cdots+\mathbf{e}_{p-1}}\mathbf{e}_1\cdots\mathbf{e}_{p-1}m_k(\mathbf{e}_p,\cdots,\mathbf{e}_{p+k-1})\mathbf{e}_{p+k}\cdots\mathbf{e}_n\ ,\quad \mathbf{e}_i\in\mathcal{H}\ .$$

Here $\mathbf{e}_1+\cdots+\mathbf{e}_{p-1}$ on (-1) denotes the degree of $\mathbf{e}_1\cdots\mathbf{e}_{p-1}$. The sign factor appears when m_k, which has degree one, passes through the $\mathbf{e}_1\cdots\mathbf{e}_{p-1}$.

Then summing up these $\mathfrak{m}_k$ for $k\geq 0$,

$$\mathfrak{m}=\mathfrak{m}_0+\mathfrak{m}_1+\mathfrak{m}_2+\cdots\ , \tag{2.4}$$

and this $\mathfrak{m}$ is the coderivation. The coderivation on the coalgebra $C(\mathcal{H})$ is always written in this form.

---

[9] One may or may not include the $\mathcal{H}^{\otimes 0}$ term for defining an A_∞-algebra. If includes, one can also define a weak A_∞-algebra uniformly, so we use this convention.

Moreover, the form of a cohomomorphism $\mathcal{F} : C(\mathcal{H}) \to C(\mathcal{H}')$ is determined by a collection of degree zero multilinear maps $f_k : \mathcal{H}^{\otimes k} \to \mathcal{H}'(k \geq 0)$ which are homogeneous of degree zero of the following form

$$\begin{aligned} \mathcal{F}(\mathbf{e}_1 \cdots \mathbf{e}_n) = \sum_{1 \leq k_1 < k_2 \cdots < k_i = n} & e^{f_0(\mathbf{1})} f_{k_1}(\mathbf{e}_1, \cdots, \mathbf{e}_{k_1}) e^{f_0(\mathbf{1})} f_{k_2 - k_1}(\mathbf{e}_{k_1+1}, \cdots, \mathbf{e}_{k_2}) e^{f_0(\mathbf{1})} \\ & \cdots e^{f_0(\mathbf{1})} f_{n-k_{i-1}}(\mathbf{e}_{k_{i-1}+1}, \cdots, \mathbf{e}_n) , \end{aligned} \tag{2.5}$$

where each $f(\cdots)$ belongs to $\mathcal{H}'$. $e^{f_0(\mathbf{1})}$ is defined by

$$e^{f_0(\mathbf{1})} = \mathbf{1} + f_0(\mathbf{1}) + f_0(\mathbf{1}) \otimes f_0(\mathbf{1}) + f_0(\mathbf{1}) \otimes f_0(\mathbf{1}) \otimes f_0(\mathbf{1}) + \cdots .$$

If $f_0(\mathbf{1}) = 0$, eq.(2.5) is simplified since $e^{f_0(\mathbf{1})} = \mathbf{1}$. Note that $\mathbf{1}$ is defined as $\mathcal{H}^{\otimes m} \otimes \mathbf{1} \otimes \mathcal{H}^{\otimes n} = \mathcal{H}^{\otimes (m+n)}$ for $m, n \geq 0$ and $m + n \geq 1$.

2.2 A_∞-algebras and A_∞-morphisms

Definition 2.5 (A_∞-algebra [106, 107]) Let $\mathcal{H}$ be a graded vector space and $C(\mathcal{H}) = \oplus_{k \geq 0} \mathcal{H}^{\otimes k}$ be its tensor coalgebra. A *weak A_∞-algebra* is a coalgebra $C(\mathcal{H})$ with a coderivation $\mathfrak{m} = \mathfrak{m}_0 + \mathfrak{m}_1 + \mathfrak{m}_2 + \cdots$ satisfying

$$(\mathfrak{m})^2 = 0 .$$

We denote the collection of multilinear maps $\{m_k\}_{k \geq 0}$ also by $\mathfrak{m}$ and the weak A_∞-algebra by $(\mathcal{H}, \mathfrak{m})$. In particular, $(\mathcal{H}, \mathfrak{m})$ is called an *A_∞-algebra* if $m_0 = 0$.

In general, a coderivation $\mathfrak{m} : C \to C$ on a coalgebra C satisfying $(\mathfrak{m})^2 = 0$ as above is called a *codifferential.* Thus, a (weak) A_∞-algebra is a *differential graded coalgebra* of the tensor coalgebra of a graded vector space $\mathcal{H}$.

For an A_∞-algebra if we act $(\mathfrak{m})^2 = (\mathfrak{m}_1 + \mathfrak{m}_2 + \cdots)^2$ on $\mathbf{e}_1 \cdots \mathbf{e}_n \in C(\mathcal{H})$, its image belongs to $\mathcal{H}^{\otimes 1} \oplus \cdots \oplus \mathcal{H}^{\otimes n}$, and the condition that the $\mathcal{H}^{\otimes 1}$ part of the image equal zero is

$$\sum_{\substack{k+l=n+1 \\ j=0,\cdots,k-1}} (-1)^{\mathbf{e}_1 + \cdots + \mathbf{e}_j} m_k(\mathbf{e}_1, \cdots, \mathbf{e}_j, m_l(\mathbf{e}_{j+1}, \cdots, \mathbf{e}_{j+l}), \mathbf{e}_{j+l+1}, \cdots, \mathbf{e}_n) = 0 , \tag{2.6}$$

where $\mathbf{e}_i$ on (-1) denotes the degree of $\mathbf{e}_i$. The collection of the identities (2.6) for $n \geq 1$ is the original definition of A_∞-algebras. On the other hand, the construction of A_∞-algebras using the tensor coalgebra $C(\mathcal{H})$ is called *the bar construction.* Actually, eq.(2.6) is equivalent (*i.e.* sufficient) to $\mathfrak{m}^2 = 0$ due to the anticommutativity of $\mathfrak{m}_i$'s. This fact is clearer in the dual language in the next section, where the nilpotent coderivation $\mathfrak{m}$ is replaced to a nilpotent differential δ on a formal (noncommutative) supermanifold. [10]

---

[10] The dual language fits the field theory. In the context of BRST-formalism, physicists usually show the nilpotency of BRST-operator δ on polynomials of fields and ghosts (and antifields) by confirming the nilpotency on each component field. This is just the dual of eq.(2.6).

Let us write down the first three constraints in eq.(2.6).

$$\begin{aligned}
&m_1^2=0 \quad , \\
&m_1(m_2(\mathbf{e}_1,\mathbf{e}_2))+m_2(m_1(\mathbf{e}_1),\mathbf{e}_2)+(-1)^{\mathbf{e}_1}m_2(\mathbf{e}_1,m_1(\mathbf{e}_2))=0 \quad , \qquad (2.7)\\
&m_2(m_2(\mathbf{e}_1,\mathbf{e}_2),\mathbf{e}_3)+(-1)^{\mathbf{e}_1}m_2(\mathbf{e}_1,m_2(\mathbf{e}_2,\mathbf{e}_3))\\
&\qquad +m_1(m_3(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3))+m_3(m_1(\mathbf{e}_1),\mathbf{e}_2,\mathbf{e}_3)+(-1)^{\mathbf{e}_1}m_3(\mathbf{e}_1,m_1(\mathbf{e}_2),\mathbf{e}_3)\\
&\qquad +(-1)^{\mathbf{e}_1+\mathbf{e}_2}m_3(\mathbf{e}_1,\mathbf{e}_2,m_1(\mathbf{e}_3))=0 \ .
\end{aligned}$$

The first equation indicates m_1 is nilpotent and $(\mathcal{H},m_1)$ defines a complex on the $\mathbb{Z}$-graded vector space $\mathcal{H}$. The second equation implies differential m_1 satisfies Leibniz rule for the product m_2. The third equation means product m_2 is associative up to the terms including m_3.

Remark 2.6 In the case $m_n=0$ for $n\geq 3$, an A_∞-algebra reduces to a differential graded (associative) algebra (dga). The differential d and the product $\bullet$ of dga $\mathfrak{g}$ correspond to m_1 and m_2, respectively. However, the product $\bullet$ of dga preserves the degree and m_2 in A_∞-algebras raises the degree by one. For this reason, when a dga $(\mathfrak{g},d,\bullet)$ is considered as an A_∞-algebra $(\mathcal{H},\mathfrak{m})$ defined in Definition 2.5, the degree in the A_∞-algebra is defined as the degree of the dga minus one. Namely, let $s:\mathfrak{g}^k\to\mathfrak{g}^{k-1}[1]=:\mathcal{H}^{k-1}$ be the isomorphism called the *suspension*, where $\mathfrak{g}^k$ is the degree k part of $\mathfrak{g}$. This [1] 'eats' one degree of $\mathfrak{g}$, and the degree of $\mathfrak{g}^{k-1}[1]$ is regarded as $(k-1)$ through the operation. Then the following diagram commutes

$$\begin{tikzcd}
\mathfrak{g}^k\otimes\mathfrak{g}^l \arrow[r, "\bullet"] \arrow[d, "s"'] & \mathfrak{g}^{k+l} \arrow[d, "s"'] \\
\mathcal{H}^{k-1}\otimes\mathcal{H}^{l-1} \arrow[r, "m_2(\ ,\ )"] & \mathcal{H}^{k+l-1}
\end{tikzcd} .$$

There are many literatures where the degree of A_∞-algebras are defined with the dga degree. Usually Witten's open string field theory [124], which has the structure of a dga, is also defined with the dga degree explained above. The degree is certainly natural from the origin of A_∞-algebras (see subsection 1.1). However, when higher products $m_3,m_4,\cdots$ are introduced, the degree given in Definition 2.2 is simpler for A_∞-algebras. For this reason, we use this convention in the present paper. The precise relation between these two conventions can be found in [27].

Definition 2.7 (A_∞-morphism) Given two weak A_∞-algebras $(\mathcal{H},\mathfrak{m})$ and $(\mathcal{H}',\mathfrak{m}')$, *a weak A_∞-morphism* $\mathcal{F}:(\mathcal{H},\mathfrak{m})\to(\mathcal{H}',\mathfrak{m}')$ is a cohomomorphism from $C(\mathcal{H})$ to $C(\mathcal{H}')$ satisfying

$$\mathcal{F}\mathfrak{m}=\mathfrak{m}'\mathcal{F}\ . \qquad (2.8)$$

In particular for two A_∞-algebras $(\mathcal{H},\mathfrak{m})$ and $(\mathcal{H}',\mathfrak{m}')$ a weak A_∞-morphism $\mathcal{F}:(\mathcal{H},\mathfrak{m})\to(\mathcal{H}',\mathfrak{m}')$ is called *an A_∞-morphism* iff $f_0=0$.

For an A_∞-morphism $\mathcal{F}:(\mathcal{H},\mathfrak{m})\to(\mathcal{H}',\mathfrak{m}')$, evaluating the condition (2.8) with $\mathbf{e}_1\cdots\mathbf{e}_n\in C(\mathcal{H})$ for $n\geq 1$, one gets a relation in $\oplus_{n'=1}^{n}\mathcal{H}'^{\otimes n'}$. Picking up the $\mathcal{H}'^{\otimes 1}$ part of the equation then yields

$$
\begin{aligned}
&\sum_{1\le k_1<k_2\cdots<k_i=n} m'_i(f_{k_1}(\mathbf{e}_1,\cdots,\mathbf{e}_{k_1}),f_{k_2-k_1}(\mathbf{e}_{k_1+1},\cdots,\mathbf{e}_{k_2})\cdots f_{n-k_{i-1}}(\mathbf{e}_{k_{i-1}+1},\cdots,\mathbf{e}_n))\\
&\quad=\sum_{k+l=n+1}\sum_{j=0}^{k-1}(-1)^{\mathbf{e}_1+\cdots+\mathbf{e}_j}f_k(\mathbf{e}_1,\cdots,\mathbf{e}_j,m_l(\mathbf{e}_{j+1},\cdots,\mathbf{e}_{j+l}),\mathbf{e}_{j+l+1},\cdots,\mathbf{e}_n)\ .
\end{aligned}
\tag{2.9}
$$

The first two constraints in (2.9) read:

$$
\begin{aligned}
m'_1(f_1(\mathbf{e}_1)) &= f_1(m_1(\mathbf{e}_1))\ ,\\
m'_2(f_1(\mathbf{e}_1),f_1(\mathbf{e}_2)) &= f_1(m_2(\mathbf{e}_1,\mathbf{e}_2))\\
&\qquad + m'_1(f_2(\mathbf{e}_1,\mathbf{e}_2)) + f_2(m_1(\mathbf{e}_1),\mathbf{e}_2) + (-1)^{\mathbf{e}_1}f_2(\mathbf{e}_1,m_1(\mathbf{e}_2))\ .
\end{aligned}
$$

In particular, the first equation implies that f_1 induces a degree zero linear map f_{1*} between the cohomologies $H_{m_1}(\mathcal{H})$ and $H_{m'_1}(\mathcal{H}')$. In the dual picture explained in the next section, $\mathcal{F}$ is identified with a nonlinear map between two supermanifolds.

Definition 2.8 (A_∞-(quasi)-isomorphism) An A_∞-morphism $\mathcal{F}=\{f_1,f_2,\cdots\}:(\mathcal{H},\mathfrak{m})\to(\mathcal{H}',\mathfrak{m}')$ is called an A_∞-*quasi-isomorphism* if f_1 induces an isomorphism between the cohomology spaces $H_{m_1}(\mathcal{H})$ and $H_{m'_1}(\mathcal{H}')$. In particular, if $f_1:\mathcal{H}\to\mathcal{H}'$ is an isomorphism, $\mathcal{F}$ is called an A_∞-*isomorphism*. Moreover, if $(\mathcal{H}',\mathfrak{m}')=(\mathcal{H},\mathfrak{m})$, we call $\mathcal{F}$ an A_∞-*automorphism*.

These are also defined in weak A_∞ level. It is clear that any A_∞-isomorphism $\mathcal{F}:(\mathcal{H},\mathfrak{m})\to(\mathcal{H}',\mathfrak{m}')$ has its inverse A_∞-isomorphism $\mathcal{F}^{-1}:(\mathcal{H}',\mathfrak{m}')\to(\mathcal{H},\mathfrak{m})$. Also, if $\mathcal{F}$ is an A_∞-quasi-isomorphism, there exists an inverse quasi-isomorphism (we denote it also by $\mathcal{F}^{-1}$) [64, 65], which will be discussed in subsection 5.3.

Remark 2.9 (Cocommutativity and L_∞-algebras) L_∞-algebras are obtained by imposing cocommutativity upon coalgebra $C(\mathcal{H})$. A coalgebra C is *cocommutative* iff there exists an operator $\tau:C\otimes C\to C\otimes C$, $\tau(x\otimes y)=(-1)^{xy}y\otimes x$ that is compatible with the coproduct,

$$\tau\triangle=\triangle\ .$$

The corresponding tensor coalgebra is $C(\mathcal{H})$ divided by the ideal generated by $\mathbf{e}_i\otimes\mathbf{e}_j-(-1)^{\mathbf{e}_i\mathbf{e}_j}\mathbf{e}_j\otimes\mathbf{e}_i$. Namely, in this case $\{\mathbf{e}_i\}$ are set to be graded commutative. An L_∞-algebra is then obtained by defining degree one codifferential (coderivation whose square is zero) so that it is compatible with the graded commutativity, that is, by graded symmetrizing each multi-linear map m_k [75] (see also [76, 24, 55], etc.).

2.3 Cyclic A_∞-structures

In this subsection A_∞-structures with cyclic symmetry are defined. We consider a graded vector space $\mathcal{H}$ equipped with an odd constant symplectic inner product. An origin of these definitions is the BV-formalism as explained in subsection 6.1. The naturalness of these definitions can be realized from the dual picture in section 4.

Definition 2.10 (Odd constant symplectic structure) Let $\mathcal{H}$ be a graded vector space. An *odd constant symplectic structure* $\omega : \mathcal{H} \otimes \mathcal{H} \to \mathbb{C}$ is a nondegenerate skewsymmetric bilinear map of degree minus one. For bases $\mathbf{e}_i, \mathbf{e}_j \in \mathcal{H}$, it is represented as

$$\omega(\mathbf{e}_i, \mathbf{e}_j) = \omega_{ij} \ .$$

Here $\omega_{ij} = 0$ if $\deg(\mathbf{e}_i) + \deg(\mathbf{e}_j) \neq 1$, since the degree of ω is minus one. Moreover $\omega_{ji} = -\omega_{ij}$ since it is skewsymmetric.

Definition 2.11 (Cyclic A_∞-algebra) Suppose a graded vector space $\mathcal{H}$ is equipped with an odd constant symplectic structure $\omega : \mathcal{H} \otimes \mathcal{H} \to \mathbb{C}$ in Definition 2.10. A triple $(\mathcal{H}, \omega, \mathfrak{m})$ is called a *cyclic A_∞-algebra* when $(\mathcal{H}, \mathfrak{m})$ is an A_∞-algebra and $\mathfrak{m}$ is cyclic with respect to ω, that is,

$$\omega(\mathbf{e}_{i_1}, m_k(\mathbf{e}_{i_2}, \cdots, \mathbf{e}_{i_{k+1}})) = (-1)^{\mathbf{e}_{i_2}} \omega(\mathbf{e}_{i_2}, m_k(\mathbf{e}_{i_3}, \cdots, \mathbf{e}_{i_{k+1}}, \mathbf{e}_{i_1}))$$

for each $k \geq 1$.

A_∞-algebras with cyclic symmetry as above are considered in the context of mathematical physics for instance in [62, 129, 26]. See [80]. Also, a homotopy extension of this cyclicity is proposed in [119].

Remark 2.12 Let us define a collection of degree zero multilinear maps $S := \{\mathcal{V}_k : \mathcal{H}^{\otimes k} \to \mathbb{C}\}_{k \geq 2}$ by

$$\mathcal{V}_{k+1}(\mathbf{e}_{i_1}, \cdots, \mathbf{e}_{i_{k+1}}) := (-1)^{\mathbf{e}_{i_1}} \omega(\mathbf{e}_{i_1}, m_k(\mathbf{e}_{i_2}, \cdots, \mathbf{e}_{i_{k+1}})) \ .$$

The cyclicity is $\mathcal{V}_{k+1}(\mathbf{e}_{i_1}, \cdots, \mathbf{e}_{i_{k+1}}) = (-1)^{\mathbf{e}_{i_1}} \mathcal{V}_{k+1}(\mathbf{e}_{i_2}, \cdots, \mathbf{e}_{i_{k+1}}, \mathbf{e}_{i_1})$. Then a cyclic A_∞-algebra $(\mathcal{H}, \omega, \mathfrak{m})$ can also be defined by triple $(\mathcal{H}, \omega, S)$. Hereafter we use both notations for a cyclic A_∞-algebra. Suppose first that S of degree zero is given. The degree of the inner product ω is then determined as minus one (odd).

Definition 2.13 (Cyclic A_∞-morphism) Let $(\mathcal{H}, \omega, S)$ and $(\mathcal{H}', \omega', S')$ be two cyclic A_∞-algebras and suppose there exists an A_∞-morphism $\mathcal{F} : (\mathcal{H}, \mathfrak{m}) \to (\mathcal{H}', \mathfrak{m}')$. We then call $\mathcal{F}$ a *cyclic A_∞-morphism* when

$$\omega'(f_1(\mathbf{e}_i), f_1(\mathbf{e}_j)) = \omega(\mathbf{e}_i, \mathbf{e}_j) \ , \tag{2.10}$$

and for fixed $n \geq 3$,

$$\sum_{k,l \geq 1,\ k+l=n} \omega'(f_k(\mathbf{e}_{i_1}, \cdots, \mathbf{e}_{i_k}), f_l(\mathbf{e}_{i_{k+1}}, \cdots, \mathbf{e}_n)) = 0 \tag{2.11}$$

holds.

Remark 2.14 We shall explain in subsection 4.5 that, in the dual language, $\mathcal{F}$ is just the morphism which preserves the constant symplectic forms and the actions.

It is clear that cyclic A_∞-algebras and cyclic A_∞-morphisms can be defined also at weak level in the same way as in the previous subsection.

2.4 Maurer-Cartan equations and deformation theory

Here we shall define Maurer-Cartan equations for A_∞-algebras and their roles in deformation theory (see [24]). Maurer-Cartan equation is an equation for elements of $\mathcal{H}$. Therefore we usually concentrate on its degree-zero part so that the equation has an usual meaning. Let us consider an element $\Phi \in \mathcal{H}^0$ where $\mathcal{H}^0$ is the degree-zero subvector space of $\mathcal{H}$. Using the basis of $\mathcal{H}^0$, $\{\mathbf{e}_i^0\}$, one can express it as $\Phi = \sum_i \mathbf{e}_i^0 \phi^i$ for $\phi^i \in \mathbb{C}$. Note that we shall define later a 'superfield' Φ as a more extended object than the one here, and consider the Maurer-Cartan equation in a similar manner as below. Alternatively, we shall discuss the situation here in a precise way in subsection 7.2. In this section we present some abstract definitions without the details.

Consider formally the following exponential map of $\Phi \in \mathcal{H}^0$

$$e^\Phi := \mathbf{1} + \Phi + \Phi\otimes\Phi + \Phi\otimes\Phi\otimes\Phi + \cdots\ . \tag{2.12}$$

$e^\Phi \in C(\mathcal{H}^0) \subset C(\mathcal{H})$ satisfies $\triangle e^\Phi = e^\Phi \otimes e^\Phi$ and such element is called a *grouplike element.*

Definition 2.15 (Maurer-Cartan equation) For an A_∞-algebra $(\mathcal{H}, \mathfrak{m})$, define

$$\mathfrak{m}_*(e^\Phi) := m_1(\Phi) + m_2(\Phi\otimes\Phi) + m_3(\Phi\otimes\Phi\otimes\Phi) + \cdots\ .$$

$\mathfrak{m}_*(e^\Phi) = 0$ is called *Maurer-Cartan equation* for $(\mathcal{H}, \mathfrak{m})$. We denote by $\mathcal{MC}(\mathcal{H}, \mathfrak{m})$ the solution space of the Maurer-Cartan equation for $(\mathcal{H}, \mathfrak{m})$.

Because $\mathfrak{m}(e^\Phi) = e^\Phi \cdot \mathfrak{m}_*(e^\Phi)\cdot e^\Phi$, $\mathfrak{m}_*(e^\Phi) = 0$ is equivalent to $\mathfrak{m}(e^\Phi) = 0$, where $\mathbf{1}$ is defined as $\mathcal{H}^{\otimes m} \otimes \mathbf{1} \otimes \mathcal{H}^{\otimes n} = \mathcal{H}^{\otimes(m+n)}$ for $m, n \geq 0$ and $m + n \geq 1$. When an A_∞-algebra is a dga, *i.e.* $m_3 = m_4 = \cdots = 0$, its Maurer-Cartan equation takes the form $m_1(\Phi) + m_2(\Phi\otimes\Phi) = 0$. It is nothing but the condition of a 'flat connection'. In the case of field theory equipped with a classical BV-structure, the theory has a cyclic A_∞-structure, and its Maurer-Cartan equation is just the equation of motion of the action (see eq.(5.26)).

Remark 2.16 The solution space $\mathcal{MC}(\mathcal{H}, \mathfrak{m})$ of the Maurer-Cartan equation for $(\mathcal{H}, \mathfrak{m})$ parameterizes deformations of original A_∞-algebra $(\mathcal{H}, \mathfrak{m})$.

Generally, for a A_∞-algebra $(\mathcal{H}, \mathfrak{m})$ and a $\mathbb{Z}$-graded vector space $\tilde{\mathcal{H}}$, suppose a cohomomorphism $\tilde{\mathcal{F}} : C(\tilde{\mathcal{H}}) \to C(\mathcal{H})$ is given. Then there exists the inverse cohomomorphism $\tilde{\mathcal{F}}^{-1} : C(\tilde{\mathcal{H}}) \to C(\mathcal{H})$ such that $\tilde{\mathcal{F}}\tilde{\mathcal{F}}^{-1} = \mathbf{1}$ and $\tilde{\mathcal{F}}^{-1}\tilde{\mathcal{F}} = \mathbf{1}$. Namely, a weak A_∞-structure $\tilde{\mathfrak{m}} = \tilde{\mathcal{F}}^{-1}\mathfrak{m}\tilde{\mathcal{F}} : C(\tilde{\mathcal{H}}) \to C(\tilde{\mathcal{H}})$ is induced. Actually, it is clear that $\tilde{\mathcal{F}}\tilde{\mathfrak{m}} = \mathfrak{m}\tilde{\mathcal{F}}$ holds.

In this situation let us consider the cohomomorphism $\tilde{\mathcal{F}} = \{\tilde{f}_0, \tilde{f}_1, \cdots\}$ with $\tilde{f}_0 = \Phi \in \mathcal{H}^0$, $\tilde{f}_1 = \mathrm{Id}$ and $\tilde{f}_k = 0$ for $k \geq 2$. The induced A_∞-structure $\tilde{\mathfrak{m}} = \{\tilde{m}_0, \tilde{m}_1, \cdots\}$ is given of the form

$$\tilde{m}_k(\mathbf{e}_{i_1}, \cdots, \mathbf{e}_{i_k}) = \sum_{l_0\geq 0,\cdots,l_k\geq 0} m_{k+l_0+\cdots+l_k}(\Phi^{\otimes l_0}, \mathbf{e}_{i_1}, \Phi^{\otimes l_1}, \cdots, \Phi^{\otimes l_{k-1}}, \mathbf{e}_{i_k}, \Phi^{\otimes l_k})$$

for $k \geq 0$. In the equation above, $\mathbf{e}_i$ in both sides are identified with each other by $\tilde{f}_1 = \mathrm{Id} : \tilde{\mathcal{H}} \to \mathcal{H}$. One can easily see, by concentrating on the case $k = 0$ in the equation above, that the induced weak A_∞-algebra $(\tilde{\mathcal{H}}, \tilde{\mathfrak{m}})$ is an A_∞-algebra if and only if $\Phi \in \mathcal{MC}(\mathcal{H}, \mathfrak{m})$.

In the case of cyclic A_∞-algebras, this means that, for each solution of the equation of motion, another A_∞-algebra is defined. Such a property is observed in classical closed string field theory, *i.e.* for (cyclic) L_∞-algebras in [94] and is related to the problem of background independence of string field theory (see the end of subsection 6.4).

The solution space $\mathcal{MC}(\mathcal{H},\mathfrak{m})$ is a subspace of the whole deformation space of A_∞-algebra $(\mathcal{H},\mathfrak{m})$ that is defined by

$$\{\mathfrak{m}_{def}: C(\mathcal{H})\to C(\mathcal{H});\ \text{degree one coderivation}\ |(\mathfrak{m}+\mathfrak{m}_{def})^2=0\}\ .$$

From string theory point of view, the A_∞-structure $\mathfrak{m}$ is related to a structure of tree open string interactions (for the case of string field theories see subsection 1.2, and for a topological string case see for instance [22]), and the deformation associated to $\mathcal{MC}(\mathcal{H},\mathfrak{m})$ corresponds to deformation that comes from condensation of open string fields.

When we are interested in the space $\mathcal{MC}(\mathcal{H},\mathfrak{m})$, it is often convenient to relate $(\mathcal{H},\mathfrak{m})$ to another A_∞-algebra $(\tilde{\mathcal{H}},\tilde{\mathfrak{m}})$. Suppose that there exists an A_∞-morphism $\tilde{\mathcal{F}}:(\tilde{\mathcal{H}},\tilde{\mathfrak{m}})\to(\mathcal{H},\mathfrak{m})$. The A_∞-morphism $\tilde{\mathcal{F}}$ then preserves the solutions of the Maurer-Cartan equations. In the context of field theory, this fact means that cyclic A_∞-morphisms preserve the equations of motions. For $\tilde{\Phi}\in\mathcal{MC}(\tilde{\mathcal{H}},\tilde{\mathfrak{m}})$, Φ is constructed as the pushforward of $\tilde{\mathcal{F}}$, a (nonlinear) coordinate transformation between two formal noncommutative supermanifolds (see the next section),

$$\Phi=\tilde{\mathcal{F}}_*(\tilde{\Phi})=\sum_{n=1}^{\infty}\tilde{f}_n(\tilde{\Phi},\cdots,\tilde{\Phi})\ . \tag{2.13}$$

It by construction satisfies $\tilde{\mathcal{F}}(e^{\tilde{\Phi}})=e^{\Phi}$. The following equality then holds,

$$\mathfrak{m}(e^{\Phi})=\mathfrak{m}\tilde{\mathcal{F}}(e^{\tilde{\Phi}})=\tilde{\mathcal{F}}\tilde{\mathfrak{m}}(e^{\tilde{\Phi}})\ ,$$

and one can immediately see that $\Phi\in\mathcal{MC}(\mathcal{H},\mathfrak{m})$ (*i.e.* $\mathfrak{m}(e^{\Phi})=0$) if $\tilde{\Phi}\in\mathcal{MC}(\tilde{\mathcal{H}},\tilde{\mathfrak{m}})$ (*i.e.* $\tilde{\mathfrak{m}}(e^{\tilde{\Phi}})=0$).

More precisely, for a given A_∞-algebra $(\mathcal{H},\mathfrak{m})$, there exists a notion of gauge transformation (Definition 7.9). It is, so to speak, an automorphism of the theory generated by infinitesimal transformations. As seen in subsection 6.1 it just corresponds to the gauge transformation in classical BV-field theory. By definition it preserves $\mathcal{MC}(\mathcal{H},\mathfrak{m})$, *i.e.* , the equation of motion. What should be considered is then, instead of $\mathcal{MC}(\mathcal{H},\mathfrak{m})$, the moduli space of A_∞-algebra $(\mathcal{H},\mathfrak{m})$, that is defined by dividing $\mathcal{MC}(\mathcal{H},\mathfrak{m})$ over the gauge transformation (see Definition 7.13). An A_∞-morphism, that preserves the solution space of Maurer-Cartan equations, in fact induces a well-defined morphism between the moduli spaces. In particular, it is known that the moduli spaces are isomorphic to each other if there exists an A_∞-quasi-isomorphism between them (Theorem 7.16). Such generality of homotopy algebras is, for instance in L_∞ case, applied by M. Kontsevich [65] to the proof of the existence of deformation quantizations [12] on Poisson manifolds and their classification.

3 Dual geometric description of homotopy algebras

We shall give the dual description of A_∞-algebras. In this picture, A_∞-algebras are understood more geometrically. The definition of the dual of a coalgebra in the present paper is given in subsection 3.1. Its geometric viewpoint is explained in subsection 3.2, where we deal with a formal noncommutative supermanifold. In subsection 3.3 we define the notion of superfield which simplify the convention in the dual picture. It will be used in later discussions. These arguments hold similarly for L_∞-algebras.

3.1 The dual of coalgebras

Let $\mathcal{H}$ be a graded vector space, and $C(\mathcal{H}) := \oplus_{n=0}^{\infty}(\mathcal{H}^{\otimes n})$ be its tensor coalgebra. The basis of $\mathcal{H}$ is denoted by $\{\mathbf{e}_i\}$, and here we define the dual basis of $\mathbf{e}_{i_1}\cdots\mathbf{e}_{i_k} \in \mathcal{H}^{\otimes k}$ by introducing a natural pairing as follows. At first, denote the dual basis of $\{\mathbf{e}_i\}$ by $\{\mathbf{e}^i\}$, and define a pairing between $\mathcal{H}$ and $\mathcal{H}^*$ as

$$\langle \mathbf{e}^i | \mathbf{e}_j \rangle = \delta^i_j \ . \tag{3.1}$$

We represent an elements of $C(\mathcal{H})$ as $g = \sum_{k=1}^{\infty} g^{i_k \cdots i_1}\mathbf{e}_{i_1}\cdots\mathbf{e}_{i_k}$, and an element of $C(\mathcal{H})^*$, the dual of $C(\mathcal{H})$ as $a = \sum_{k=1}^{\infty} a_{i_1\cdots i_k}\mathbf{e}^{i_k}\cdots\mathbf{e}^{i_1}$. Generalizing the above pairing between $\mathcal{H}$ and $\mathcal{H}^*$ (3.1), here the pairing between $C(\mathcal{H})$ and $C(\mathcal{H})^*$ is defined as

$$\langle \mathbf{e}^{i_k}\cdots\mathbf{e}^{i_1} | \mathbf{e}_{j_1}\cdots\mathbf{e}_{j_l} \rangle = \epsilon^{i_1\cdots i_k}_{j_1\cdots j_l}, \tag{3.2}$$

where $\epsilon^{i_1\cdots i_k}_{j_1\cdots j_l}$ is defined to be zero if $k \neq l$ and $\epsilon^{i_1\cdots i_k}_{j_1\cdots j_k} = \delta^{i_1}_{j_1}\cdots\delta^{i_k}_{j_k}$ if $k = l$. In addition we define $\langle \mathbf{1}|\mathbf{1}\rangle = 1$. Moreover, for $a_1, \cdots, a_n \in C(\mathcal{H})^*$ and $g_1, \cdots, g_n \in C(\mathcal{H})$, the pairing of n-tensor is given by

$$\langle a_1 \otimes \cdots \otimes a_n | g_1 \otimes \cdots \otimes g_n \rangle = \langle a_1|g_1\rangle \cdots \langle a_n|g_n\rangle \ .$$

Since now we have obtained the pairing between $C(\mathcal{H})$ and its dual $C(\mathcal{H})^*$, we can translate operations on $C(\mathcal{H})$ into those on $C(\mathcal{H})^*$. For the coproduct $\triangle$ on $C(\mathcal{H})$, the product m on $C(\mathcal{H})^*$ is defined as

$$\langle m(a \otimes b)|g\rangle = \langle a \otimes b|\triangle g\rangle \ , \tag{3.3}$$

the derivation δ corresponding to the coderivation $\mathfrak{m}$ is defined as

$$\langle \delta(a)|g\rangle = \langle a|\mathfrak{m}(g)\rangle \ , \tag{3.4}$$

and homomorphism $\mathcal{F}^*$ corresponds to the cohomomorphism $\mathcal{F}$ from $C(\mathcal{H})$ to another tensor algebra $C(\mathcal{H}')$ is determined as

$$\langle \mathcal{F}^*(a)|g\rangle = \langle a|\mathcal{F}(g)\rangle \ . \tag{3.5}$$

Because $g \in C(\mathcal{H})$ and $a \in C(\mathcal{H}')^*$, the homomorphism $\mathcal{F}^*$ is a map from $C(\mathcal{H}')^*$ to $C(\mathcal{H})^*$. Therefore $\mathcal{F}^*$ is in fact the pullback of $\mathcal{F}$. Here we write the elements of $C(\mathcal{H})$ on the left hand side and the elements of $C(\mathcal{H})^*$ on the right hand side. The operations on $C(\mathcal{H}')$ or $C(\mathcal{H}')^*$ are distinguished by attaching $'$ to them. The above definitions of the operations on $C(\mathcal{H})^*$

translate various conditions for the operations on $C(\mathcal{H})$ into those on $C(\mathcal{H})^*$ as follows. The coassociativity of $\triangle$ is equivalent to the associativity of m :

$$\begin{array}{rcl} \langle m(m(a\otimes b)\otimes c)|g\rangle & = & \langle a\otimes b\otimes c|(\triangle\otimes\mathbf{1})\triangle(g)\rangle \\ & & \parallel \\ \langle m(a\otimes m(b\otimes c))|g\rangle & = & \langle a\otimes b\otimes c|(\mathbf{1}\otimes\triangle)\triangle g\rangle \end{array} . \tag{3.6}$$

The condition that $\mathfrak{m}$ is the coderivation is translated into the Leibniz rule for δ:

$$\begin{array}{rcl} \langle \delta\cdot m(a\otimes b)|g\rangle & = & \langle a\otimes b|\triangle\cdot\mathfrak{m}(g)\rangle \\ & & \parallel \\ \langle m(\delta\otimes\mathbf{1}+\mathbf{1}\otimes\delta)(a\otimes b)|g\rangle & = & \langle a\otimes b|(\mathfrak{m}\otimes\mathbf{1}+\mathbf{1}\otimes\mathfrak{m})\triangle g\rangle \end{array} . \tag{3.7}$$

The condition that $\mathcal{F} : C(\mathcal{H}) \to C(\mathcal{H}')$ is a cohomomorphism is rewritten as the one that $\mathcal{F}^* : C(\mathcal{H}')^* \to C(\mathcal{H})^*$ is a homomorphism:

$$\begin{array}{rcl} \langle \mathcal{F}^*\cdot m'(a\otimes b)|g\rangle & = & \langle a\otimes b|\triangle'\cdot\mathcal{F}(g)\rangle \\ & & \parallel \\ \langle m(\mathcal{F}^*(a)\otimes\mathcal{F}^*(b))|g\rangle & = & \langle a\otimes b|(\mathcal{F}\otimes\mathcal{F})\triangle g\rangle \end{array} . \tag{3.8}$$

$(\mathcal{H},\mathfrak{m})$ is an A_∞-algebra means that $(C(\mathcal{H})^*,\delta)$ is a complex on the dual:

$$0=\langle\delta\cdot\delta(a)|g\rangle=\langle a|\mathfrak{m}\cdot\mathfrak{m}(g)\rangle=0\ . \tag{3.9}$$

Finally the condition that $\mathcal{F}$ is an A_∞-morphism is translated into the equivariance of $\mathcal{F}^*$:

$$\begin{array}{rcl} \langle \delta\cdot\mathcal{F}^*(a)|g\rangle & = & \langle a|\mathcal{F}\cdot\mathfrak{m}(g)\rangle \\ & & \parallel \\ \langle \mathcal{F}^*\cdot\delta'(a)|g\rangle & = & \langle a|\mathfrak{m}'\cdot\mathcal{F}(g)\rangle \end{array} . \tag{3.10}$$

The above statement will be realized with some graphs. [11] In the above explanation, the elements of $C(\mathcal{H})$ are written in the left hand side of the pairings (ket), and the elements of the dual algebra $C(\mathcal{H})^*$ are in the right hand side (bra). Here, for the algebra on the left hand side, we represent the product m, the derivation δ, and the homomorphism $\mathcal{F}^*$ as $m = \supset\!\!-$, $\delta = -\boxed{\delta}-$, $\mathcal{F}^* = -\boxed{\mathcal{F}^*}-$. According to the operations of the algebra from left, the lines of the graphs are connected to the right direction. In other words, the operations on the algebra $C(\mathcal{H})^*$ in the left hand side from left yields the flow from the left to the right on the lines of the graphs. Next, for the coalgebra $C(\mathcal{H})$ in the right hand side, we represent the coproduct $\triangle$, the coderivation $\mathfrak{m}$, and the cohomomorphism $\mathcal{F}$ as $\triangle = \supset\!\!-$, $\mathfrak{m} = -\boxed{\mathfrak{m}}-$, $\mathcal{F} = -\boxed{\mathcal{F}}-$, and define the orientation of the operation from the right to the left on the lines of the graphs. Lastly, in order to distinguish the left and right in the pairings, we introduce $\langle\ |\ \rangle$ between the algebra $C(\mathcal{H})$ and the coalgebra $C(\mathcal{H})^*$.

The definition of the algebra $C(\mathcal{H})^*$ dual to the coalgebra $C(\mathcal{H})$ (3.3)(3.4)(3.5) are written graphically as follows. The graphs in both sides of the equations represent the $\mathbb{C}$ valued

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[11] The graphs used below is different from that in the body of this paper. In fact, a line denotes the flow of an element of $\mathcal{H}$ in the body of this paper, but the line used below denotes an element of $C(\mathcal{H})$.

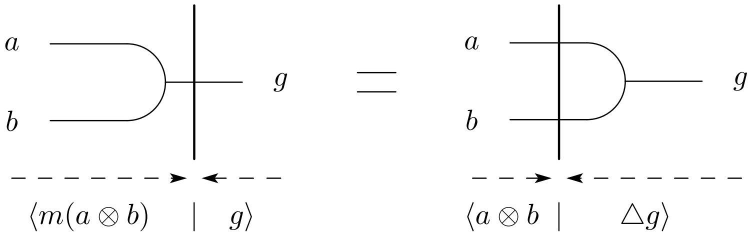

Figure 7: $\langle m(a\otimes b)|g\rangle = \langle a\otimes b|\triangle g\rangle$ eq.(3.3)

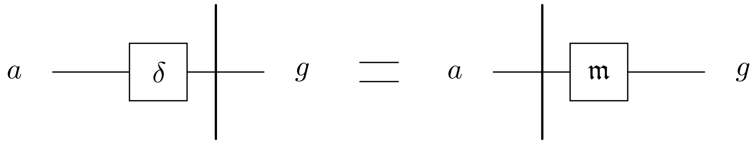

Figure 8: $\langle \delta(a)|g\rangle = \langle a|\mathfrak{m}(g)\rangle$ eq.(3.4)

pairings. The arrow on the dashed line in Figure 7 denotes the orientation of the operations in both sides. The m is defined so that the pairing is invariant when the | on the right hand side of Figure 7 is moved to the left. Then the $\supset\!-$ is m on the left of |, and it becomes $\triangle$ on the right of |. Similarly, in Figure 8, the $-\!\boxed{\delta}\!-$ on the left of | becomes $-\!\boxed{\mathfrak{m}}\!-$ on the right and the $-\!\boxed{\mathfrak{m}}\!-$ is $-\!\boxed{\delta}\!-$ when is transferred to the left. The situation is similar for $-\!\boxed{\mathcal{F}^*}\!-$ and $-\!\boxed{\mathcal{F}}\!-$ (Figure 9).

From the rewriting above, the following dualities can be understood naturally by using graphs; $\mathfrak{m}$ is a coderivation vs. the δ is a derivation (3.7), the $\mathcal{F}$ is a cohomomorphism vs. the $\mathcal{F}^*$ is a homomorphism (3.8), the $(\mathcal{H},\mathfrak{m})$ is an A_∞-algebra vs. the $(C(\mathcal{H})^*,\delta)$ is a complex(3.9), and the $\mathcal{F}$ is an A_∞-morphism vs. the $\mathcal{F}^*$ is δ-equivariant (3.10). For instance, eq.(3.8) is shown as Figure 10.

3.2 Formal noncommutative supermanifolds

In this subsection, we represent explicitly m, δ and $\mathcal{F}^*$, which correspond to $\triangle$, $\mathfrak{m}$ and $\mathcal{F}$, respectively, and realize them geometrically on the algebra $C(\mathcal{H})^*$ dual to the $C(\mathcal{H})$. For the coassociative coproduct

$$\triangle(\mathbf{e}_1\cdots\mathbf{e}_n)=\sum_{k=1}^{n-1}(\mathbf{e}_1\cdots\mathbf{e}_k)\otimes(\mathbf{e}_{k+1}\cdots\mathbf{e}_n)\ ,$$

the corresponding associative product m defined in eq.(3.3) are written as

$$m((\mathbf{e}^{i_k}\cdots\mathbf{e}^{i_1})\otimes(\mathbf{e}^{j_l}\cdots\mathbf{e}^{j_1}))=\mathbf{e}^{j_l}\cdots\mathbf{e}^{j_1}\mathbf{e}^{i_k}\cdots\mathbf{e}^{i_1}\ . \tag{3.11}$$

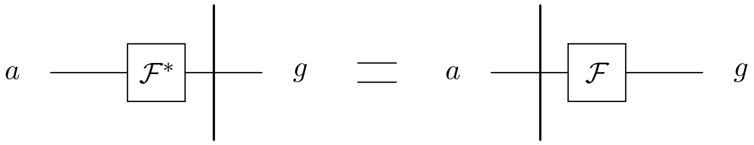

Figure 9: $\langle \mathcal{F}^*(a)|g\rangle = \langle a|\mathcal{F}(g)\rangle$ eq.(3.5)

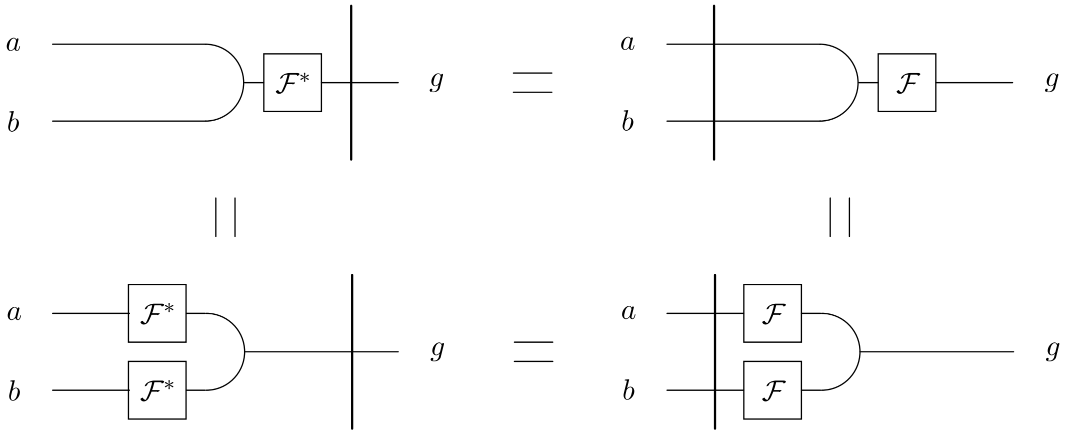

Figure 10: $\langle \mathcal{F}^* \cdot m(a \otimes b)|g\rangle = \langle m(\mathcal{F}^*(a) \otimes \mathcal{F}^*(b))|g\rangle$ eq.(3.8)

For $a = \sum_{k=1}^{\infty} a_{i_1\cdots i_k}\mathbf{e}^{i_k}\cdots\mathbf{e}^{i_1}$ and $b = \sum_{l=1}^{\infty} b_{j_1\cdots j_l}\mathbf{e}^{j_l}\cdots\mathbf{e}^{j_1}$, $m(a\otimes b)$ becomes

$$m((\sum_{k=1}^{\infty} a_{i_1\cdots i_k}\mathbf{e}^{i_k}\cdots\mathbf{e}^{i_1}) \otimes (\sum_{l=1}^{\infty} b_{j_1\cdots j_l}\mathbf{e}^{j_l}\cdots\mathbf{e}^{j_1})) = \sum_n (a\cdot b)_{m_1\cdots m_n}\mathbf{e}^{m_n}\cdots\mathbf{e}^{m_1}$$

$$(a\cdot b)_{m_1\cdots m_n} = \sum_{p=1}^{n-1} \epsilon_{m_1\cdots m_n}^{i_1\cdots i_p j_1\cdots j_{n-p}} a_{i_1\cdots i_p} b_{j_1\cdots j_{n-p}} \ .$$

It is easily seen that by the above definition of m, $(a\cdot b)_{m_1\cdots m_n} = \langle m(a\otimes b)|\mathbf{e}_{m_1}\cdots\mathbf{e}_{m_n}\rangle = \langle a\otimes b|\triangle(\mathbf{e}_{m_1}\cdots\mathbf{e}_{m_n})\rangle$ holds.

$a, b \in C(\mathcal{H})^*$ can be regarded as the polynomial functions on the graded vector space $\mathcal{H}$. The dual basis $\{\mathbf{e}^i\}$ is thought of as the coordinates (coordinate functions) of vector space $\mathcal{H}$, though it is graded and noncommutative when the product structure is also considered. Hereafter we change the notation and denote $\mathbf{e}^i$ by ϕ^i. $C(\mathcal{H})^*$ is also replaced by $C(\phi)$ and its element is represented as

$$a(\phi) = \sum_{k=1}^{\infty} a_{i_1\cdots i_k}\phi^{i_k}\cdots\phi^{i_1} \ .$$

The coordinate functions $\{\phi^i\}$ will then be treated as fields in the sense of field theory. The pair of $\mathbb{Z}$-graded vector space $\mathcal{H}$ and the algebra of formal power series of the coordinates $C(\phi)$ on $\mathcal{H}$ is called *formal supermanifold* [3, 65]. We call so, though this may be an infinitesimal neighborhood or germ of a more general global supermanifold. Though usually the term 'super' indicates $\mathbb{Z}_2$-

graded, we use it for $\mathbb{Z}$-graded object. The term 'formal' is that in formal power series or in formal geometry. In our situation, the coordinates are associative but noncommutative, so this is a formal noncommutative supermanifold. On the other hand, the dual picture of an L_∞-algebra is described by a formal (commutative) supermanifold. Namely, all the arguments in this paper can be translated into those for L_∞-algebras by imposing the graded commutativity upon $\{\phi^i\}$ corresponding to the graded commutativity in the L_∞-algebra (Remark 2.9).

- *coderivation*

Next, for a coderivation $\mathfrak{m} = \mathfrak{m}_1 + \mathfrak{m}_2 + \cdots$,

$$\mathfrak{m}_k(\mathbf{e}_1 \cdots \mathbf{e}_n) = \sum_{p=1}^{n-k} (-1)^{\mathbf{e}_1+\cdots+\mathbf{e}_p} \mathbf{e}_1 \cdots \mathbf{e}_{p-1} m_k(\mathbf{e}_p, \cdots, \mathbf{e}_{p+k-1}) \mathbf{e}_{p+k} \cdots \mathbf{e}_n \ , \quad \mathbf{e}_i \in \mathcal{H} \ , \tag{3.12}$$

we construct δ which is the dual to $\mathfrak{m}$. By the definition of δ (3.4), one sees that a derivation corresponding to the coderivation may be constructed separately for k. Let us express $m_k : \mathcal{H}^{\otimes k} \to \mathcal{H}$ as

$$m_k(\mathbf{e}_{i_1}, \cdots, \mathbf{e}_{i_k}) = \mathbf{e}_j c^j_{i_1 \cdots i_k} \ , \qquad c^j_{i_1 \cdots i_k} \in \mathbb{C} \tag{3.13}$$

and then $\delta_k : C(\mathcal{H})^* \to C(\mathcal{H})^*$,

$$\delta_k = \frac{\overleftarrow{\partial}}{\partial \phi^j} c^j_{i_1 \cdots i_k} \phi^{i_k} \cdots \phi^{i_1}$$

is a derivation. Here we identify the span of the coordinate $\{\phi^i\}$ with $\mathcal{H}^*$ and replace $\mathbf{e}^i$ to ϕ^i. The derivation δ is constructed as

$$\delta = \delta_1 + \delta_2 + \cdots = \sum_{k=1}^{\infty} \frac{\overleftarrow{\partial}}{\partial \phi^j} c^j_{i_1 \cdots i_k} \phi^{i_k} \cdots \phi^{i_1} \ . \tag{3.14}$$

It is regarded as an (odd) *formal vector field* on the formal noncommutative pointed supermanifold. The formal manifold with such δ is called *Q-manifold* in [3]. [12] Note that the condition that $\mathfrak{m}_k$ is a coderivation is replaced to that δ_k satisfies the Leibniz rule on the polynomials of ϕ^i's. For instance the operation of δ on $(\phi^3\phi^2\phi^1)$ is defined as

$$\delta_k(\phi^3\phi^2\phi^1) = \phi^3\phi^2 c^1_{i_1\cdots i_k}\phi^{i_k}\cdots\phi^{i_1} + (-1)^{\mathbf{e}_1}\phi^3 c^2_{i_1\cdots i_k}\phi^{i_k}\cdots\phi^{i_1}\phi^1 + (-1)^{\mathbf{e}_1+\mathbf{e}_2} c^3_{i_1\cdots i_k}\phi^{i_k}\cdots\phi^{i_1}\phi^2\phi^1 \ .$$

The sign arises when the δ with degree one passes through some elements which have their degree.

In field theories equipped with a classical BV-structure, this δ is just the BV-BRST transformation as seen in subsection 6.1. Note that in this case $\{\phi^i\}$ consists of both fields and antifields.

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[12] This Q does not correspond to the BRST operator Q in the body of this paper but δ. δ in this paper is written as Q in [3].

Remark 3.1 When $\mathfrak{m}$ satisfies $\mathfrak{m}\cdot\mathfrak{m}=0$, we have relations between $\mathfrak{m}_k$. Rewriting m_k using eq.(3.13) yields relations between $c^j_{i_1\cdots i_k}$. On the other hand, in the dual language, the condition $\mathfrak{m}\cdot\mathfrak{m}=0$ is equivalent to $\delta\cdot\delta=0$. Calculating $\delta\cdot\delta$ and concentrating on the terms of n powers of ϕ^i lead to

$$\begin{aligned}\sum_{k+l=n+1}\delta_l\cdot\delta_k &= \left(\frac{\overleftarrow{\partial}}{\partial\phi^i}c^i_{i_1\cdots i_k}\phi^{i_k}\cdots\phi^{i_1}\right)\frac{\overleftarrow{\partial}}{\partial\phi^j}c^j_{j_1\cdots j_l}\phi^{j_l}\cdots\phi^{j_1}\\ &=\frac{\overleftarrow{\partial}}{\partial\phi^i}\sum_{k+l=n+1}\sum_{m=1}^{k}(-1)^{\mathbf{e}_{i_1}+\cdots+\mathbf{e}_{i_{m-1}}}c^i_{i_1\cdots i_k}c^{i_m}_{j_1\cdots j_l}\phi^{i_k}\cdots\phi^{i_{m+1}}\left(\phi^{j_l}\cdots\phi^{j_1}\right)\phi^{i_{m-1}}\cdots\phi^{i_1}\ .\end{aligned}$$

The coefficient of $\phi^n\cdots\phi^1$ then reads

$$0=\sum_{\substack{k+l=n+1\\ m=0,\cdots,k-1}}(-1)^{\mathbf{e}_1+\cdots+\mathbf{e}_m}c^i_{1\cdots m,i_m,m+l+1\cdots n}c^{i_m}_{m+1\cdots m+l}\ . \tag{3.15}$$

This is exactly the relation $\mathfrak{m}\cdot\mathfrak{m}=0$ (or eq.(2.6)) rewritten with $\{c^i_{i_1\cdots i_k}\}$.

- *cohomomorphism*

In the terminology of the formal supermanifold, a homomorphism corresponding to a cohomomorphism $\mathcal{F}$ are constructed as follows. For two graded vector spaces $\mathcal{H},\mathcal{H}'$ and a cohomomorphism $\mathcal{F}:C(\mathcal{H})\to C(\mathcal{H}')$ defined in eq.(2.5), let us now express f_n as

$$f_n(\mathbf{e}_{i_1},\cdots,\mathbf{e}_{i_n})=\mathbf{e}_{j'}f^{j'}_{i_1\cdots i_n}\ ,\qquad f^{j'}_{i_1\cdots i_n}\in\mathbb{C}$$

for $n\geq 0$. The homomorphism $\mathcal{F}^*$ actually gives the pullback from $C(\mathcal{H}')^*$, the formal power series ring on $\mathcal{H}'$, to $C(\mathcal{H})^*$. Moreover $\mathcal{F}^*:C(\mathcal{H}')^*\to C(\mathcal{H})^*$ is induced from $\mathcal{F}_*$ below

$$\begin{array}{cccc}\mathcal{F}_*: & \mathcal{H} & \to & \mathcal{H}'\\ & \phi & \mapsto & \phi'=\mathcal{F}_*(\phi)\end{array}\ ,\quad \phi^{j'}=\mathcal{F}^{j'}_*(\phi)=f^{j'}+f^{j'}_i\phi^i+f^{j'}_{i_1i_2}\phi^{i_2}\phi^{i_1}+\cdots+f^{j'}_{i_1\cdots i_n}\phi^{i_n}\cdots\phi^{i_1}+\cdots\ , \tag{3.16}$$

where $\{\phi^i\}$ and $\{\phi^{i'}\}$ are the coordinates on $\mathcal{H}$ and $\mathcal{H}'$, respectively. Namely, for an element $a(\phi'):=\sum_{k=1}^{\infty}a_{i'_1\cdots i'_k}\phi^{i'_k}\cdots\phi^{i'_1}\in C(\mathcal{H}')^*$, $\mathcal{F}^*(a(\phi'))=a(\mathcal{F}_*(\phi))$ holds. One can see that the cohomomorphism $\mathcal{F}$ is, in the dual geometric picture, a nonlinear map $\mathcal{F}_*$ from a formal supermanifold $\mathcal{H}$ to $\mathcal{H}'$. If $f^{j'}=0$, the $\mathcal{F}_*$ preserves the origin.

- *A_∞-morphism*

The condition that this $\mathcal{F}$ is an A_∞-morphism is equivalent to the statement that this map $\mathcal{F}_*$ between two formal supermanifolds is compatible with the actions of δ and δ' on both sides, *i.e.* $\mathcal{F}_*$ is a morphism between Q-manifolds. For any $a(\phi')\in C(\mathcal{H}')^*$, the condition is

$$\mathcal{F}^*\delta'(a(\phi'))=\delta\mathcal{F}^*a(\phi')\ , \tag{3.17}$$

and is written explicitly as

$$\mathcal{F}^*\left(a(\phi')\frac{\overleftarrow{\partial}}{\partial\phi^{j'}}c^{j'}(\phi')\right)=a(\mathcal{F}_*(\phi))\frac{\overleftarrow{\partial}}{\partial\phi^j}c^j(\phi)\ ,$$

where we expressed $\delta = \frac{\overleftarrow{\partial}}{\partial \phi^j} c^j(\phi)$. Because $a(\mathcal{F}_*(\phi))\frac{\overleftarrow{\partial}}{\partial \phi^j} c^j(\phi) = \mathcal{F}^* \left(a(\phi')\frac{\overleftarrow{\partial}}{\partial \phi^{j'}} \right) \frac{\phi^{j'}\overleftarrow{\partial}}{\partial \phi^j} c^j(\phi)$ in the right hand side, we get

$$\mathcal{F}^* \left(c^{j'}(\phi') \right) = \frac{\phi^{j'}\overleftarrow{\partial}}{\partial \phi^j} c^j(\phi) \ . \tag{3.18}$$

We can see that when δ and $\mathcal{F}^*$ are given and $\mathcal{F}^*$ has its inverse, then δ' is induced as $c^{j'}(\phi') = (\mathcal{F}^{-1})^* \left(\frac{\phi^{j'}\overleftarrow{\partial}}{\partial \phi^j} c^j(\phi) \right)$. This is the dual description of $\mathfrak{m}' = \mathcal{F}\mathfrak{m}\mathcal{F}^{-1}$ when $\mathcal{F}$ is an A_∞-isomorphism.

The dual description of cyclicity will be discussed in the next section.

3.3 Superfield and mixed description

Definition 3.2 (superfield) For basis $\{\mathbf{e}_i\}$ of $\mathcal{H}$ and the dual basis $\{\phi^i\}$, we define $\Phi := \mathbf{e}_i \phi^i \in \mathcal{H} \otimes \mathcal{H}^*$ and call it the *superfield*.

Since the degree of ϕ^i is minus the degree of $\mathbf{e}_i$, the superfield Φ has degree zero. The roles are in fact similar to those of superfield in supersymmetric field theory, though some more extended notions are included. Note that the term 'superfield' does not mean that Φ is a field (function) on our supermanifold. In the case of string field theory it is called the string field.

It is useful to incorporate the coalgebra description and its dual. The multilinear map $m_k : \mathcal{H}^{\otimes k} \to \mathcal{H}$ is extended to operation on $\Phi^{\otimes k}$ as

$$m_k(\Phi, \cdots, \Phi) = \mathbf{e}_j c^j_{i_1 \cdots i_k} \phi^{i_k} \cdots \phi^{i_1} \ .$$

In other words, $m_k(\Phi, \cdots, \Phi) \in \mathcal{H} \otimes (\mathcal{H}^*)^{\otimes k}$ is defined by this equation. Since $\mathbf{e}_j$ is identified with $\frac{\overleftarrow{\partial}}{\partial \phi^j}$, $m_k(\Phi, \cdots, \Phi)$ is identified with δ_k. Similarly, for multilinear map $f_k : \mathcal{H}^{\otimes k} \to \mathcal{H}'$ which defines a cohomomorphism or A_∞-morphism $\mathcal{F}$, its operation on superfields is defined by

$$f_k(\Phi, \cdots, \Phi) = e_{j'} f^{j'}_{i_1 \cdots i_k} \phi^{i_k} \cdots \phi^{i_1} \ .$$

$\mathcal{F}_*$ in eq.(3.16) is then represented by

$$\Phi' = \mathcal{F}_*(\Phi) = f_1(\Phi) + f_2(\Phi, \Phi) + f_3(\Phi, \Phi, \Phi) + \cdots \ .$$

Polynomial functions $C(\phi)$ also have superfield description. For $a_{i_1 \cdots i_k} \phi^{i_k} \cdots \phi^{i_1} \in C(\phi)$, let us define a multilinear map $a_k : \mathcal{H}^{\otimes k} \to \mathbb{C}$ such that $a_k(\mathbf{e}_{i_1}, \cdots, \mathbf{e}_{i_k}) = a_{i_1 \cdots i_k}$. Any element in $C(\phi)$ is then expressed as

$$a_k(\Phi, \cdots, \Phi) = a_{i_1 \cdots i_k} \phi^{i_k} \cdots \phi^{i_1} \ .$$

In the next section we shall also consider cyclic functions, which is defined by imposing cyclic condition upon $a_{i_1 \cdots i_k} \in \mathbb{C}$ (Definition 4.1). Namely, they are described by a cyclic multilinear map $a_k : \mathcal{H}^{\otimes k} \to \mathbb{C}$.

Thus, superfield Φ can be thought of as an expression of general element in $\mathcal{H}$ and also play the role of complete system of $\mathcal{H}$. One can see that in this superfield description one can discuss the dual side without the indices.

Instead of superfield $\Phi \in \mathcal{H}\otimes\mathcal{H}^*$, one can consider more generally $\mathcal{H}$ over associative graded algebra $C(\tilde{\phi})$ generated freely by $\{\tilde{\phi}^i\}$, though we do not use it later in this paper. An element of the $C(\tilde{\phi})$-module $\mathcal{H}$ is represented as $A(\tilde{\phi}) = \mathbf{e}_i A^i(\tilde{\phi})$ where $A^i(\tilde{\phi}) \in C(\tilde{\phi})$. The degree of $A(\tilde{\phi})$ is just the sum $\deg(\mathbf{e}_i) + \deg(A^i(\tilde{\phi}))$. $C(\mathcal{H})$ is then extended to $C(\mathcal{H}\otimes C(\tilde{\phi}))$. The coproduct is defined just in a similar way as that on $C(\mathcal{H})$. Furthermore, the operations of any coderivation $\mathfrak{m} : C(\mathcal{H}) \to C(\mathcal{H})$ and any cohomomorphism $\mathcal{F} : C(\mathcal{H}) \to C(\mathcal{H}')$ are naturally extended to operations on $C(\tilde{\phi})$-module. It is convenient to rewrite each element in $C(\mathcal{H}\otimes C(\tilde{\phi}))$ into that of the form in $C(\mathcal{H})\otimes C(\tilde{\phi})$ as follows,

$$\mathbf{e}_{i_1}A_1^{i_1}(\tilde{\phi})\cdots\mathbf{e}_{i_n}A_n^{i_n}(\tilde{\phi}) = (-1)^{\sum_{k=1}^{n-1} A_k^{i_k}(\sum_{l=k+1}^{n}(A_l^{i_l}+\mathbf{e}_{i_l}))}\mathbf{e}_{i_1}\cdots\mathbf{e}_{i_n}(A_n^{i_n}(\tilde{\phi})\cdots A_1^{i_1}(\tilde{\phi}))\ ,$$

where $A_k^{i_k}$ in the sign factor in the left hand side indicates $\deg(A_k^{i_k}(\tilde{\phi}))$. The sign factor is determined as the Kostul sign that arises when $A_{i_k}^{i_k}$ passes through $\mathbf{e}_{i_{k+1}}\cdots\mathbf{e}_{i_n}A_{i_n}^{i_n}\cdots A_{i_{k+1}}^{i_{k+1}}$ for $k = n-1, n-2, \cdots, 1$. The expression of the right hand side enables us to extend any operation on $C(\mathcal{H})$ to that on $C(\mathcal{H}\otimes C(\tilde{\phi}))$. Though we related elements in $C(\mathcal{H}\otimes C(\tilde{\phi}))$ to that in $C(\mathcal{H})\otimes C(\tilde{\phi})$, we should not consider all elements in $C(\mathcal{H})\otimes C(\tilde{\phi})$. Only those that comes from $C(\mathcal{H}\otimes C(\tilde{\phi}))$ have a tree structure and can have an A_∞-structure, and so on.

4 Odd symplectic geometry on formal noncommutative supermanifolds

A constant symplectic structure of a cyclic A_∞-algebra is a constant symplectic structure on a formal noncommutative supermanifold. Such noncommutative symplectic geometry is discussed in [61]. In this section we shall discuss nonconstant symplectic structures. They can be defined naturally from the physics of open strings. After considering the constant ones in subsection 4.1, we shall first define general nonconstant ones in subsection 4.2. Such geometry is discussed by A. Schwarz [92] in case of graded commutative supermanifolds, where the existence of the Darboux theorem is known. In [92] the body (even coordinates parts) of the supermanifolds are treated global. However, in noncommutative case, local properties have never been discussed enough. Therefore we shall discuss the local properties of nonconstant symplectic and Poisson structures. We show a key lemma (Lemma 4.8), the Poincaré's lemma on the formal noncommutative supermanifolds. Due to the lemma, we examine the properties of symplectic diffeomorphisms in subsection 4.3, and show the Darboux theorem on the formal noncommutative supermanifolds (Theorem 4.15) in subsection 4.4. The study of the formal noncommutative symplectic supergeometry is directly related to the notion of cyclic A_∞-algebras. We look back over cyclic A_∞-algebras from these dual pictures in subsection 4.5.

4.1 The constant symplectic structures

Consider a $\mathbb{Z}$-graded vector space $\mathcal{H}$ equipped with a constant odd symplectic structure $\omega(\mathbf{e}_i,\mathbf{e}_j) = \omega_{ij}$ in Definition 2.10. It defines a symplectic structure on formal noncommutative supermanifold of $\mathcal{H}$. Its inverse is defined by

$$\omega_{ij}\omega^{jk} = \omega^{kj}\omega_{ji} = \delta_i^k$$

and it defines an odd Poisson structure as seen below. The graded antisymmetry implies

$$\omega_{ji} = -\omega_{ij}\ , \qquad \omega^{ji} = -\omega^{ij} = -(-1)^{(i+1)(j+1)}\omega^{ij}\ .$$

As above, in this section we often denote $\deg(\mathbf{e}_i)$ simply by i instead of $\mathbf{e}_i$. The existence of the nondegenerate symplectic structure depends on the structure of $\mathcal{H}$ but is natural in the context of the BV-formalism in field theory.

Next, we define an algebra of functions to construct an odd Poisson algebra. In the previous section we considered an associative noncommutative polynomial algebra $C(\mathcal{H})^* = C(\phi)$ on a formal noncommutative supermanifold. In this section, we need to consider additional structure on $C(\phi)$. The idea of the following definition is that explained in subsection 1.5.

Definition 4.1 (Function on a formal noncommutative supermanifold) For ϕ^i the dual coordinate of $\mathbf{e}_i$, let us consider an element $\frac{1}{k}a_{i_1\cdots i_k}\phi^{i_k}\cdots\phi^{i_1} \in C(\phi)$ whose coefficient $a_{i_1\cdots i_k}$ is cyclic,

$$a_{i_1\cdots i_k} = (-1)^{(i_k+\cdots+i_2)i_1}a_{i_2\cdots i_k i_1}\ .$$

Such elements have the following property

$$\begin{aligned}\frac{1}{k}a_{i_1\cdots i_k}\phi^{i_k}\cdots\phi^{i_1} &= \frac{1}{k}a_{i_2\cdots i_k i_1}\phi^{i_1}\phi^{i_k}\cdots\phi^{i_2}\\ &= (-1)^{(i_k+\cdots+i_2)i_1}\frac{1}{k}a_{i_1\cdots i_k}\phi^{i_1}\phi^{i_k}\cdots\phi^{i_2}\ .\end{aligned}$$

That is, the coordinates ϕ^i are regarded cyclic. We denote by $C(\phi)_c$ the subgroup of $C(\phi)$. We also consider the free tensor algebra of $C(\phi)_c$ and denote it by $TC(\phi)_c$. The free tensor product is defined to be graded commutative and denoted by $\bullet$. For instance for $A, B \in C(\phi)_c$, $B \bullet A = (-1)^{AB} A \bullet B$ holds.

The cyclic symmetry for $C(\phi)_c$ is just the property of open string disk as explained in subsection 1.5. This is similar to a 'trace' in the terminology of noncommutative geometry [20]. The restriction $C(\phi) \to C(\phi)_c$ (given by cyclic symmetrization) is regarded as a trace which keeps all other informations of $C(\phi)$. Considering $TC(\phi)_c$ are also natural from the viewpoints of the BV-formalism. As an element in $C(\phi)_c$ is regarded as an S^1 as in eq.(1.15), an element in $TC(\phi)_c$ can be characterized by a multiple copy of S^1. However we do not need $TC(\phi)_c$ essentially in this paper. The essential necessity of it is related to 'homotopy algebras of quantum open strings' if it would be defined.

Definition 4.2 (Odd Poisson structure (Gerstenhaber structure)) A *Gerstenhaber algebra* is an algebra equipped with degree zero associative product $\bullet$ and degree one bracket $(\ ,\ )$ satisfying the following equations:

(a) $(B, A) = -(-1)^{(A+1)(B+1)}(A, B)$,

(b) $(A, B \bullet C) = (A, B) \bullet C + (-1)^{(A+1)B} B \bullet (A, C)$,

(c) $(-1)^{(A+1)(C+1)}((A, B), C) + \textit{cyclic} = 0$

where A, B and C are elements of the algebra. A, B and C on (-1) denote the degree of A, B and C, respectively.

An odd Poisson algebra can then be constructed from the constant symplectic structure in Definition 2.10. The odd Poisson bracket can be defined so that it fits the picture in eq.(1.16).

Definition 4.3 (Constant Poisson structure) On $TC(\phi)_c$ let us consider a Poisson bracket written as follows,

$$(\ , \ ) = \frac{\overleftarrow{\partial}}{\partial\phi^i}\omega^{ij}\frac{\overrightarrow{\partial}}{\partial\phi^j} \ . \tag{4.1}$$

For two elements of $C(\phi)_c$, the bracket is defined explicitly as

$$\begin{aligned}
&\frac{1}{k}\mathcal{V}_{i_1\cdots i_k}\phi^{i_k}\cdots\phi^{i_1}\frac{\overleftarrow{\partial}}{\partial\phi^i}\omega^{ij}\frac{\overrightarrow{\partial}}{\partial\phi^j}\frac{1}{l}\mathcal{V}_{j_1\cdots j_l}\phi^{j_l}\cdots\phi^{j_1}\\
&:= \frac{1}{k+l-2}\mathcal{V}_{i_1\cdots i_k}\omega^{i_1 j_l}\mathcal{V}_{j_1\cdots j_l}\left(\phi^{i_k}\cdots\phi^{i_2}\phi^{j_{l-1}}\cdots\phi^{j_1} + cyclic\right)\\
&= \frac{1}{k+l-2}\left(\mathcal{V}_{ii_l\cdots i_{k+l-2}}\omega^{ij}\mathcal{V}_{i_1\cdots i_{l-1}j} + cyclic\right)\phi^{i_{k+l-2}}\cdots\phi^{i_1} \ .
\end{aligned}$$

In the second equality, the '*cyclic*' means that $\phi^{i_k}\cdots\phi^{i_2}\phi^{j_{l-1}}\cdots\phi^{j_1}$ is moved cyclic such as

$$\begin{aligned}
&\phi^{i_k}\cdots\phi^{i_2}\phi^{j_{l-1}}\cdots\phi^{j_1}\\
&\longrightarrow (-1)^{(i_k+\cdots+i_2+j_{l-1}+\cdots+j_2)j_1}\phi^{j_1}\phi^{i_k}\cdots\phi^{i_2}\phi^{j_{l-1}}\cdots\phi^{j_2}\\
&\longrightarrow \cdots
\end{aligned}$$

and these $(k+l-2)$ terms are then summed up in $(\cdots)$. In the third line, '*cyclic*' indicates that $i_1\cdots i_{k+l-2}$ is moved cyclic with appropriate sign and the resulting $(k+l-2)$ terms are then summed up. More explicitly, for $b_{i_1\cdots i_{k+l-2}} := \mathcal{V}_{ii_l\cdots i_{k+l-2}}\omega^{ij}\mathcal{V}_{i_1\cdots i_{l-1}j} \in \mathbb{C}$, the coefficient of $\phi^{i_{k+l-2}}\cdots\phi^{i_1}$ in the third line is written as

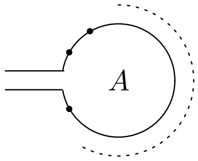
$$\frac{1}{k+l-2}\sum_{k'=1}^{k+l-2}(-1)^{(i_1+\cdots+i_{k'-1})(i_{k'}+\cdots+i_{k+l-2})}b_{i_{k'}\cdots i_{k+l-2}\cdot i_1\cdots i_{k'-1}} \ .$$

By definition, this coefficient is cyclic with respect to the indices $i_1\cdots i_{k+l-2}$.

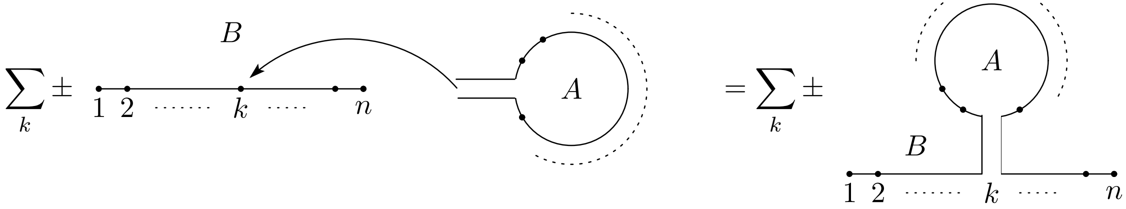
The cyclicity is a natural property of open string, where operators $\mathbf{e}_{i_1}\cdots\mathbf{e}_{i_{k+l-2}}$ are inserted on a boundary of an open string disk S^1.

As in commutative case, this constant Poisson bracket actually satisfies the Jacobi identity. It follows from Lemma 4.12, but it can also be shown by writing pictures as in subsection 1.5. Consequently, $(C(\phi)_c, (\ , \ ))$ forms a Lie algebra. Moreover we can extend the bracket on $TC(\phi)_c$ naturally by using Definition 4.2 (b). Thus, $(TC(\phi)_c, (\ , \ ))$ forms a Gerstenhaber algebra.

Remark 4.4 (Additional structure) Note that the property of Definition 4.2(b) holds also for $A \in C(\phi)_c$ and $B, C \in C(\phi)$ with the replacement of the product $\bullet$ with the usual associative product in $C(\phi)$. This implies that $(A, \ )$ and also $(\ , A)$ act as a derivation on $C(\phi)$. Pictorially this fact can be seen as follows. For instance for $B \in C(\phi)$ and $A \in C(\phi)_c$, the operation of $(\ , A)$ is figured as

$(\ , A) =$ 

where the double line  corresponds to ω^{ij}. (B, A) is then expressed as



.

Namely, the interval of B becomes also an interval after the insertion of S^1 of A.

We can also extend ω^{ij} to be nonconstant and define a natural class of nonconstant Poisson structures on a formal noncommutative supermanifold. They are defined as closed two-forms on the formal noncommutative supermanifold as seen in the next subsection.

4.2 The symplectic and Poisson structures

We can extend the constant odd Poisson structure to nonconstant ones.

Definition 4.5 (Covariant odd Poisson bracket) On $TC(\phi)_c$ we define the following degree one nondegenerate bracket

$$(A, B) = \sum_{ij,IJ} (-1)^{(B-j)J} \omega^{ij}_{JI} \left(\frac{A\overleftarrow{\partial}}{\partial \phi^i} \phi^I \frac{\overrightarrow{\partial} B}{\partial \phi^j} \phi^J \right)_c \ , \tag{4.2}$$

where I, J are multi-indices for polynomials of coordinates ϕ^i and $_c$ denotes the operation of cyclic symmetrization as in Definition 4.3. The nondegeneracy is equivalent to the nondegeneracy of $\omega_{ij,\emptyset\emptyset}$. The coefficients are required to satisfy

$$\omega^{ji}_{IJ} = -(-1)^{(i+1)(j+1)+IJ} \omega^{ij}_{JI}$$

so that the bracket has the property of Definition 4.2 (a). We then call eq.(4.2) an odd Poisson bracket if it satisfies the Jacobi identity (Definition 4.2 (c)).

The geometric picture of this definition is figured in eq.(1.17).

We would like to know the condition that the Jacobi identity holds. We shall discuss it by translating these Poisson structure into symplectic side. For considering symplectic geometry on formal noncommutative supermanifolds, we need vector fields, differential forms, and so on.

Definition 4.6 (Hamiltonian vector field) Using the properties of derivation in Definition 4.2 (b) and Remark 4.4, we define a Hamiltonian vector field. For a Hamiltonian $B \in C(\phi)_c$, we denote the corresponding Hamiltonian vector field by

$$\delta_B = (\ , B) \ .$$

Note that δ_B has degree $B+1$. The operation of $\delta_B : C(\phi) \to C(\phi)$ is defined by

$$\begin{aligned}&\delta_B \left(a_{i_1 \cdots i_n} \phi^{i_n} \cdots \phi^{i_1} \right) \\ &\quad = \sum_{j=1}^{n} a_{i_1 \cdots i_{j-1} j i_{j+1} \cdots i_n} (-1)^{(B+1)(i_1+\cdots+i_{j-1})} \phi^{i_n} \cdots \phi^{j+1} \left((-1)^{(B-k)J} \omega^{jk}_{JI} \phi^I \frac{\overrightarrow{\partial} B}{\partial \phi^k} \phi^J \right) \phi^{j-1} \cdots \phi^{i_1}\end{aligned}$$

for $a_{i_1 \cdots i_n} \phi^{i_n} \cdots \phi^{i_1} \in C(\phi)$. One can see that this is a natural extension of the derivation in Remark 4.4. The operation $\delta_B : C(\phi)_c \to C(\phi)_c$ is then defined by cyclic symmetrizing the equation above. Namely, one obtains

$$\delta_B A := (A, B)$$

for $A \in C(\phi)_c$.

Definition 4.7 (Exterior derivative and differential form) We first extend each coordinate ϕ^i by exterior derivative d as

$$0 \mapsto \phi^i \mapsto d\phi^i \mapsto 0 \ .$$

For d we introduce another degree $\sharp$. We assign $\sharp(d) = 1$. We denote by φ the coordinate ϕ or its exterior derivative $d\phi$. A differential form is then defined of the form

$$a(\varphi) := \frac{1}{k} a_{\mu_1 \cdots \mu_k} \varphi^{\mu_k} \cdots \varphi^{\mu_1} \ .$$

The space of differential forms is then denoted by $C(\varphi)$. Denote $\sharp(\varphi_\mu) =: \sharp_\mu$ $(= 0$ or $1)$ and then the degree of differential form is simply the sum

$$\sharp(\varphi^{\mu_k} \cdots \varphi^{\mu_1}) = \sharp_{\mu_k} + \cdots + \sharp_{\mu_1} \ .$$

The space of l-th differential forms is denoted by $C(\varphi)^{\langle l \rangle}$. $C(\varphi)$ is then the direct sum of $C(\varphi)^{\langle l \rangle}$ for $l \geq 0$. We further consider cyclic elements of $C(\varphi)$, which we denote by $C(\varphi)_c$. Suppose that, for $a(\varphi) \in C(\varphi)$ above, φ^{μ_k} is ϕ^{i_k} or $d\phi^{i_k}$. We then say $a(\varphi) \in C(\varphi)$ is cyclic, $a(\varphi) \in C(\varphi)_c$, iff the coefficient satisfies

$$a_{\mu_1 \cdots \mu_k} = (-1)^{\mathbf{e}_{i_1}(\mathbf{e}_{i_2}+\cdots+\mathbf{e}_{i_k}) + \sharp_{\mu_1}(\sharp_{\mu_2}+\cdots+\sharp_{\mu_k})} a_{\mu_2 \cdots \mu_k \mu_1} \ .$$

Namely, we count the degree for $\mathcal{H}$ and that for differential forms independently. The space of cyclic l-th differential forms is denoted by $C(\varphi)_c^{\langle l \rangle}$. The action of exterior derivative is naturally extended to that on $C(\varphi)^{\langle l \rangle}$ and $C(\varphi)_c^{\langle l \rangle}$. Especially, $d : C(\varphi)_c^{\langle l \rangle} \to C(\varphi)_c^{\langle l+1 \rangle}$ is given by

$$da(\varphi) = \frac{1}{k} \sum_{i=1}^{k} (-1)^{\sharp_{\mu_1}+\cdots+\sharp_{\mu_{i-1}}} a_{\mu_1 \cdots \mu_k} \varphi^{\mu_k} \cdots d\varphi^{\mu_i} \cdots \varphi^{\mu_1} \ . \tag{4.3}$$

Note that $d\varphi^{\mu_i} = 0$ iff $\sharp_{\mu_i} = 1$. In the expression above it is clear that $(d)^2 = 0$. Thus a complex $(C(\varphi)_c^{\langle \cdot \rangle}, d)$ is given. We also consider $(TC(\varphi)_c^{\langle \cdot \rangle}, d)$ with a natural extension.

One can write eq.(4.3) in a cyclic expression as

$$da(\varphi)=\left(\frac{1}{k}\sum_{i=1}^{k}(-1)^{\sharp_{\mu_1}+\cdots+\sharp_{\mu_{i-1}}}a_{\mu_1\cdots\check{\mu}_i\cdots\mu_k}\right)\varphi^{\mu_k}\cdots\varphi^{\mu_1}\ .$$

Here $\check{\mu}_i$ denotes the index corresponding to $d^{-1}\varphi_{\mu_i}$. That is, $a_{\mu_1\cdots\check{\mu}_i\cdots\mu_k}=0$ if $\sharp_{\mu_i}=0$. Thus d can be regarded as operation on the coefficient such as $(da)_{\mu_1\cdots\mu_k}=\sum_{i=1}^{k}(-1)^{\sharp_{\mu_1}+\cdots+\sharp_{\mu_{i-1}}}a_{\mu_1\cdots\check{\mu}_i\cdots\mu_k}$. In this expression the complex is realized as an analogue of Cech cohomology. Note that in noncommutative case $C(\varphi)^{\langle l\rangle}$ and $C(\varphi)_c^{\langle l\rangle}$ do not vanish trivially for any $l\geq 0$ even if the supermanifold is finite dimensional.

Lemma 4.8 (Poincaré's lemma) *The cohomology with respect to d is trivial.*

proof. One can construct a homotopy operator $d^{-1}:C(\varphi)_c^{\langle l+1\rangle}\to C(\varphi)_c^{\langle l\rangle}$ which satisfies

$$dd^{-1}+d^{-1}d=\mathrm{Id}\ .$$

It is constructed explicitly as

$$d^{-1}a(\varphi)=\frac{1}{k}\sum_{i=1}^{k}(-1)^{\sharp_{\mu_1}+\cdots+\sharp_{\mu_{i-1}}}a_{\mu_1\cdots\mu_k}\varphi^{\mu_k}\cdots(d^{-1}\varphi^{\mu_i})\cdots\varphi^{\mu_1}\ .$$

This fact completes the proof. ■

The local triviality of the deRham complex implies that this noncommutative geometry provides some natural extension of usual commutative geometry. We shall use this fact for later subsections. We shall see that the symplectic diffeomorphism discussed in the next subsection is related to the first deRham cohomology, whereas, the Darboux theorem in subsection 4.4 is equivalent to the triviality of the second deRham cohomology.

Remark 4.9 (Compatibility with transformations) Let $(\mathcal{H},TC(\phi)_c)$ and $(\mathcal{H}',TC(\phi')_c)$ be two formal noncommutative supermanifolds and $\mathcal{F}^*:TC(\phi')_c\to TC(\phi)_c$ a pullback induced by a map $\phi^{i'}=f(\phi)=f^{i'}_{j}+f^{i'}_{j}\phi^{j}+f^{i'}_{j_1j_2}\phi^{j_2}\phi^{j_1}+\cdots$. For $a(\phi')\in C(\phi')_c$, $\mathcal{F}^*(a(\phi))$ is written explicitly as

$$\mathcal{F}^*(a(\phi'))=(\ a(f(\phi))\ )_c$$

where $_c$ is the operation of cyclic symmetrization as in Definition 4.3. An exterior derivative d on $TC(\varphi')_c$ is related to that on $TC(\varphi)_c$ by

$$\mathcal{F}^*(d\phi^{i'})=d\phi^{i}\frac{\overrightarrow{\partial}f(\phi)}{\partial\phi^{i}}\quad(=\frac{f(\phi)\overleftarrow{\partial}}{\partial\phi^{i}}d\phi^{i})\ .$$

By using this relation, $\mathcal{F}^*:TC(\phi')_c\to TC(\phi)_c$ can be extended naturally to $\mathcal{F}^*:TC(\varphi')_c\to TC(\varphi)$. $\mathcal{F}^*:(TC(\varphi')_c,d)\to(TC(\varphi)_c,d)$ is then compatible with the exterior derivatives on both sides. That is,

$$\mathcal{F}^*d=d\mathcal{F}^*\ .$$

Definition 4.10 (Odd symplectic structure) The *odd symplectic structure* on a formal noncommutative supermanifold is defined by degree minus one closed 2-form in $C(\varphi)_c$. [13] We represent it as

$$\omega=\sum_{ij,IJ}\omega_{ji,JI}\left(\phi^I d\phi^i\phi^J d\phi^j\right)_c\ ,$$

where I and J are multi-indices. Note that, by the definition of $C(\varphi)_c$, the polynomial that consists of ϕ's and $d\phi$'s are cyclic. We then denote by $_c$ that $\phi^I d\phi^i\phi^J d\phi^j$ is defined to be cyclic symmetrized. From this, the coefficient satisfies

$$\omega_{ij,IJ}=-\omega_{ji,JI}\ .$$

In order to relate the odd Poisson structure and the odd symplectic structure, we need to define the contraction of forms with vector fields.

Definition 4.11 (Contraction) For an element in $C(\varphi)^k$, the contraction with k vector fields are defined by

$$\begin{aligned}&\phi^{I_1}d\phi^{i_1}\phi^{I_2}d\phi^{i_2}\cdots d\phi^{i_k}\phi^{I_{k+1}}(\delta_{A_1},\cdots,\delta_{A_k})\\ &\quad\phi^{I_1}(-1)^{(A_1+1)(I_2+i_2+\cdots+i_k+I_{k+1})}(d\phi^{i_1},\delta_{A_1})\phi^{I_2}(-1)^{(A_2+1)(I_3\cdots+i_k+I_{k+1})}(d\phi^{i_2},\delta_{A_2})\\ &\qquad\qquad\cdots\cdots(-1)^{(A_k+1)I_{k+1}}(d\phi^{i_k},\delta_{A_k})\phi^{I_{k+1}}\ .\end{aligned}$$

When we write $\delta_A=\frac{\overleftarrow{\partial}}{\partial\phi^i}A^i(\phi)$, $(d\phi^i,\delta_A)=A^i(\phi)$.

The contraction of an element in $C(\varphi)_c^{\langle k\rangle}$ with k vector fields follows from the formula above. Any element in $C(\varphi)_c^{\langle k\rangle}$ is written in the form $a(\varphi):=a_{I_k,i_k,\cdots,I_1,i_1}\phi^{I_1}d\phi^{i_1}\phi^{I_2}d\phi^{i_2}\cdots d\phi^{i_k}$ where $a_{I_k,i_k,\cdots,I_1,i_1}\in\mathbb{C}$ has a condition of cyclicity. The contraction with k vector fields is then given by

$$a(\varphi)(\delta_{A_1},\cdots,\delta_{A_k})=a_{I_k,i_k,\cdots,I_1,i_1}\left(\phi^{I_1}d\phi^{i_1}\phi^{I_2}d\phi^{i_2}\cdots d\phi^{i_k}(\delta_{A_1},\cdots,\delta_{A_k})\right)_c\ .$$

One can see that the definition of contraction is well-defined, that is, the cyclicity of $C(\varphi)_c$ is compatible with the cyclicity of $C(\phi)_c$. By the cyclic permutation of k vector fields, one gets

$$a(\varphi)(\delta_{A_2},\cdots,\delta_{A_k},\delta_{A_1})=(-1)^{k-1}(-1)^{(A_1+1)(\sum_{l=2}^k(A_l+1))}a(\varphi)(\delta_{A_1},\cdots,\delta_{A_k})\ .$$

The Poisson bracket and the symplectic structure is then related to each other by

$$(A,B)=\omega(\delta_A,\delta_B)\ . \tag{4.4}$$

The odd Poisson structure and the odd symplectic structure are inverse to each other. Explicitly in component language, these are related by

$$\sum_{i,I+J=K,I'+J'=K'}\omega_{ji,JI}\omega^{ik}_{J'I'}(-1)^{I'+J'+kJ'}=\sum_{i,I+J=K,I'+J'=K'}\omega^{ki}_{I'J'}\omega_{ij,IJ}(-1)^{iI'}=\delta_{K,\emptyset}\delta_{K',\emptyset}\delta^k_j\ .$$

---

[13] It is not defined as an closed element of $TC(\varphi)_c$. The definition above is natural from the viewpoints of open string physics.

Note that $\omega_{ji,JI}$ is uniquely determined when $\omega^{ik}_{J'I'}$ is given and vice versa. This fact can easily be checked by induction with respect to the powers of ϕ.

Lemma 4.12 *An odd bracket (,) of the form in eq.(4.2) satisfies the Jacobi identity iff the corresponding two-form ω is closed. Namely, (,) is an odd Poisson bracket iff ω is an odd symplectic form.*

proof. It follows from calculating $d\omega(\delta_A, \delta_B, \delta_C)$ for $A, B, C \in C(\phi)_c$ directly and using the correspondence (4.4). ■

Remark 4.13 $\delta_{C(\phi)_c}$ is an algebraic homomorphism between the Gerstenhaber algebra and the Lie super algebra defined as follows.

$$\delta_{(B,A)} = [\delta_A, \delta_B] \ , \qquad [\delta_A, \delta_B] := \delta_A\delta_B - (-1)^{(A+1)(B+1)}\delta_B\delta_A \ .$$

4.3 The symplectic diffeomorphisms and the Hamiltonian flows

It is known in the usual commutative situation that the infinitesimal symplectic diffeomorphisms are generated by Hamiltonian vector fields. Namely, the following holds

$$(A, B) + \delta_\epsilon(A, B) = (A + \delta_\epsilon A, B + \delta_\epsilon B)$$

up to ϵ^2. This fact holds also in the noncommutative situation.

Proposition 4.14 *Any infinitesimal symplectic diffeomorphisms on a formal noncommutative symplectic supermanifold can be expressed as a Hamiltonian vector fields.*

proof. Let us define inner product of $a(\varphi) = a_{I_k,i_k,\cdots,I_1,i_1}\phi^{I_1}d\phi^{i_1}\phi^{I_2}d\phi^{i_2}\cdots d\phi^{i_k} \in C(\varphi)^{\langle k\rangle}_c$ with vector field $\delta_A = \frac{\overleftarrow{\partial}}{\partial\phi^j}A^j(\phi)$ by

$$\iota_{(\delta_A)}a(\varphi) = a_{I_k,i_k,\cdots,I_1,i_1}\phi^{I_1}d\phi^{i_1}\phi^{I_2}d\phi^{i_2}\cdots A^k(\phi) \ .$$

It is shown directly that the infinitesimal transformation of $a(\varphi) \in C(\varphi)^{\langle k\rangle}_c$ by δ_A is then Lie derivative,

$$(d\iota_{\delta_A} + \iota_{\delta_A}d)a(\varphi) \ .$$

Here assume that an infinitesimal transformation $\delta_\epsilon := \frac{\overleftarrow{\partial}}{\partial\phi^i}\epsilon^i(\phi)$ preserves the symplectic form ω,

$$0 = 2(d\iota_{\delta_\epsilon} + \iota_{\delta_\epsilon}d)\omega \ .$$

Because $d\omega = 0$, the second term is dropped out. Thus, the condition of the vector field δ_ϵ to preserve the symplectic form is

$$0 = d\iota_{\delta_\epsilon}\omega = d\left(\sum_{ij,IJ}\omega_{ji,JI}\phi^I d\phi^i\phi^J\epsilon^i(\phi)\right)_c \ .$$

By Lemma 4.8 d-closed forms are d-exact, so $\sum_{ij,IJ}\omega_{ji,JI}\left(\phi^I d\phi^i \phi^J \epsilon^i(\phi)\right)_c$ is written as $d\epsilon$ for some degree one element $\epsilon \in C(\phi)_c$. Namely, for any $A \in C(\phi)_c$

$$\omega(\delta_A, \delta_\epsilon) = (d\epsilon)(\delta_A)$$

holds. Since the left hand side is $\delta_\epsilon A$ and the right hand side is $(\epsilon, A) = (A, \epsilon)$, one gets

$$\delta_\epsilon = (\ ,\epsilon)\ .$$

■

Note that the integral of this infinitesimal symplectic diffeomorphism is written of the form

$$e^{(\ ,\epsilon)} = \mathbf{1} + \sum_{k\geq 1}\frac{1}{k!}(\ ,\epsilon)^k \tag{4.5}$$

where $(\ ,\epsilon)^k$ acts on $A \in TC(\phi)_c$ as $(\cdots((A,\epsilon),\epsilon),\cdots,\epsilon)$. In fact, from the properties of the Poisson bracket, one can check the transformation satisfies

(a) $e^{(\ ,\epsilon)}AB = e^{(\ ,\epsilon)}A \cdot e^{(\ ,\epsilon)}B\ , \qquad A, B \in C(\phi),\ AB \in C(\phi),$

(a') $e^{(\ ,\epsilon)}A \bullet B = e^{(\ ,\epsilon)}A \bullet e^{(\ ,\epsilon)}B\ , \qquad A, B \in TC(\phi)_c,$

(b) $e^{(\ ,\epsilon)}(A,B) = (e^{(\ ,\epsilon)}A, e^{(\ ,\epsilon)}B)\ , \qquad A, B \in TC(\phi)_c\ .$

Condition (a) implies that the finite transformation is a homomorphism induced by a coordinate transformation. Condition (a') and (b) just mean that the transformation preserves the product $\bullet$ and symplectic form, respectively.

4.4 The Darboux theorem for noncommutative odd symplectic structures

Theorem 4.15 *Any symplectic form on a formal noncommutative supermanifold can be transformed to be constant by a coordinate transformation.*

proof. Let ω be any symplectic form on a formal noncommutative supermanifold. We shall consider to transform it to be constant from the lower power of the coordinates ϕ. Suppose now that ω is transformed to be constant up to k powers of ϕ. We then consider the transformation of the form

$$\phi^i \longrightarrow \phi^i + f^i(\phi)\ , \qquad f^i(\phi) := f^i_{i_1\cdots i_{k+1}}\phi^{i_{k+1}}\cdots\phi^{i_1}\ .$$

By this transformation, ω is transformed to

$$\begin{aligned}&\omega_{ji,\emptyset\emptyset}d\phi^i d\phi^j + \sum_{|I+J|=k}\omega_{ji,JI}\left(\phi^I d\phi^i \phi^J d\phi^j\right)_c + \cdots\\ &\qquad\longrightarrow \omega_{ji,\emptyset\emptyset}d\phi^i d\phi^j + \sum_{|I+J|=k}\omega_{ji,JI}\left(\phi^I d\phi^i \phi^J d\phi^j\right)_c + 2d\left(\omega_{ji,\emptyset\emptyset}f^i(\phi)d\phi^j)\right)_c + \cdots\end{aligned}$$

Here note that since ω is closed separately with respect to the powers of ϕ, $\omega_{ji,JI}\phi^I d\phi^i \phi^J d\phi^j$ is closed and furthermore exact due to Lemma 4.8. Therefore it can be canceled by $2d\left(\omega_{ji,\emptyset\emptyset}f^i(\phi)d\phi^j)\right)$

with appropriate f^i since the constant part $\omega_{ji,\emptyset\emptyset}$ is nondegenerate. Thus, ω is transformed to be constant up to $(k+1)$ powers of ϕ. Repeating this process completes the proof. ■

Since the Poincaré's lemma (Lemma 4.8) holds, one can also extend the above results to a noncommutative version of Darboux-Weinstein's theorem [123]. In contrast, one can also prove Proposition 4.14 in the previous subsection by power expansion as in Theorem 4.15.

4.5 Cyclic A_∞-algebras from the dual pictures

Here we shall reconsider the meaning of cyclic A_∞-algebras from the viewpoints of odd symplectic geometry on formal noncommutative supermanifolds.

Let us consider any degree zero function $S \in C(\phi)_c$ which has critical point at the origin $\phi = 0$ on a formal noncommutative supermanifolds. Such a function can be written as

$$S = \frac{1}{2}\mathcal{V}_{i_1 i_2}\phi^{i_2}\phi^{i_1} + \sum_{k\geq 3}\frac{1}{k}\mathcal{V}_{i_1\cdots i_k}\phi^{i_k}\cdots\phi^{i_1} \tag{4.6}$$

where $\mathcal{V}_{i_1\cdots i_k} \in \mathbb{C}$ and $S \in C(\phi)_c$. As seen later, this is just the action of field theory.

For a given covariant odd Poisson structure in Definition 4.5, let us consider the Hamiltonian vector field of S

$$\delta = (\ ,S)\ . \tag{4.7}$$

By definition it has degree one. It is furthermore nilpotent iff S satisfies

$$(S,S) = 0\ , \tag{4.8}$$

where $(\ ,\ )$ is a covariant odd Poisson structure in Definition 4.5. This fact follows from the Jacobi identity of $(\ ,\ )$. Namely, the Hamiltonian vector field δ of S satisfying $(S,S) = 0$ defines an A_∞-structure. We remark that eq.(4.8) corresponds to the classical BV-master equation explained later. Let us call such a triple $(C(\phi)_c, \omega, S)$ or equivalently $(C(\phi)_c, \omega, \delta)$ a *pre cyclic A_∞-supermanifold*, where ω is the (covariant) odd symplectic structure (Definition 4.10) corresponding to the given covariant odd Poisson structure (Definition 4.5).

On the other hand, by Theorem 4.15 one can transform any odd symplectic formal noncommutative supermanifolds to the one where the symplectic structure ω is constant, *i.e.* , a skew symmetric bilinear form on $\mathcal{H}$. In this case, the $(k+1)$-power terms of S just correspond to the A_∞-structure m_k such as

$$c^j_{i_1\cdots i_k} = (-1)^{\mathrm{e}_m}\omega^{jm}\mathcal{V}_{mi_1\cdots i_k}\ , \qquad k \geq 2\ , \tag{4.9}$$

where ω^{jm} is constant. Note that the equation above is equivalent to the one in Remark 2.12. The A_∞-odd vector field is then written as

$$\delta = (\ ,S) = \sum_{k=1}^{\infty}\frac{\overleftarrow{\partial}}{\partial\phi^j}c^j_{i_1\cdots i_k}\phi^{i_k}\cdots\phi^{i_1}\ .$$

Thus, any pre cyclic A_∞-supermanifold $(C(\phi)_c, \omega, S)$ is isomorphic to the one with a constant symplectic structure where the $(k+1)$-power terms of S just correspond to the A_∞-structure

m_k. We call such a pre cyclic A_∞-supermanifold $(C(\phi)_c, \omega, S)$ a *cyclic A_∞-supermanifold.* This is exactly the dual description of the cyclic A_∞-algebra $(\mathcal{H}, \omega, S)$ in Definition 2.11.

Next, let us consider the relation between two pre cyclic A_∞-supermanifolds $(C(\phi)_c, \omega, S)$ and $(C(\phi')_c, \omega', S')$. Suppose that there exists a morphism $\mathcal{F}_*$ (eq.(3.16)) preserving the origins ($f^{j'} = 0$) and the symplectic forms,

$$\mathcal{F}^*\omega' = \omega \ .$$

Proposition 4.16 *The following two statements are equivalent;*

- *$\mathcal{F}$ preserves the value of the action $S = \mathcal{F}^* S'$.*
- *$\mathcal{F} : (\mathcal{H}, \mathfrak{m}) \to (\mathcal{H}', \mathfrak{m}')$ is an A_∞-morphism.*

Note that here f_1 may not be an isomorphism. This equivalence follows from the fact that the symplectic structures on both sides are non-degenerate. We call $\mathcal{F} : (C(\phi)_c, \omega, S) \to (C(\phi')_c, \omega', S')$ preserving the origins, the symplectic forms $\mathcal{F}^*\omega' = \omega$ and the actions $S = \mathcal{F}^* S'$ a *morphism between the (pre) cyclic A_∞-supermanifolds.*

In this situation, both pre cyclic A_∞-supermanifolds $(C(\phi)_c, \omega, S)$ and $(C(\phi')_c, \omega', S')$ are isomorphic to some cyclic A_∞-supermanifolds $(C(\phi)_c, \tilde{\omega}, \tilde{S})$ and $(C(\phi')_c, \tilde{\omega'}, \tilde{S'})$ where $\tilde{\omega}$ and $\tilde{\omega'}$ are constant. Let $\tilde{\mathcal{F}} : (C(\phi)_c, \tilde{\omega}, \tilde{S}) \to (C(\phi)_c, \omega, S)$ and $\tilde{\mathcal{F}'} : (C(\phi')_c, \tilde{\omega'}, \tilde{S'}) \to (C(\phi')_c, \omega', S')$ be two isomorphisms. The composition map $(\tilde{\mathcal{F}'})^{-1}\mathcal{F}\tilde{\mathcal{F}} : (C(\phi)_c, \tilde{\omega}, \tilde{S}) \to (C(\phi')_c, \tilde{\omega'}, \tilde{S'})$ is then an A_∞-morphism preserving the constant symplectic structures. Thus, there exists a functor from the category of pre cyclic A_∞-supermanifolds to the category of cyclic A_∞-supermanifolds.

A morphism between two cyclic A_∞-supermanifolds is in fact the dual description of a cyclic A_∞-morphism in Definition 2.13. For two cyclic A_∞-supermanifolds $(C(\phi)_c, \omega, S)$, $(C(\phi')_c, \omega', S')$ and a morphism $\mathcal{F}_* : (C(\phi)_c, \omega, S) \to (C(\phi'), \omega', S')$, the condition $\mathcal{F}^*\omega' = \omega$ is written as

$$\left(\sum_{k',l'} \omega'_{k'l'} d(\mathcal{F}_*(\phi^{l'})) d(\mathcal{F}_*(\phi^{k'})) \right)_c = \sum_{i,j} \omega'_{ij} d\phi^j d\phi^i \ .$$

Reading the coefficient of each term of polynomial fields separately and dualizing back, one gets $\omega'(f_1(\mathbf{e}_i), f_1(\mathbf{e}_j)) = \omega(\mathbf{e}_i, \mathbf{e}_j)$ (eq.2.10) and

$$\sum_{k,l\geq 1,\ k+l=n} \sum_{cyc(i,j)} \omega'(f_k(\cdots, \mathbf{e}_{j_q}, \mathbf{e}_i, \mathbf{e}_{i_1}, \cdots), f_l(\cdots, \mathbf{e}_{i_p}, \mathbf{e}_j, \mathbf{e}_{j_1}, \cdots)) = 0 \qquad (4.10)$$

for fixed $n \geq 3$, $p \geq 0$ and $q = n - p - 2 \geq 0$, where $cyc(i,j)$ indicates all possible cyclic permutations for $\{\mathbf{e}_i, \mathbf{e}_{i_1}, \cdots, \mathbf{e}_{i_p}, \mathbf{e}_j, \mathbf{e}_{j_1}, \cdots, \mathbf{e}_{j_q}\}$ but keeping $\mathbf{e}_i$ in $f_k(\cdots)$ and $\mathbf{e}_j$ in $f_l(\cdots)$. However, eq.(4.10) includes overlapping equivalent identities. For instance, when we consider the case of order $\{\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_{j_1}, \cdots, \mathbf{e}_{j_{n-2}}\}$, the summation $cyc(i,j)$ drops out and eq.(4.10) simply gives

$$\sum_{k,l\geq 1,\ k+l=n} \omega'(f_k(\mathbf{e}_{j_l}, \cdots, \mathbf{e}_{j_{n-2}}, \mathbf{e}_i), f_l(\mathbf{e}_j, \mathbf{e}_{j_1}, \cdots, \mathbf{e}_{j_{l-1}})) = 0 \ .$$

Namely, the condition (4.10) just reduces to eq.(2.11) and the condition of morphisms between cyclic A_∞-supermanifolds is actually equivalent to the condition of cyclic A_∞-morphisms in Definition 2.13.

5 The minimal model theorem

The following theorem is one of the key theorems in homotopy algebra.

Theorem 5.1 (Minimal model theorem [50]) *Given any A_∞-algebra $(\mathcal{H},\mathfrak{m})$, let $\mathcal{H}^p$ be the cohomology of $\mathcal{H}$ with respect to m_1. Then there necessarily exists an A_∞-algebra $(\mathcal{H}^p,\tilde{\mathfrak{m}}^p)$ and an A_∞-quasi-isomorphism from $(\mathcal{H}^p,\tilde{\mathfrak{m}}^p)$ to $(\mathcal{H},\mathfrak{m})$.*

The purpose of this section is to clarify and to develop basic properties of A_∞-algebras around this theorem. For the construction of minimal models of A_∞-structures, in particular of dgas, various versions of homological perturbation theory (HPT) have been developed, for instance by [35, 40, 36, 37, 38, 45]. On the other hand, it was mentioned in [65] that there exists another stronger version of the minimal model theorem, which we call the decomposition theorem. In subsection 5.1 we shall show explicitly the decomposition theorem (Theorem 5.4). A similar result is obtained independently in [72] in the framework of a closed model category [89]. The decomposition theorem includes the minimal model theorem and implies various basic properties of homotopy algebras. In subsection 5.2 we show the decomposition theorem for cyclic A_∞-algebras. The decomposition theorem guarantees the existence of an inverse A_∞-quasi-isomorphism of an A_∞-quasi-isomorphism (Theorem 5.17) as stated in [65]. We shall explain it in subsection 5.3. Though the minimal model theorem follows from the decomposition theorem, the proof relies on inductive arguments and the form of the minimal model is not explicit. On the other hand, for any A_∞-algebra, its minimal model can be given explicitly and more recently in [66] in terms of some Feynman diagrams. The Feynman diagram expression provides us with intuitive understanding of the minimal model, though it is equivalent to minimal models given by formula as in the traditional homological perturbation theory (see [38, 45, 82]). We demonstrate in subsection 5.4 that it arises naturally from the issue of finding the solutions of the Maurer-Cartan equation for an A_∞-algebra [53]. Subsection 5.5 presents the cyclic A_∞ version, which is directly related to section 6, where we shall show that the minimal cyclic algebra is derived by semiclassical perturbation theory of field theory in the BV-formalism.

5.1 The decomposition theorem for A_∞-algebras

Definition 5.2 (Minimal A_∞-algebra) An A_∞-algebra $(\mathcal{H},\mathfrak{m})$ is called *minimal* if $m_1=0$ on $\mathcal{H}$.

Definition 5.3 An A_∞-algebra $(\mathcal{H},\mathfrak{m})$ is called *linear contractible* if $m_k=0$ for $k\geq 2$ and $Q=m_1$ has trivial cohomology.

The following theorem holds.

Theorem 5.4 (Decomposition theorem for A_∞-algebras) *Any A_∞-algebra is A_∞-isomorphic to the direct sum of a minimal A_∞-algebra and a linear contractible A_∞-algebra.*

This subsection is devoted to proving this theorem based on the strategy presented in [65]. We shall prove it in terms of formal noncommutative supermanifolds. However the result holds even in the case that the dual side is not well-defined in a strict sense, since the proof can be translated into that on the corresponding coalgebra side (see Remark 5.7).

For Q the coboundary operator of the complex $(\mathcal{H}, Q := m_1)$, we first give the analogue of the Hodge-Kodaira decomposition; consider a degree minus one linear map $Q^+ : \mathcal{H} \to \mathcal{H}$ satisfying

$$QQ^+ + Q^+Q + P = \mathbf{1} \tag{5.1}$$

on $\mathcal{H}$ such that $P^2 = P$ and $QP = 0$ hold. Namely, P is the projection onto the cohomology of $(\mathcal{H}, Q)$. By definition $PQ = 0$ and $QQ^+Q = Q$ hold, which lead to $(QQ^+)^2 = QQ^+$, $(Q^+Q)^2 = Q^+Q$ and

$$(QQ^+)(Q^+Q) = (Q^+Q)(QQ^+) = 0\ , \quad P(Q^+Q) = (Q^+Q)P = 0\ , \quad P(QQ^+) = (QQ^+)P = 0\ .$$

In physical terms, QQ^+, Q^+Q and P are the projections onto Q-trivial states, unphysical states and physical states, respectively. We denote $QQ^+ = P^t$ and $Q^+Q = P^u$ and express the decomposition of $\mathcal{H}$ as

$$\mathcal{H} = \mathcal{H}^t \oplus \mathcal{H}^u \oplus \mathcal{H}^p\ , \qquad \mathcal{H}^p := P\mathcal{H}\ , \quad \mathcal{H}^t := P^t\mathcal{H}\ , \quad \mathcal{H}^u := P^u\mathcal{H}\ . \tag{5.2}$$

In other words, we give a splitting of the complex $(\mathcal{H}, Q)$:

$$\begin{tikzcd}(\mathcal{H}, Q) \arrow[r, shift left, "\pi"] & (\mathcal{H}^p, 0) \arrow[l, shift left, "\iota"]\end{tikzcd}$$

with a contracting homotopy $Q^+ : \mathcal{H} \to \mathcal{H}$ such that $\mathbf{1} - P = QQ^+ + Q^+Q$ for $P := \iota \circ \pi$ (we shall sometimes omit $\circ$). Here 0 is the zero differential on $\mathcal{H}^p$. As above, we shall often denote the image of $\mathcal{H}^p$ by ι also by $\mathcal{H}^p$. By definition $P^2 = P$ holds. Also, since $\iota : (\mathcal{H}^p, 0) \to (\mathcal{H}, Q)$ and $\pi : (\mathcal{H}, Q) \to (\mathcal{H}^p, 0)$ are chain maps, $QP = PQ = 0$ hold. Thus, these data give the decomposition (5.2). Note that this set-up is a special case (in particular the differential on $\mathcal{H}^p$ is zero) of the *strong deformation retract* or SDR which is the starting point of various versions of homological perturbation theory (see [35, 40, 36, 37, 38, 45]).

We decompose the dual coordinates ϕ^i of $\mathcal{H}$ into x^i, y^j and p^k, the dual coordinates corresponding to the basis of $\mathcal{H}^t$, $\mathcal{H}^u$ and $\mathcal{H}^p$, respectively.

Our goal for the proof of Theorem 5.4 is to construct a local diffeomorphism around the origin of the formal noncommutative supermanifold such that x^i and y^j span the contractible directions, whereas p^k is a coordinate of the minimal part.

For the first step, we should examine the properties of the cohomology with respect to Q on formal noncommutative supermanifolds.

Definition 5.5 (δ_1-complex) Let $C(\phi)^k$ be the space of associative noncommutative polynomial functions on a formal noncommutative supermanifold of total degree k. $\delta_1 : C(\phi)^k \to C(\phi)^{k+1}$, which is the dual of Q on $\mathcal{H}$, then defines a complex. We denote it by $(C(\phi)^\star, \delta_1)$ and call the δ_1-complex.

Lemma 5.6 (δ_1-cohomology) *Let us decompose the coordinates ϕ^i into x^i, y^j, p^k where x^i, y^j and p^k are the dual coordinates corresponding to the basis of $\mathcal{H}^t$, $\mathcal{H}^u$ and $\mathcal{H}^p$, respectively. For each element $a_{i_1\cdots i_k}\phi^{i_k}\cdots\phi^{i_1}\in C(\phi)^\star$, its cohomology part is represented as*

$$a_{i_1\cdots i_k}p^{i_k}\cdots p^{i_1}=\left(a_{i_1\cdots i_k}\phi^{i_k}\cdots\phi^{i_1}\right)\Big|_{x=y=0}\ . \tag{5.3}$$

We denote by $C(p)^\star$ the space of such elements.

proof. This follows from the fact that one can construct a degree minus one homotopy operator H^* on $C(\phi)^\star$ such that

$$\mathrm{Id}-P=\delta_1 H^*+H^*\delta_1\ ,$$

where P is the projection corresponding to eq.(5.3). This H^* will be constructed explicitly later in Lemma 7.4. [14] ■

One may notice that this corresponds to the dual version (supermanifold description) of so-called the *tensor trick* (see [36, 37, 38, 45]).

proof of Theorem 5.4. The A_∞-odd vector field is then written as

$$\begin{aligned}&\delta=\delta_1+\delta_2+\cdots\ ,\\ &\delta_1=\frac{\overleftarrow{\partial}}{\partial x^i}c^i_j y^j\ ,\ \cdots\ ,\quad \delta_n=\frac{\overleftarrow{\partial}}{\partial\phi^i}c^i_{k_1\cdots k_n}\phi^{k_n}\cdots\phi^{k_1}\end{aligned}$$

for $n\geq 2$. Note that δ_1 acts nontrivially on the contractible part only. Now we would like to bring δ_n, $n\geq 2$, to the form $\frac{\overleftarrow{\partial}}{\partial p^i}c^i_{k_1\cdots k_n}p^{k_n}\cdots p^{k_1}$ by a coordinate transformation. Suppose now that δ_n is such a form for $n\leq l$. By the $(l+1)$ powers-part of the A_∞-condition, we have

$$\frac{\overleftarrow{\partial}}{\partial x^i}\left(c_x{}^i_{j_1\cdots j_{l+1}}\phi^{j_{l+1}}\cdots\phi^{j_1}\right)\frac{\overleftarrow{\partial}}{\partial x^k}c^k_j y^j+\frac{\overleftarrow{\partial}}{\partial x^i}c^i_j c_y{}^j_{j_1\cdots j_{l+1}}\phi^{j_{l+1}}\cdots\phi^{j_1}=0\ , \tag{5.4}$$

$$\frac{\overleftarrow{\partial}}{\partial y^i}\left(c_y{}^i_{j_1\cdots j_{l+1}}\phi^{j_{l+1}}\cdots\phi^{j_1}\right)\frac{\overleftarrow{\partial}}{\partial x^k}c^k_j y^j=0\ , \tag{5.5}$$

$$\frac{\overleftarrow{\partial}}{\partial p^i}\left(c_p{}^i_{j_1\cdots j_{l+1}}\phi^{j_{l+1}}\cdots\phi^{j_1}\right)\frac{\overleftarrow{\partial}}{\partial x^k}c^k_j y^j+\sum_{n\geq 2}\delta_n\delta_{l+2-n}=0\ , \tag{5.6}$$

where $c_x{}^i_{j_1\cdots j_{l+1}}\phi^{j_{l+1}}\cdots\phi^{j_1}\in C(\phi)$ is the coefficient of δ_{l+1} with respect to $\frac{\overleftarrow{\partial}}{\partial x^i}$, and so on. We sometimes ignore the upper indices i and indicate by c_x, c_y and c_p these coefficients in $C(\phi)$. One can see that c_y is δ_1-closed from eq.(5.5). Moreover, eq.(5.4) implies c_y does not have δ_1-cohomology since the first term includes y^j and the second term does not. Therefore c_y is δ_1-exact due to Lemma 5.6. Alternatively, the first and second terms of eq.(5.6) are independent of each other since the first term includes y^j and second term does not. Thus, c_p is δ_1-closed.

We would like to remove c_x, c_y and the δ_1-exact part of c_p. Note that in fact we know a transformation which removes c_x and c_y. It is given by the A_∞-quasi-isomorphism in subsection

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[14] More precisely, we will construct an operator H in Lemma 7.4 and H^* is just the dual of H.

5.4. However it is instructive to construct the transformation inductively without assuming this knowledge.

Consider the following coordinate transformation

$$\phi^i = \phi'^i + f^i(\phi) , \qquad f^i(\phi) = f^i_{k_1 \cdots k_{l+1}} \phi'^{k_{l+1}} \cdots \phi'^{k_1} . \tag{5.7}$$

Its inverse transformation is $\phi'^i = \phi^i - f^i(\phi) + \cdots$. We write the transformation (5.7) separately with respect to x, y and p such as

$$x^i = x'^i + f^i_x(\phi) , \qquad y^j = y'^j + f^j_y(\phi) , \qquad p^k = p'^k + f^k_p(\phi) .$$

δ is then transformed as (see eq.(3.18))

$$\delta \;\longrightarrow\; \delta + \frac{\overleftarrow{\partial}}{\partial x^i} c^i_j f^j_y(\phi) - \frac{\overleftarrow{\partial}}{\partial \phi^k} \left(f^k(\phi) \frac{\overleftarrow{\partial}}{\partial x^i} c^i_j y^j \right) + \cdots .$$

The conditions for the correction terms $f^i(\phi)$ to cancel δ_{l+1} except its minimal part are

$$c^i_j f^j_y(\phi) - f^i_x(\phi) \frac{\overleftarrow{\partial}}{\partial x^k} c^k_j y^j + {c_x}^i_{j_1 \cdots j_{l+1}} \phi^{j_{l+1}} \cdots \phi^{j_1} = 0 , \tag{5.8}$$

$$- f^i_y(\phi) \frac{\overleftarrow{\partial}}{\partial x^i} c^i_j y^j + {c_y}^i_{j_1 \cdots j_{l+1}} \phi^{j_{l+1}} \cdots \phi^{j_1} = 0 , \tag{5.9}$$

$$- f^i_p(\phi) \frac{\overleftarrow{\partial}}{\partial x^i} c^i_j y^j + {c_p}^i_{j_1 \cdots j_{l+1}} \phi^{j_{l+1}} \cdots \phi^{j_1} = (\text{polynomial of } p\text{'s}) . \tag{5.10}$$

We shall perform the transformation by dividing it into two steps. First, find a solution of eq.(5.8) and eq.(5.9) for f_x and f_y which removes c_x and c_y. There exist ambiguities for the solution; one solution is given by $f_x = 0$ and $f^j_y = -\bar{c}^j_k c^k_{x, j_1 \cdots j_{l+1}} \phi^{j_{l+1}} \cdots \phi^{j_1}$. It is clear that it satisfies eq.(5.8). In order to confirm eq.(5.9) one may employ eq.(5.4), an A_∞-condition for δ under the induction hypothesis. Next, c_p is δ_1-closed as stated above and eq.(5.10) implies the δ_1-exact part is removed by f_p. The remaining δ_1-cohomology part of c_p then forms the minimal A_∞-structure. This completes the induction. ■

Remark 5.7 In the proof above, one can see that the coordinate transformation $f^i(\phi)$ was constructed only from Q^+ and $\mathfrak{m}$. Therefore, the proof can be rewritten purely on the coalgebra side and is independent of whether the dual supermanifold description is strictly well-defined or not (see [56]).

Given an A_∞-algebra $(\mathcal{H}, \mathfrak{m})$, let us denote by $(\mathcal{H}_{dc}, \mathfrak{m}_{dc})$ a direct sum of a minimal A_∞-algebra and a linear contractible A_∞-algebra obtained by Theorem 5.4 and $(\mathcal{H}^p_{dc}, \mathfrak{m}^p_{dc})$ its minimal part. One can obtain an A_∞-isomorphism $\mathcal{F}_{dc} : (\mathcal{H}_{dc}, \mathfrak{m}_{dc}) \to (\mathcal{H}, \mathfrak{m})$ by composing inductively A_∞-isomorphisms (5.7). As stated above, the A_∞-isomorphism is not unique so the decomposed model $(\mathcal{H}_{dc}, \mathfrak{m}_{dc})$ is not unique. Moreover, even if $(\mathcal{H}_{dc}, \mathfrak{m}_{dc})$ is fixed, the A_∞-isomorphism $\mathcal{F}_{dc} : (\mathcal{H}_{dc}, \mathfrak{m}_{dc}) \to (\mathcal{H}, \mathfrak{m})$ is not unique since there exists gauge transformations (see Definition 7.9).

In the situation above, both the inclusion $\iota : \mathcal{H}^p_{dc} \to \mathcal{H}_{dc}$ and the projection $\pi : \mathcal{H}_{dc} \to \mathcal{H}^p_{dc}$ such that $\iota \circ \pi = P$ naturally extend to A_∞-quasi-isomorphisms by setting the leading linear maps of the A_∞-quasi-isomorphisms to be ι and π, and their higher multilinear maps zero. We write them also as $\iota : (\mathcal{H}^p_{dc}, \mathfrak{m}^p_{dc}) \to (\mathcal{H}_{dc}, \mathfrak{m}_{dc})$ and $\pi : (\mathcal{H}_{dc}, \mathfrak{m}_{dc}) \to (\mathcal{H}^p_{dc}, \mathfrak{m}^p_{dc})$. Then one can see that $\mathcal{F}_{dc} \circ \iota$ is an A_∞-quasi-isomorphism from $(\mathcal{H}^p_{dc}, \mathfrak{m}^p_{dc})$ to $(\mathcal{H}, \mathfrak{m})$ and an inverse A_∞-quasi-isomorphism is given by $\pi \circ (\mathcal{F}_{dc})^{-1} : (\mathcal{H}, \mathfrak{m}) \to (\mathcal{H}^p_{dc}, \mathfrak{m}^p_{dc})$.

Although the existence of a minimal model for any A_∞-algebra is guaranteed from Theorem 5.4, the minimal model is not unique as stated above. A trivial ambiguity is the one given by an A_∞-isomorphism transforming a minimal A_∞-algebra to another one. On this point, from Theorem 5.4, one can state the following.

Corollary 5.8 *For an A_∞-algebra $(\mathcal{H}, \mathfrak{m})$, suppose that it has a minimal model $(\mathcal{H}^p, \tilde{\mathfrak{m}}^p)$ and an A_∞-quasi-isomorphism $\tilde{\mathcal{F}}^p : (\mathcal{H}^p, \tilde{\mathfrak{m}}^p) \to (\mathcal{H}, \mathfrak{m})$ are given. Then, there exist a decomposed A_∞-algebra $(\mathcal{H}_{dc}, \mathfrak{m}_{dc})$ whose minimal part is the minimal A_∞-algebra $(\mathcal{H}^p, \tilde{\mathfrak{m}}^p)$ and an A_∞-isomorphism $\mathcal{F}_{dc} : (\mathcal{H}_{dc}, \mathfrak{m}_{dc}) \to (\mathcal{H}, \mathfrak{m})$. Equivalently, the A_∞-quasi-isomorphism $\tilde{\mathcal{F}}^p : (\mathcal{H}^p, \tilde{\mathfrak{m}}^p) \to (\mathcal{H}, \mathfrak{m})$ can be described by the composition $\mathcal{F}_{dc} \circ \iota$, where $\iota : (\mathcal{H}^p, \tilde{\mathfrak{m}}^p) \to (\mathcal{H}_{dc}, \mathfrak{m}_{dc})$ is the inclusion.*

Corollary 5.9 *Given a minimal model $(\mathcal{H}^p, \tilde{\mathfrak{m}}^p)$ of an A_∞-algebra $(\mathcal{H}, \mathfrak{m})$ and an A_∞-quasi-isomorphism $\tilde{\mathcal{F}}^p : (\mathcal{H}^p, \tilde{\mathfrak{m}}^p) \to (\mathcal{H}, \mathfrak{m})$, there exists an inverse A_∞-quasi-isomorphism $(\tilde{\mathcal{F}}^p)^{-1} : (\mathcal{H}, \mathfrak{m}) \to (\mathcal{H}^p, \tilde{\mathfrak{m}}^p)$.*

Corollary 5.10 (Uniqueness of minimal A_∞-algebras) *For an A_∞-algebra $(\mathcal{H}, \mathfrak{m})$, its minimal A_∞-algebra is unique up to an isomorphism on $\mathcal{H}^p$.*

proof. These three corollaries are shown at the same time as follows. For an arbitrary decomposed A_∞-algebra $(\mathcal{H}'_{dc}, \mathfrak{m}'_{dc})$ of an A_∞-algebra $(\mathcal{H}, \mathfrak{m})$ and an A_∞-isomorphism $\mathcal{F}'_{dc} : (\mathcal{H}'_{dc}, \mathfrak{m}'_{dc}) \to (\mathcal{H}, \mathfrak{m})$, let us consider the following diagram

$$\begin{tikzcd}
(\mathcal{H}, \mathfrak{m}) \arrow[r, shift left, "(\mathcal{F}'_{dc})^{-1}"] & (\mathcal{H}'_{dc}, \mathfrak{m}'_{dc}) \arrow[l, shift left, "\mathcal{F}'_{dc}"] \arrow[d, shift left, "\pi'"] \\
(\mathcal{H}^p, \tilde{\mathfrak{m}}^p) \arrow[u, "\tilde{\mathcal{F}}^p"] \arrow[r, "\mathcal{F}^p"] & (\mathcal{H}'^p_{dc}, \mathfrak{m}'^p_{dc}) \arrow[u, shift left, "\iota'"]
\end{tikzcd} .$$

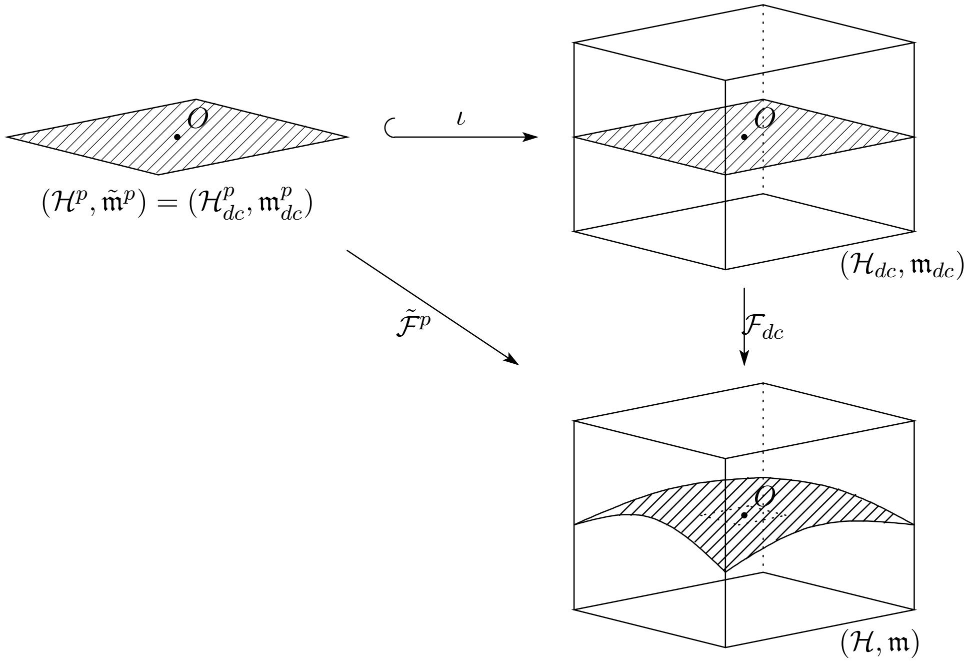
Note that $\mathcal{F}'_{dc}$ has its inverse A_∞-isomorphism $(\mathcal{H}'_{dc})^{-1}$ and ι' has its inverse A_∞-quasi-isomorphism π'. The composition $\pi' \circ (\mathcal{F}'_{dc})^{-1} \circ \tilde{\mathcal{F}}^p$ then gives an A_∞-quasi-isomorphism from $(\mathcal{H}^p, \tilde{\mathfrak{m}}^p)$ to $(\mathcal{H}'^p_{dc}, \mathfrak{m}'^p_{dc})$. Denote this composition map by $\mathcal{F}^p : (\mathcal{H}^p, \tilde{\mathfrak{m}}^p) \to (\mathcal{H}'^p_{dc}, \mathfrak{m}'^p_{dc})$. Since $\mathcal{H}^p = \mathcal{H}'^p_{dc}$, it is not only a quasi-isomorphism but also an isomorphism. On the other hand, one can consider an A_∞-algebra $(\mathcal{H}_{dc}, \mathfrak{m}_{dc})$ which is the direct sum of the minimal A_∞-algebra $(\mathcal{H}^p, \tilde{\mathfrak{m}}^p)$ and the contractible part of $(\mathcal{H}'_{dc}, \mathfrak{m}'_{dc})$. Clearly, there exists an A_∞-isomorphism $\mathcal{F} : (\mathcal{H}_{dc}, \mathfrak{m}_{dc}) \to (\mathcal{H}'_{dc}, \mathfrak{m}'_{dc})$ which is the natural extension of $\mathcal{F}^p$. Thus, the composition of $\iota : (\mathcal{H}^p, \tilde{\mathfrak{m}}^p) \to (\mathcal{H}_{dc}, \mathfrak{m}_{dc})$ with $\mathcal{F}$ yields an A_∞-quasi-isomorphism $\mathcal{F} \circ \iota : (\mathcal{H}^p, \tilde{\mathfrak{m}}^p) \to (\mathcal{H}'_{dc}, \mathfrak{m}'_{dc})$. This leads to Corollary 5.8; $\tilde{\mathcal{F}}^p = \mathcal{F}_{dc} \circ \iota$ where $\mathcal{F}_{dc} := \mathcal{F}'_{dc} \circ \mathcal{F}$.

It is clear that the existence of inverse quasi-isomorphisms (Corollary 5.9) follows from Corollary 5.8. Namely, an inverse quasi-isomorphism of $\mathcal{F}_{dc} \circ \iota$ is given by $\pi \circ \mathcal{F}_{dc}$.

The uniqueness of the minimal model (Corollary 5.10) then follows from Corollary 5.9. Since Corollary 5.9 implies the existence of an A_∞-quasi-isomorphism between any two minimal models and moreover the quasi-isomorphism is an isomorphism, one can see that there exists an A_∞-isomorphism between any two minimal models of an A_∞-algebra $(\mathcal{H}, \mathfrak{m})$. ■

We shall construct a minimal model explicitly by using Feynman graphs in Definition 5.18 in subsection 5.4. Corollary 5.10 then implies that the explicit minimal model is unique up to A_∞-isomorphisms on $\mathcal{H}^p$.

Remark 5.11 (Geometric realization) One can realize the results around the decomposition theorem above geometrically as follows. An A_∞-algebra $(\mathcal{H}, \mathfrak{m})$ is equivalent to a formal noncommutative supermanifold with an A_∞-odd vector field δ. Denote $\delta = \delta_1 + \delta_\bullet$ where δ_1 is the dual of m_1 and $\delta_\bullet$ is the rest. First we fix the basis of the formal noncommutative supermanifold from δ and its homotopy operator (Q^+) and decompose $\mathcal{H} = \mathcal{H}^p \oplus \mathcal{H}^t \oplus \mathcal{H}^u$ as a graded vector space. Theorem 5.4 implies that there exists a nonlinear coordinate transformation so that the vector field $\delta_\bullet$ flows along the $\mathcal{H}^p$ direction and does not depend on $\mathcal{H}^t \oplus \mathcal{H}^u$ direction. This fact guarantees the existence of the minimal model. Moreover for a minimal model $(\mathcal{H}^p, \tilde{\mathfrak{m}}^p)$, an A_∞-quasi-isomorphism $\tilde{\mathcal{F}}^p : (\mathcal{H}^p, \tilde{\mathfrak{m}}^p) \to (\mathcal{H}, \mathfrak{m})$ can be regarded as an embedding of a hypersurface $\mathcal{H}^p$ into $\mathcal{H}$ as in Figure 5.11, though it is a formal noncommutative graded hypersurface. It follows from the condition on the A_∞-morphisms $\tilde{\mathcal{F}}^p \tilde{\mathfrak{m}}^p = \mathfrak{m}\tilde{\mathcal{F}}^p$ that on the hypersurface in $\mathcal{H}$ the A_∞-odd vector field $\delta = \delta_\bullet$ is tangent to it. [15] Corollary 5.8 then implies



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[15] Note that δ_1 vanishes on the hypersurface.

that such an embedding can necessarily be given by the inclusion ι and a nonlinear coordinate transformation $\mathcal{F}_{dc}$ preserving the tangent space at the origin O. Thus, one can see that many ordinary geometric intuitions are valid in formal noncommutative graded situations.

5.2 The decomposition theorem for cyclic A_∞-algebras

In this subsection we prove the decomposition theorem for cyclic A_∞-algebras. For a given cyclic A_∞-algebra $(\mathcal{H},\omega,\mathfrak{m})$, the proof requires a homotopy operator which gives an orthogonal decomposition of $\mathcal{H}$ with respect to the inner product ω. Let us begin with arbitrary homotopy operator Q^+ which defines a Hodge-Kodaira decomposition of $\mathcal{H}$

$$QQ^+ + Q^+Q + P = \mathbf{1}\ .$$

There are ambiguities of the choice of Q^+; $\mathcal{H}^t = QQ^+\mathcal{H}$ is unique, but $\mathcal{H}^p = P\mathcal{H}$ is unique modulo $\mathcal{H}^t$ and $\mathcal{H}^u = Q^+Q\mathcal{H}$ is unique modulo $\mathcal{H}^t \oplus \mathcal{H}^p$. Consider the odd symplectic inner product ω of a cyclic A_∞-algebra

$$\omega_{ij} = \omega(\mathbf{e}_i, \mathbf{e}_j)\ .$$

From the cyclicity $\omega(\mathcal{H}, Q\mathcal{H}) = -\omega(Q\mathcal{H},\mathcal{H})$ holds, which implies the following properties independent of the ambiguities; if $\mathbf{e}_j \in \mathcal{H}^t$ then $\omega_{ij} = 0$ for $\mathbf{e}_i \in \mathcal{H}^t \oplus \mathcal{H}^p$, and if $\mathbf{e}_j \in \mathcal{H}^t \oplus \mathcal{H}^p$ then $\omega_{ij} = 0$ for $\mathbf{e}_i \in \mathcal{H}^t$. If we denote the block element of matrix $\{\omega_{ij}\}$ where $\mathbf{e}_i \in \mathcal{H}^u$ and $\mathbf{e}_j \in \mathcal{H}^p$ as ω_{up} and similar for the other eight block elements, the matrix $\{\omega_{ij}\}$ is represented as the left hand side of eq.(5.11). This implies that by basis transformation corresponding to the ambiguity in Q^+ the inner product ω is decomposed as the right hand side

$$\{\omega_{ij}\} = \begin{pmatrix} \omega_{uu} & \omega_{up} & \omega_{ut} \\ \omega_{pu} & \omega_{pp} & \omega_{pt} \\ \omega_{tu} & \omega_{tp} & \omega_{tt} \end{pmatrix} = \begin{pmatrix} \omega_{uu} & \omega_{up} & \omega_{ut} \\ \omega_{pu} & \omega_{pp} & 0 \\ \omega_{tu} & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 0 & \omega_{ut} \\ 0 & \omega_{pp} & 0 \\ \omega_{tu} & 0 & 0 \end{pmatrix}\ . \tag{5.11}$$

Thus, there exists a homotopy operator Q^+, which defines the state space $\mathcal{H}^u$, so that ω is decomposed orthogonally.

Definition 5.12 (Homotopy operator of Q compatible with ω) Given a cyclic A_∞-algebra $(\mathcal{H},\omega,\mathfrak{m})$, let $QQ^+ + Q^+Q + P = \mathbf{1}$ be a Hodge-Kodaira decomposition of $\mathcal{H}$. We call Q^+ a homotopy operator of Q compatible with ω if $\omega(\mathcal{H}^p,\mathcal{H}^u) = \omega(\mathcal{H}^u,\mathcal{H}^u) = 0$. (When $(\mathcal{H},\omega,\mathfrak{m})$ is a cyclic A_∞-algebra, $\omega(\mathcal{H}^t,\mathcal{H}^p) = \omega(\mathcal{H}^t,\mathcal{H}^t) = 0$ is automatically satisfied.)

Note that a homotopy operator Q^+ of Q compatible with ω satisfies

$$\omega(\mathbf{1}\otimes Q^+) = \omega(Q^+\otimes \mathbf{1})\ , \qquad \omega(\mathbf{1}\otimes P) = \omega(P\otimes \mathbf{1})\ . \tag{5.12}$$

In particular, the first equation follows from $\omega(\mathbf{1}\otimes Q^+) = \omega(QQ^+\otimes Q^+) = \omega(Q^+\otimes QQ^+) = \omega(Q^+\otimes \mathbf{1})$. As stated above, any cyclic A_∞-algebra $(\mathcal{H},\omega,\mathfrak{m})$ has a homotopy operator of Q compatible with ω. It satisfies additional conditions compared with a homotopy operator defining the usual Hodge-Kodaira decomposition; however, it is still not unique. The remaining ambiguity is just related to the choice of the gauge fixing (Definition 6.3) when the propagator (Definition 6.4) is constructed in the BV-formalism.

The properties of the δ_1-complex are similar to those in the previous non-cyclic case.

Definition 5.13 (Cyclic δ_1-complex) Let $C(\phi)^k_c$ be the space of associative noncommutative polynomial cyclic functions on a formal noncommutative supermanifold of total degree k. The action δ_1 on it satisfies the Leibniz rule and so preserves the cyclicity. $\delta_1 : C(\phi)^k_c \to C(\phi)^{k+1}_c$ then defines a complex. We denote it by $(C(\phi)^\star_c, \delta_1)$ and call it the cyclic δ_1-complex.

Lemma 5.14 (Cyclic δ_1-cohomology) *For each element $a_{i_1\cdots i_k}\phi^{i_k}\cdots\phi^{i_1} \in C(\phi)^\star_c$, its cohomology part is represented as*

$$a_{i_1\cdots i_k}p^{i_k}\cdots p^{i_1} = \left(a_{i_1\cdots i_k}\phi^{i_k}\cdots\phi^{i_1}\right)\Big|_{x=y=0} .$$

We denote by $C(p)^\star_c$ the space of such elements.

proof. The corresponding homotopy operator on $C(\phi)^\star_c$ can be obtained by the cyclic symmetrization of the homotopy operator in Lemma 5.6. ■

Theorem 5.15 (Decomposition theorem for cyclic A_∞-algebra) *Any cyclic A_∞-algebra is cyclic A_∞-isomorphic to the direct sum of a minimal cyclic A_∞-algebra and a linear contractible cyclic A_∞-algebra.*

Note that this implies the symplectic form is also decomposed into the direct sum.

proof. Let us represent the A_∞-odd vector field of a cyclic A_∞ algebra $(\mathcal{H}, \omega, S)$ as

$$\delta = (\ , S) , \qquad S = S_2 + S_3 + \cdots , \qquad S_k = \frac{1}{k}\mathcal{V}_{i_1\cdots i_k}\phi^{i_k}\cdots\phi^{i_1} .$$

As in the proof of Theorem 5.4, the strategy of the proof is to construct a cyclic A_∞-isomorphism so that the induced cyclic A_∞-structure $(\mathcal{H}, \omega, S')$ is of the form $S' = S_2 + S'_3 + S'_4 + \cdots$ with $S'_k \in C(p)^\star_c$ for $k \geq 3$. We choose the coordinates of linear contractible direction as x and y, and minimal direction as p. Suppose that S_k belongs to $C(p)^k_c \subset C(\phi)^k_c$ up to l, that is, $S_k = \frac{1}{k}\mathcal{V}_{i_1\cdots i_k}p^{i_k}\cdots p^{i_1}$ for $3 \leq k \leq l$. The $(l+1)$ powers-part of the A_∞-condition $\frac{1}{2}(S,S) = 0$ implies

$$0 = S_{l+1}\frac{\overleftarrow{\partial}}{\partial x^i}c^i_j y^j + \frac{1}{2}\sum_{k\geq 3}(S_k, S_{l+3-k}) .$$

The first term and second term are independent because the first one includes y^j and the second one does not. Thus, S_{l+1} is δ_1-closed. Recall that by Proposition 4.14 any coordinate transformation preserving a constant symplectic structure can be written in the form in eq.(4.5). The transformation

$$S \longrightarrow e^{(\ ,\epsilon(\phi))}S(\phi) = S(\phi) - \epsilon(\phi)\frac{\overleftarrow{\partial}}{\partial x^i}c^i_j y^j + \cdots , \qquad \epsilon(\phi) = \epsilon_{i_1\cdots i_{l+1}}\phi^{i_{l+1}}\cdots\phi^{i_1}$$

can cancel the exact part of S_{l+1} by an appropriate $\epsilon(\phi)$ and we can transform S_{l+1} to be an element of $C(p)_c^{l+1}$. Thus the statement of this theorem can be proved by induction. ■

This procedure is similar to homological perturbation theory in [44].

Note that this result is stronger than Theorem 5.4. Namely, it claims that any cyclic A_∞-algebra can be transformed to the direct sum of a minimal and a contractible one with the cyclicity (or equivalently the constant symplectic structure) being preserved.

Corollary 5.16 *For a cyclic A_∞-quasi-isomorphism $\mathcal{F}^p$ from a minimal cyclic A_∞-algebra to another cyclic A_∞-algebra, there exists an inverse cyclic A_∞-quasi-isomorphism.*

proof. The proof is just the same as for Corollary 5.9.

5.3 Existence of the inverse of A_∞-quasi-isomorphisms

The decomposition theorem in the previous subsections gives clear understanding for the properties of homotopy algebras. One usage is to prove the following.

Theorem 5.17 (Existence of the inverse quasi-isomorphism) *Let $(\mathcal{H},\mathfrak{m})$ and $(\mathcal{H}',\mathfrak{m}')$ be two A_∞-algebras and assume that an A_∞-quasi-isomorphism $\mathcal{F}$ from $(\mathcal{H},\mathfrak{m})$ to $(\mathcal{H}',\mathfrak{m}')$ is given. Then there exists an inverse A_∞-quasi-isomorphism $(\mathcal{F})^{-1}:(\mathcal{H}',\mathfrak{m}')\to(\mathcal{H},\mathfrak{m})$.*

The outline of Theorem 5.17 is presented by M. Kontsevich in [65] for L_∞-case. The author noticed the necessity of the decomposition theorem, instead of minimal model theorem, for the proof of Theorem 5.17 by M. Akaho [2].

proof. First, we transform both A_∞-algebras $(\mathcal{H},\mathfrak{m})$ and $(\mathcal{H}',\mathfrak{m}')$ to their minimal models $(\mathcal{H}^p,\tilde{\mathfrak{m}}^p)$ and $(\mathcal{H}'^p,\tilde{\mathfrak{m}'}^p)$ by A_∞-quasi-isomorphisms $\tilde{\mathcal{F}}^p$ and $\tilde{\mathcal{F}'}^p$ in Theorem 5.4. $\tilde{\mathcal{F}}^p$ and $\tilde{\mathcal{F}'}^p$ have their inverse quasi-isomorphisms, and the A_∞-quasi-isomorphism $\mathcal{F}^p$ from $(\mathcal{H}^p,\tilde{\mathfrak{m}}^p)$ to $(\mathcal{H}'^p,\tilde{\mathfrak{m}'}^p)$ is then given by the composition $(\tilde{\mathcal{F}'}^p)^{-1}\circ\mathcal{F}\circ\tilde{\mathcal{F}}^p$

$$\begin{tikzcd}
(\mathcal{H},\mathfrak{m}) \arrow[r, "\mathcal{F}"] \arrow[d, shift left, "(\tilde{\mathcal{F}}^p)^{-1}"] & (\mathcal{H}',\mathfrak{m}') \arrow[d, shift left, "(\tilde{\mathcal{F}'}^p)^{-1}"] \\
(\mathcal{H}^p,\tilde{\mathfrak{m}}^p) \arrow[u, shift left, "\tilde{\mathcal{F}}^p"] \arrow[r, shift left, "\mathcal{F}^p"] & (\mathcal{H}'^p,\tilde{\mathfrak{m}'}^p) \arrow[u, shift left, "\tilde{\mathcal{F}'}^p"] \arrow[l, shift left, "(\mathcal{F}^p)^{-1}"]
\end{tikzcd}$$

so that the diagram commutes. Because the quasi-isomorphism $\mathcal{F}^p$ is isomorphism, it has its inverse, and one can obtain an A_∞-quasi-isomorphism as $\tilde{\mathcal{F}}^p\circ(\mathcal{F}^p)^{-1}\circ(\tilde{\mathcal{F}'}^p)^{-1}$. ■

In the situation $(\mathcal{H},\mathfrak{m})=(\mathcal{H}^p,\tilde{\mathfrak{m}}^p)$, $(\mathcal{H}',\mathfrak{m}')=(\mathcal{H},\mathfrak{m})$ and $\mathcal{F}'=\tilde{\mathcal{F}}^p$ the statement of this theorem reduces to Corollary 5.9.

It is clear that this theorem holds also for cyclic A_∞-algebras.

5.4 Maurer-Cartan equations, Feynman graphs and the minimal model theorem

For a given A_∞-algebra $(\mathcal{H},\mathfrak{m})$, the existence of its minimal A_∞-algebra was shown in subsection 5.1. The existence was proved inductively and an explicit form of the minimal A_∞-algebra is unclear. In [66] (see also [38, 45, 82] and [46] for L_∞ case) the explicit form is presented by using Feynman graphs. In this subsection we shall discuss the 'meaning' of the minimal A_∞-algebra given explicitly in [66].

Here we construct the A_∞-morphism $\{\tilde{f}^p_k\}$ and A_∞-structure $\{\tilde{m}^p_k\}$ with $k \geq 2$ naturally as the problem of finding the solutions for the Maurer-Cartan equation. This explanation of the minimal model theorem is inspired by a lecture by K. Fukaya [23] (see [24]). We shall then prove that the $(\mathcal{H}^p, \tilde{\mathfrak{m}}^p)$ and $\tilde{\mathcal{F}}$ are indeed an A_∞-algebra and an A_∞-quasi-isomorphism, respectively, for the sake of completeness. The procedure of finding the solution is quite natural and standard, and so similar procedures can be found in various problems. In the context of cyclic A_∞/L_∞-algebras for open/closed string field theory, the Maurer-Cartan equations are the equations of motions of the actions, and the procedure relates to the way of finding some classical solutions (L_∞ case [84, 71]) or constructing the tachyon potential (A_∞ case [83], see also [53]).

Consider solving the Maurer-Cartan equation (MC-eq.) for an A_∞-algebra:

$$\sum_{k\geq 1} m_k(\Phi) = 0 \ . \tag{5.13}$$

Hereafter we often use a shorthand notation $m_k(\Phi)$ for $m_k(\Phi, \cdots, \Phi)$ as above. This is an extended MC-eq. of that in Definition 2.15 in the sense that here we take $\Phi = e_i\phi^i \in \mathcal{H}\otimes\mathcal{H}^*$ the superfield in Definition 3.2, that is, we include $\mathbf{e}_i$ of any degree and then ϕ^i is the formal noncommutative coordinate. As in the previous subsections, consider the Hodge-Kodaira decomposition

$$QQ^+ + Q^+Q + P = \mathbf{1} \qquad (Q := m_1)$$

and decompose $\mathcal{H}$ as $\mathcal{H} = \mathcal{H}^t \oplus \mathcal{H}^u \oplus \mathcal{H}^p$ where $\mathcal{H}^p := P\mathcal{H}$, $\mathcal{H}^t := QQ^+\mathcal{H}$ and $\mathcal{H}^u := Q^+Q\mathcal{H}$. As seen in subsection 6.1, in the case of field theory, Q^+ can be regarded as the propagator and also plays the role of the gauge fixing. P is then the projection onto the physical states. [16]

Here we assume that Φ is sufficiently 'small'; smaller than the radius of convergence, or equivalently Φ is assumed to be multiplied by the corresponding small parameter $\hbar << 1$, or instead the $\hbar$ is treated as a formal parameter. Then the solution is almost the solution of $Q(\Phi) = 0$, in the sense that $Q(\Phi) \sim \mathcal{O}(\hbar^2)$. Since the solutions for eq.(5.13) are preserved under the gauge transformation $\delta_\alpha\Phi = Q(\alpha) + m_2(\alpha, \Phi) + m_2(\Phi, \alpha) + \cdots \sim Q(\alpha)$, we will find the gauge fixed solutions $Q^+\Phi = 0$. We express the gauge fixed Φ as $\Phi|_{gf} = \Phi^p + \Phi^u$ where $\Phi^p \in \mathcal{H}^p$ and $\Phi^u \in \mathcal{H}^u$. As explained below, Φ^u can be solved recursively for the power of Φ^p. Because here we regard that Φ^p is 'small', one can define a degree by the power of Φ^p. By substituting $\Phi|_{gf} = \Phi^p + \Phi^u$ into Φ, the MC-eq. (5.13) turns out to be

$$Q(\Phi^u) + \sum_{k\geq 2} m_k(\Phi^p + \Phi^u) = 0 \ , \tag{5.14}$$

---

[16] The arguments in this subsection can be generalized by relaxing the condition that QQ^+, Q^+Q and P are projections. Namely, for a given A_∞-algebra, other quasi-isomorphic A_∞-algebras can be constructed by a tree graphical way as in this subsection at least an identity $QQ^+ + Q^+Q + P = \mathbf{1}$ is given. As an application of such a generalized construction to SFT, see section 6 of [53].

and acting by Q^+ on both sides of this equation yields

$$\Phi^u = -\sum_{k\geq 2} Q^+ m_k(\Phi^p + \Phi^u) \ . \tag{5.15}$$

As will be presented explicitly later, Φ^u is expressed in terms of the powers of Φ^p by substituting the right hand side of eq.(5.15) into Φ^u in the right hand side of eq.(5.15) itself recursively. However not all $\Phi|_{gf} = \Phi^p + \Phi^u$ expressed in terms of Φ^p give the solution of eq.(5.13) because eq.(5.15) is derived from 'Q^+ acting via eq.(5.13)'. In order to find Φ^u which is the solution of eq.(5.13), we substitute eq.(5.15) in the MC-eq.(5.13) once again,

$$\begin{aligned} 0 &= Q(\Phi^p + \Phi^u) + \sum_{k\geq 2} m_k(\Phi) \\ &= (Q^+Q + P - 1)\sum_{k\geq 2} m_k(\Phi) + \sum_{k\geq 2} m_k(\Phi) \\ &= Q^+Q\sum_{k\geq 2} m_k(\Phi) + \sum_{k\geq 2} P m_k(\Phi) \end{aligned} \tag{5.16}$$

and we can get a condition (obstruction) for Φ^p. The first term in the third line of eq.(5.16) vanishes due to MC-eq.(5.13) because $Q\sum_{k\geq 2} m_k(\Phi) = -QQ(\Phi) = 0$, and a condition for Φ^p is derived as

$$\sum_{k\geq 2} P m_k(\Phi^p + \Phi^u) = 0 \ . \tag{5.17}$$

The minimal A_∞-algebra $(\mathcal{H}^p, \tilde{\mathfrak{m}}^p)$ and the A_∞-quasi-isomorphism $\tilde{\mathcal{F}}^p : (\mathcal{H}^p, \tilde{\mathfrak{m}}^p) \to (\mathcal{H}, \tilde{\mathfrak{m}})$ are given by eq.(5.17) and eq.(5.15), respectively. As mentioned above, a nonlinear map from $\mathcal{H}^p$ to $\mathcal{H}$ is obtained by substituting the right hand side of eq.(5.15) into the right hand side of the equation $\Phi|_{gf} = \Phi^p + \Phi^u$ recursively. Here we want to distinguish the element of $\mathcal{H}^p$ from that of $\mathcal{H}$, so we rewrite $\Phi^p \in \mathcal{H}^p$ as $\tilde{\Phi}^p \in \mathcal{H}^p$. Let us represent the nonlinear map by a collection of multilinear maps $\tilde{f}^p_l : (\mathcal{H}^p)^{\otimes l} \to \mathcal{H}$ as

$$\Phi|_{gf} = \tilde{f}^p_1(\tilde{\Phi}^p) + \tilde{f}^p_2(\tilde{\Phi}^p, \tilde{\Phi}^p) + \tilde{f}^p_3(\tilde{\Phi}^p, \tilde{\Phi}^p, \tilde{\Phi}^p) + \cdots \tag{5.18}$$

where $\tilde{f}^p_1 : \mathcal{H}^p \to \mathcal{H}$ the inclusion map. Alternatively, eq.(5.17) is also expressed as an equation for $\tilde{\Phi}^p$ by substituting eq.(5.18) into eq.(5.17). Let us write it also by a collection of multilinear maps $\tilde{m}^p_l : (\mathcal{H}^p)^{\otimes l} \to \mathcal{H}^p$ as

$$\iota \sum_{k\geq 2} \tilde{m}^p_k(\tilde{\Phi}^p) = 0 \ , \tag{5.19}$$

where $\iota : \mathcal{H}^p \to \mathcal{H}$ is the inclusion. Once $\tilde{m}^p_k(\tilde{\Phi}^p)$ and $\tilde{f}^p_k(\tilde{\Phi}^p)$ are obtained, $\tilde{m}^p_k(\mathbf{e}^p_1, \cdots, \mathbf{e}^p_k)$ and $\tilde{f}^p_k(\mathbf{e}^p_1, \cdots, \mathbf{e}^p_k)$ can be obtained immediately as the coefficient of $\phi^k \cdots \phi^1$, where $\mathbf{e}^p_i, i = 1, \cdots k$ are bases of $\mathcal{H}^p$ and $\tilde{\Phi}^p = \mathbf{e}^p_i \phi^i$. As shown in the end of this subsection, $(\mathcal{H}^p, \tilde{\mathfrak{m}}^p := \{\tilde{m}^p_k\}_{k\geq 2})$ forms a minimal A_∞-algebra. The equation (5.19) is just the Maurer-Cartan equation for $(\mathcal{H}^p, \tilde{\mathfrak{m}}^p)$. $\tilde{\mathcal{F}}^p := \{\tilde{f}^p_l\}_{l\geq 1}$ is then an A_∞-quasi-isomorphism from $(\mathcal{H}^p, \tilde{\mathfrak{m}}^p)$ to $(\mathcal{H}, \mathfrak{m})$.

From the field theory point of view the above result means that if the expectation value of physical states satisfying the Maurer-Cartan equation (5.19) is given, the solution of the equations of motions for field theory (5.13) is obtained by the A_∞-quasi-isomorphism (5.18).

Here we summarize the arguments above and define the A_∞-structure and A_∞-quasi-isomorphism precisely.

Definition 5.18 (An explicit minimal model) Given an A_∞-algebra $(\mathcal{H},\mathfrak{m})$, assume that we have a splitting of the complex $(\mathcal{H},Q)$ such that $\iota:\mathcal{H}^p\to\mathcal{H}$ is the inclusion, $\pi:\mathcal{H}\to\mathcal{H}^p$ is the projection and $Q^+:\mathcal{H}\to\mathcal{H}$ is the contracting homotopy $\mathbf{1}-P=QQ^++Q^+Q$ for $P=\iota\circ\pi$. Then, an A_∞-structure $\tilde{\mathfrak{m}}^p$ on the cohomology $\mathcal{H}^p$ and an A_∞-quasi-isomorphism $\tilde{\mathcal{F}}^p:(\mathcal{H}^p,\tilde{\mathfrak{m}}^p)\to(\mathcal{H},\mathfrak{m})$ are constructed as follows. $\tilde{\mathcal{F}}^p=\{\tilde{f}^p_l:(\mathcal{H}^p)^{\otimes l}\to\mathcal{H}\}_{l\geq 1}$ is defined recursively with respect to k as

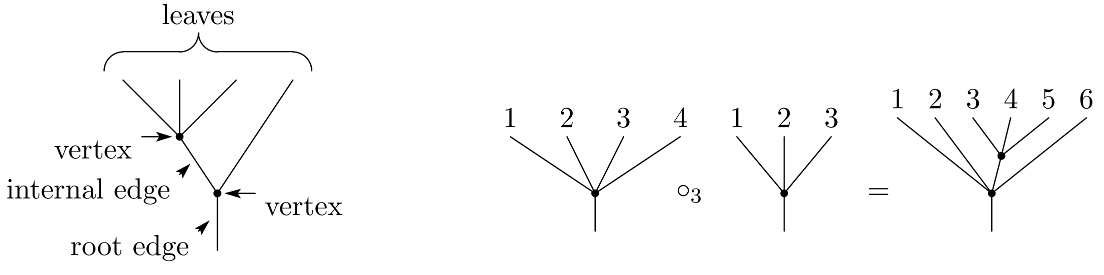
$$\tilde{f}^p_k=-Q^+\sum_{i\geq 2}\sum_{1\leq k_1<k_2\cdots<k_i=k}m_i(\tilde{f}^p_{k_1}\otimes\tilde{f}^p_{k_2-k_1}\otimes\cdots\otimes\tilde{f}^p_{k-k_{i-1}})$$

with $\tilde{f}^p_1:=\iota:\mathcal{H}^p\to\mathcal{H}$. $\tilde{\mathfrak{m}}^p=\{\tilde{m}^p_k:(\mathcal{H}^p)^{\otimes k}\to\mathcal{H}\}_{k\geq 2}$ is then given by

$$\tilde{m}^p_k=\pi\sum_{i\geq 2}\sum_{1\leq k_1<k_2\cdots<k_i=k}m_i(\tilde{f}^p_{k_1}\otimes\tilde{f}^p_{k_2-k_1}\otimes\cdots\otimes\tilde{f}^p_{k-k_{i-1}})\ .$$

It is also convenient to present an alternative description of Definition 5.18 in terms of rooted planar tree graphs. It is related to Feynman graphs in field theory in section 6.

Definition 5.19 (Rooted planar tree graph) A rooted planar tree graph is a simply connected rooted planar tree without loops. It consists of vertices, internal edges and external edges. Both ends of an internal edge are on two vertices. An external edge has one end on a vertex and another end is free. The number of incident edges at a vertex is greater than three. The term 'planar' means the cyclic order of edges at each vertex is distinguished. A rooted planar tree has a *root* that is a free end of an external edge. The external edge is called the *root edge*. The vertex on which the root edge ends is the *root vertex*. The free ends of the remaining external edges are called the *leaves*. We call a rooted planar tree that has k leaves a k-tree and denote it by Γ_k. We denote by G_k the set of k-trees.



(a) (b)

Figure 11: (a). Notation for planar rooted tree. The above one is a 4-tree. (b). An example of grafting, grafting of 3-corolla to 4-corolla along leaf 3.

For k-tree Γ_k, l-tree Γ_l and an integer $1 \le i \le k$, the *grafting* of l-tree to k-tree along leaf i is given by identifying the root edge of the Γ_l with the i-th leaf of Γ_k (see Figure 11 (b)). The resulting $(k+l-1)$-tree is denoted by $\Gamma_k \circ_i \Gamma_l$.

G_1 has only one element $|$ that has no vertex. A k-tree that has only one vertex is called a *k-corolla.* Any other tree, that has more than one vertices, is obtained by grafting corollas.

Definition 5.20 (Minimal model; an alternative description) Let us define a map $\tilde{f} : G_k \to (\mathcal{H}^{\otimes k} \to \mathcal{H})$ as follows. To the 1-tree (that has no vertex) we associate the identity operator $\mathrm{Id} : \mathcal{H} \to \mathcal{H}$. To a k-corolla, we associate $-Q^+ m_k : \mathcal{H}^{\otimes k} \to \mathcal{H}$. For any k-tree Γ_k and l-tree Γ_l, denote the associated endomorphisms by $\tilde{f}_{\Gamma_k} \in (\mathcal{H}^{\otimes k} \to \mathcal{H})$ and $\tilde{f}_{\Gamma_l} \in (\mathcal{H}^{\otimes l} \to \mathcal{H})$. $\tilde{f}$ is then defined so that it is compatible with the grafting of the trees. Namely, to $\Gamma_k \circ_i \Gamma_l$ we associate

$$\tilde{f}_{\Gamma_k \circ_i \Gamma_l} = \tilde{f}_{\Gamma_k} \circ \left(\mathbf{1}^{\otimes(i-1)} \otimes \tilde{f}_{\Gamma_l} \otimes \mathbf{1}^{\otimes(k-i)} \right) : \mathcal{H}^{\otimes(k+l-1)} \to \mathcal{H} \ .$$

Thus, for any k-tree, $\tilde{f}_{\Gamma_k}$ is defined. The A_∞-quasi-isomorphism $\tilde{\mathcal{F}}^p = \{\tilde{f}^p_k\}_{k \ge 1}$ is then defined by

$$\tilde{f}^p_k = \sum_{\Gamma_k \in G_k} \tilde{f}_{\Gamma_k} \circ (\iota)^{\otimes k} \ ,$$

where $\iota : \mathcal{H}^p \to \mathcal{H}$ is the inclusion.

The minimal A_∞-structure $\tilde{\mathfrak{m}}^p$ is defined by using another map $\tilde{m} : G_k \to (\mathcal{H}^{\otimes k} \to \mathcal{H})$.

To the 1-tree we associate the differential $Q : \mathcal{H} \to \mathcal{H}$. To a k-corolla, we associate $m_k : \mathcal{H}^{\otimes k} \to \mathcal{H}$. To any k-tree Γ_k denote the associated endomorphisms by $\tilde{m}_{\Gamma_k} \in (\mathcal{H}^{\otimes k} \to \mathcal{H})$. For a grafting $\Gamma_k \circ_i \Gamma_l$ we associate

$$\tilde{m}_{\Gamma_k \circ_i \Gamma_l} = \tilde{m}_{\Gamma_k} \circ \left(\mathbf{1}^{\otimes(i-1)} \otimes \tilde{f}_{\Gamma_l} \otimes \mathbf{1}^{\otimes(k-i)} \right) : \mathcal{H}^{\otimes(k+l-1)} \to \mathcal{H} \ .$$

Thus, for any tree graph, $\tilde{m}$ is defined so that it is compatible with this grafting. $\tilde{\mathfrak{m}}^p$ is then given by

$$\tilde{m}^p_k = \pi \circ \sum_{\Gamma_k \in G_k} \tilde{m}_{\Gamma_k} \circ (\iota)^{\otimes k} \ ,$$

where $\pi : \mathcal{H} \to \mathcal{H}^p$ is the projection. Note that $\tilde{m}^p_1$ automatically vanishes.

As a result, for a given l-tree Γ_l, $\tilde{m}^p_{\Gamma_l}$ is given by attaching m_k to each vertex that has $(k+1)$-incident edges, $-Q^+$ to each internal edge, ι to each leave and π to the root edge. $\tilde{f}^p_{\Gamma_l}$ is also given in the same way but replacing π on the root edge to $-Q^+$.

An explicit example is given in Figure 12. In the order of the graphs in Figure 12, we have

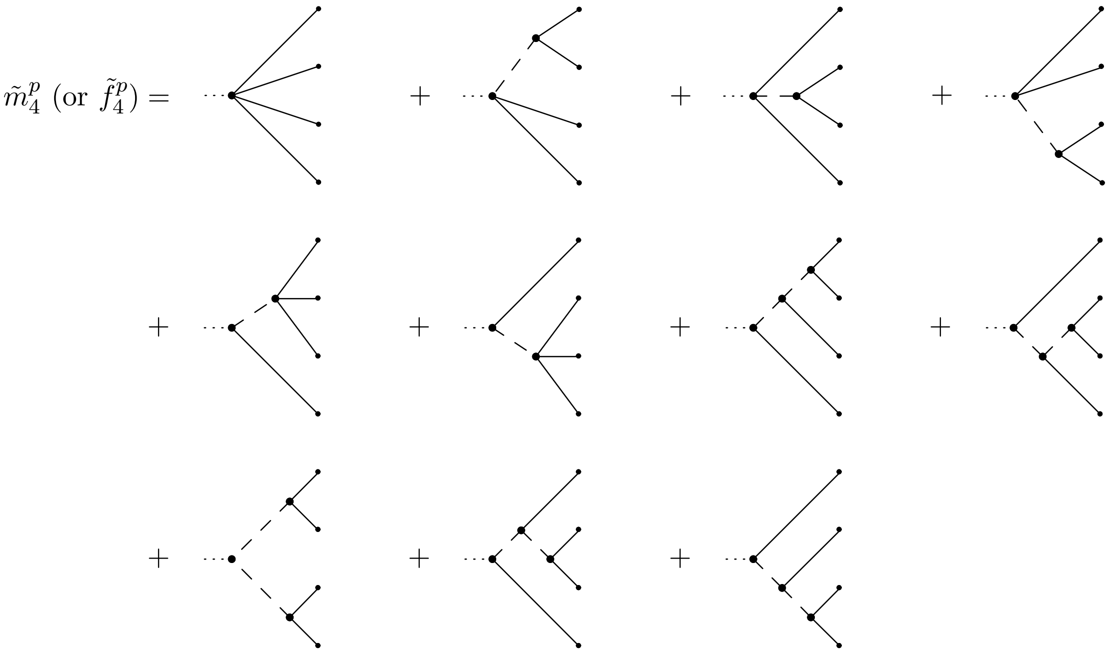


Figure 12: For example $\tilde{m}^p_4$ and $\tilde{f}^p_4$ are given. The large dots represent the vertices $\{m_k\}$. The dashed lines denote the internal edges and we attach $-Q^+$ on them. The dotted line on each graph is the root edge, on which we attach π for $\tilde{m}^p_k$ and $-Q^+$ for $\tilde{f}^p_k$. For $\tilde{m}^p_4$ and $\tilde{f}^p_4$, all such 4-trees are summed up with weight +1.

$$
\begin{aligned}
\tilde{m}^p_4(\mathbf{e}^p_1,\mathbf{e}^p_2,\mathbf{e}^p_3,\mathbf{e}^p_4) &= \pi\circ m_4(\mathbf{e}^p_1,\mathbf{e}^p_2,\mathbf{e}^p_3,\mathbf{e}^p_4) + \pi\circ m_3(-Q^+m_2(\mathbf{e}^p_1,\mathbf{e}^p_2),\mathbf{e}^p_3,\mathbf{e}^p_4)\\
&+ \pi\circ m_3(\mathbf{e}^p_1,-Q^+m_2(\mathbf{e}^p_2,\mathbf{e}^p_3),\mathbf{e}^p_4) + \pi\circ m_3(\mathbf{e}^p_1,\mathbf{e}^p_2,-Q^+m_2(\mathbf{e}^p_3,\mathbf{e}^p_4))\\
&+ \pi\circ m_2(-Q^+m_3(\mathbf{e}^p_1,\mathbf{e}^p_2,\mathbf{e}^p_3),\mathbf{e}^p_4) + \pi\circ m_2(\mathbf{e}^p_1,-Q^+m_3(\mathbf{e}^p_2,\mathbf{e}^p_3,\mathbf{e}^p_4))\\
&+ \pi\circ m_2(-Q^+m_2(-Q^+m_2(\mathbf{e}^p_1,\mathbf{e}^p_2),\mathbf{e}^p_3),\mathbf{e}^p_4)\\
&+ \pi\circ m_2(\mathbf{e}^p_1,-Q^+m_2(-Q^+m_2(\mathbf{e}^p_2,\mathbf{e}^p_3),\mathbf{e}^p_4))\\
&+ \pi\circ m_2(-Q^+m_2(\mathbf{e}^p_1,\mathbf{e}^p_2),-Q^+m_2(\mathbf{e}^p_3,\mathbf{e}^p_4))\\
&+ \pi\circ m_2(-Q^+m_2(\mathbf{e}^p_1,-Q^+m_2(\mathbf{e}^p_2,\mathbf{e}^p_3)),\mathbf{e}^p_4)\\
&+ \pi\circ m_2(\mathbf{e}^p_1,-Q^+m_2(\mathbf{e}^p_2,-Q^+m_2(\mathbf{e}^p_3,\mathbf{e}^p_4)))\ ,
\end{aligned}
$$

and $\tilde{f}^p_4$ is obtained similarly but replaced each π on the outgoing line to $-Q^+$.

We emphasize that the definition above is derived essentially by using eq.(5.15) recursively.

$$
\Phi|_{gf} = \tilde{\Phi}^p - Q^+\sum_{k\geq 2} m_k(\Phi|_{gf})\ .
$$

Let us extend the equation above to

$$
\Phi = \tilde{\Phi} - Q^+\sum_{k\geq 2} m_k(\Phi)\ , \tag{5.20}
$$

where $\tilde{\Phi}\in\mathcal{H}$. One can construct an A_∞-isomorphism $\tilde{\Phi}\mapsto\Phi$ recursively in a similar way as in Definition 5.18. Then one can get another A_∞-algebra as the pullback of $(\mathcal{H},\mathfrak{m})$ by this isomorphism. Let us denote this A_∞-algebra by $(\mathcal{H},\tilde{\mathfrak{m}})$ and the A_∞-isomorphism by $\tilde{\mathcal{F}}$: $(\mathcal{H},\tilde{\mathfrak{m}})\to(\mathcal{H},\mathfrak{m})$. The explicit forms of $\tilde{\mathfrak{m}}:=\{\tilde{m}_k\}_{k\ge 1}$ and $\tilde{\mathcal{F}}:=\{\tilde{f}_k\}_{k\ge 1}$ are given in the terminology in Definition 5.20 as

$$\tilde{f}_k=\sum_{\Gamma_k\in G_k}\tilde{f}_{\Gamma_k}\quad(k\ge 1)\ ,\qquad \tilde{m}_1=m_1=Q\ ,\quad \tilde{m}_k=P\circ\sum_{\Gamma_k\in G_k}\tilde{m}_{\Gamma_k}\quad(k\ge 2)\ .$$

In particular, one has $\tilde{f}_1=\mathrm{Id}$. Note that these are the operations not on $\mathcal{H}^p$ but on $\mathcal{H}$. That is, $\tilde{m}_k:\mathcal{H}^{\otimes k}\to\mathcal{H}^p\subset\mathcal{H}$ and $\tilde{f}_k:\mathcal{H}^{\otimes k}\to\mathcal{H}$ do not vanish generally even if one of the $\mathcal{H}$ in $\mathcal{H}^{\otimes k}$ includes an element in $\mathcal{H}^t\oplus\mathcal{H}^u$. For $\iota:\mathcal{H}^p\to\mathcal{H}$ the inclusion map, we have the relations $\tilde{m}^p_k=\pi\circ\tilde{m}_k\circ(\iota)^{\otimes k}$ and $\tilde{f}^p_k=\tilde{f}_k\circ(\iota)^{\otimes k}$ for $k\ge 2$.

Remark 5.21 Since $\tilde{\mathcal{F}}:(\mathcal{H},\tilde{\mathfrak{m}})\to(\mathcal{H},\mathfrak{m})$ is not only an A_∞-quasi-isomorphism but also an A_∞-isomorphism, it has its inverse A_∞-isomorphism $(\tilde{\mathcal{F}})^{-1}:(\mathcal{H},\mathfrak{m})\to(\mathcal{H},\tilde{\mathfrak{m}})$, which is given by

$$\begin{array}{cccl}(\tilde{\mathcal{F}})^{-1}_* & : & \mathcal{H} & \longrightarrow \ \mathcal{H}\\ & & \Phi & \mapsto \ \tilde{\Phi}:=\Phi+Q^+\sum_{k\ge 2}m_k(\Phi)\ .\end{array}$$

In the rest of this subsection we shall give a proof of the statement below.

Lemma 5.22 *$(\mathcal{H}^p,\tilde{\mathfrak{m}}^p)$ and $(\mathcal{H},\tilde{\mathfrak{m}})$ are in fact A_∞-algebras and $\tilde{\mathcal{F}}^{(p)}$ (by $\tilde{\mathcal{F}}^{(p)}$ we denote $\tilde{\mathcal{F}}$ or $\tilde{\mathcal{F}}^p$) is an A_∞-morphism.*

proof. The fact that $(\mathcal{H}^p,\tilde{\mathfrak{m}}^p)$ is an A_∞-algebra and $\tilde{\mathcal{F}}^p$ is an A_∞-quasi-isomorphism between $(\mathcal{H}^p,\tilde{\mathfrak{m}}^p)$ to $(\mathcal{H},\mathfrak{m})$ immediately follows from the fact that $(\mathcal{H},\tilde{\mathfrak{m}})$ is an A_∞-algebra and $\tilde{\mathcal{F}}$ is an A_∞-quasi-isomorphism between $(\mathcal{H},\tilde{\mathfrak{m}})$ to $(\mathcal{H},\mathfrak{m})$. The latter fact is explicitly described by the equations on $\mathcal{H}$ as

$$\mathfrak{m}\tilde{\mathcal{F}}=\tilde{\mathcal{F}}\tilde{\mathfrak{m}}\ ,\qquad(\tilde{\mathfrak{m}})^2=0\ .$$

The former is obtained by restricting these equations to $\mathcal{H}^p$, that is, $\mathfrak{m}\,\tilde{\mathcal{F}}\,\iota=\tilde{\mathcal{F}}\,\tilde{\mathfrak{m}}\,\iota$ and $(\tilde{\mathfrak{m}})^2\,\iota=0$ on $\mathcal{H}^p$. Therefore we will prove the latter fact. In order to see this, it is enough to confirm the following two facts : $\mathfrak{m}\tilde{\mathcal{F}}=\tilde{\mathcal{F}}\tilde{\mathfrak{m}}$ and $(\tilde{\mathfrak{m}})^2=0$ on $\mathcal{H}$. These are shown at the same time by checking

$$\tilde{\mathfrak{m}}=\tilde{\mathcal{F}}^{-1}\mathfrak{m}\tilde{\mathcal{F}}\tag{5.21}$$

since $\tilde{\mathcal{F}}^{-1}\tilde{\mathcal{F}}=\mathbf{1}$ and $\tilde{\mathcal{F}}\tilde{\mathcal{F}}^{-1}=\mathbf{1}$. We shall show it below in the dual picture. Let δ be A_∞-odd vector fields corresponding to $\mathfrak{m}$. Recall that $(\tilde{\mathcal{F}})^{-1}_*(\tilde{\Phi})=\Phi+Q^+\sum_{k\ge 2}m_k(\Phi,\cdots,\Phi)$, and we write it as

$$\tilde{\phi}^i=\phi^i+\bar{c}^i_j\sum_{k\ge 2}c^j_{i_1\cdots i_k}\phi^{i_k}\cdots\phi^{i_1}\ ,\tag{5.22}$$

where $Q^+\mathbf{e}_j=\mathbf{e}_i\bar{c}^i_j$. The coordinate transformation corresponding to $\tilde{\mathcal{F}}_*:\tilde{\Phi}\to\Phi$ is obtained by using

$$\phi^i=\tilde{\phi}^i-\bar{c}^i_j\sum_{k\ge 2}c^j_{i_1\cdots i_k}\phi^{i_k}\cdots\phi^{i_1}\tag{5.23}$$

recursively.

Let $\tilde{\delta}$ be the A_∞-odd vector field dual to $\tilde{\mathcal{F}}^{-1}\mathfrak{m}\tilde{\mathcal{F}}$ in the right hand side of eq.(5.21). From now we rewrite δ as this $\tilde{\delta}$ using eq.(5.22) and eq.(5.23), and show that the $\tilde{\delta}$ is in fact the A_∞-odd vector field dual to $\tilde{\mathfrak{m}}$ defined by eq.(5.20). For

$$\delta = \frac{\overleftarrow{\partial}}{\partial\phi^i}\left(c^i_j\phi^j + \sum_{k\geq 2} c^i_{i_1\cdots i_k}\phi^{i_k}\cdots\phi^{i_1}\right) \ ,$$

$\tilde{\delta}$ is obtained by

$$\tilde{\delta} = \frac{\overleftarrow{\partial}}{\partial\tilde{\phi}^i}\frac{\tilde{\phi}^i\overleftarrow{\partial}}{\partial\phi^l}\left(c^l_j\phi^j + \sum_{k\geq 2} c^l_{i_1\cdots i_k}\phi^{i_k}\cdots\phi^{i_1}\right) \ . \tag{5.24}$$

Here $\frac{\tilde{\phi}^i\overleftarrow{\partial}}{\partial\phi^l} = \delta^i_k + \bar{c}^i_j\sum_{k\geq 2} c^j_{i_1\cdots i_k}\left(\phi^{i_k}\cdots\phi^{i_1}\right)\frac{\overleftarrow{\partial}}{\partial\phi^l}$ by using eq.(5.22) and all ϕ's are supposed to be substituted into by eq.(5.23). Thus eq.(5.24) is further rewritten as

$$\begin{aligned}\tilde{\delta} =& \frac{\overleftarrow{\partial}}{\partial\tilde{\phi}^i}c^i_j\left(\tilde{\phi}^j - \bar{c}^j_l\sum_{n\geq 2} c^l_{j_1\cdots j_n}\phi^{j_n}\cdots\phi^{j_1}\right)\\ &+\frac{\overleftarrow{\partial}}{\partial\tilde{\phi}^i}\sum_{k\geq 2} c^i_{i_1\cdots i_k}\phi^{i_k}\cdots\phi^{i_1}\\ &+\frac{\overleftarrow{\partial}}{\partial\tilde{\phi}^i}\bar{c}^i_j\sum_{k\geq 2} c^j_{i_1\cdots i_k}\left(\phi^{i_k}\cdots\phi^{i_1}\right)\frac{\overleftarrow{\partial}}{\partial\phi^l}\sum_{n\geq 1} c^l_{j_1\cdots j_n}\phi^{j_n}\cdots\phi^{j_1} \ .\end{aligned} \tag{5.25}$$

Note that, by the A_∞-condition for δ, the equation of the third line is replaced by

$$\frac{\overleftarrow{\partial}}{\partial\tilde{\phi}^i}\bar{c}^i_j\left(-c^j_l\right)\sum_{n\geq 1} c^l_{j_1\cdots j_n}\phi^{j_n}\cdots\phi^{j_1} \ .$$

Thus one can get

$$\tilde{\delta} = \frac{\overleftarrow{\partial}}{\partial\tilde{\phi}^i}c^i_j\tilde{\phi}^j + \frac{\overleftarrow{\partial}}{\partial\tilde{\phi}^i}\sum_{k\geq 2}(\delta^i_l - c^i_j\bar{c}^j_l - \bar{c}^i_j c^j_l)\sum_{k\geq 2} c^l_{i_1\cdots i_k}\phi^{i_k}\cdots\phi^{i_1} \ .$$

The second term of the first line, the term of the second line and the one of the third line in eq.(5.25) are gathered as the second term above. $(\delta^i_l - c^i_j\bar{c}^j_l - \bar{c}^i_j c^j_l)$ restricts the indices l to those in $\mathcal{H}^p$, *i.e.* , the dual description of P. Therefore by substituting eq.(5.23) recursively in the equation above one can see that $\tilde{\delta}$ above is just the dual expression of $\tilde{\mathfrak{m}}$. Thus the proof is completed. ■

Another proof in terms of the 'superfield' Φ is presented in [53].

5.5 Minimal cyclic A_∞-algebras and Feynman graphs

There exists an explicit construction of the minimal model also for cyclic A_∞-algebras. In this subsection we shall see that the arguments in the previous subsection are applicable directly to cyclic version.

Our starting point is the (extended) Maurer-Cartan equation (5.13) in the previous subsection. One can see that the equation is identified with $\delta = 0$. As stated in subsection 4.5, a cyclic A_∞-algebra $(\mathcal{H}, \omega, \mathfrak{m})$ has a degree zero cyclic function, the action $S \in C(\phi)_c$ (eq.(4.6)). The A_∞-odd vector field is then given by $\delta = (\ , S)$. Since the odd constant Poisson bracket $(\ ,\ )$ is nondegenerate, it is just equivalent to the equation of motion of the action,

$$\frac{\overrightarrow{\partial}}{\partial\phi^j} S = 0\ . \tag{5.26}$$

The action is expressed in terms of the superfield Φ as

$$S(\Phi) = \sum_{k\geq 2} \frac{1}{k} \mathcal{V}_k(\Phi, \cdots, \Phi) = \frac{1}{2}\omega(\Phi, Q\Phi) + \sum_{k\geq 3} \frac{1}{k}\omega(\Phi, m_{k-1}(\Phi, \cdots, \Phi))\ , \tag{5.27}$$

where $\mathcal{V}_k(\mathbf{e}_{i_1}, \cdots, \mathbf{e}_{i_k}) = \mathcal{V}_{i_1\cdots i_k}$.

Lemma 5.23 (A minimal cyclic A_∞-algebra) *For a given cyclic A_∞-algebra $(\mathcal{H}, \omega, \mathfrak{m})$, suppose we have a Hodge-Kodaira decomposition of $\mathcal{H}$ with a homotopy operator Q^+ of Q compatible with ω. Then, construct the explicit minimal A_∞-algebra $(\mathcal{H}^p, \tilde{\mathfrak{m}}^p)$ of the A_∞-algebra $(\mathcal{H}, \mathfrak{m})$ and the A_∞-quasi-isomorphism $\tilde{\mathcal{F}}^p : (\mathcal{H}^p, \tilde{\mathfrak{m}}^p) \to (\mathcal{H}, \mathfrak{m})$ in Definition 5.18. Next, define a symplectic structure $\tilde{\omega}^p$ on $\mathcal{H}^p$ by restricting the symplectic structure ω on $\mathcal{H}$ to $\mathcal{H}^p$, that is, $\tilde{\omega}^p := \omega(\iota\otimes\iota)$ for $\iota : \mathcal{H}^p \to \mathcal{H}$.*

- *(a) The A_∞-structure $\tilde{\mathfrak{m}}^p$ is cyclic with respect to $\tilde{\omega}^p$, that is, $(\mathcal{H}^p, \tilde{\omega}^p, \tilde{\mathfrak{m}}^p)$ defines a minimal cyclic A_∞-algebra.*
- *(b) $\tilde{\mathcal{F}}^p : (\mathcal{H}^p, \tilde{\omega}^p, \tilde{\mathfrak{m}}^p) \to (\mathcal{H}, \omega, \mathfrak{m})$ is a cyclic A_∞-quasi-isomorphism.*

proof. Recall that, for each $k \geq 2$, $\tilde{m}^p_k$ is defined so that it is associated to the summation over all rooted planar k-trees. The sum of all k-trees is cyclic, that is, invariant with respect to the cyclic permutations of the root and the leaves (see also subsection 6.3). Statement (a) follows from this fact together with the conditions (5.12). For statement (b), by the definition of cyclic A_∞-morphisms (Definition 2.13), it is sufficient to show that $\tilde{\mathcal{F}}^p$ preserves the symplectic structures $\tilde{\omega}^p$ and ω. $(\tilde{\mathcal{F}}^p)^*\omega$ is written as [17]

$$\Big((\tilde{\mathcal{F}}^p)^*\omega\Big)_{ij} = \frac{\overrightarrow{\partial}\phi^k}{\partial\tilde{p}^i}\omega_{kl}\frac{\phi^l\overleftarrow{\partial}}{\partial\tilde{p}^j} = (-1)^{\mathbf{e}^p_i}\omega\left(\frac{\overrightarrow{\partial}}{\partial\tilde{p}^i}\Phi, \Phi\frac{\overleftarrow{\partial}}{\partial\tilde{p}^j}\right)\ ,$$

where $\tilde{p}^i$ is the dual coordinate of the basis vector $\mathbf{e}^p_i \in \mathcal{H}^p$. Since $\Phi = \tilde{\Phi}^p - Q^+\sum_{k\geq 2} m_k(\Phi)$ and the image of Q^+ vanishes in the symplectic inner product in the right hand side of the above equation, the right hand side becomes $(-1)^{\mathbf{e}^p_i}\omega(\frac{\overrightarrow{\partial}}{\partial\tilde{p}^i}\tilde{\Phi}^p, \tilde{\Phi}^p\frac{\overleftarrow{\partial}}{\partial\tilde{p}^j}) = \tilde{\omega}^p_{ij}$. Thus it has been shown that the map $\tilde{\mathcal{F}}^p$ preserves the symplectic structures. ■

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[17] The equation below is actually equivalent to the one for odd Poisson structure. The equivalence follows from $\omega_{ij}\omega^{jk} = \delta^k_i$ and $\tilde{\omega}^p_{ij}\tilde{\omega}^{p,jk} = \delta^k_i$.

These facts together with Proposition 4.16 further imply that the action of the corresponding minimal cyclic A_∞-algebra

$$\tilde{S}(\tilde{\Phi}^p)=\sum_{k\geq 2}\frac{1}{k+1}\omega^p(\tilde{\Phi}^p,\tilde{m}^p_k(\tilde{\Phi}^p)) \tag{5.28}$$

can also be obtained by the pullback of $S(\Phi)$ by $\tilde{\mathcal{F}}^p$

$$\tilde{S}(\tilde{\Phi}^p)=(\tilde{\mathcal{F}}^p)^*S(\Phi)=\Big(\ S((\tilde{\mathcal{F}}^p)_*(\tilde{\Phi}^p))\ \Big)_c\ . \tag{5.29}$$

It is also interesting to show the equality $\tilde{S}(\tilde{\Phi}^p)=(\tilde{\mathcal{F}}^p)^*S(\Phi)$ directly; $(\tilde{\mathcal{F}}^p)^*S(\Phi)$ coincides with $\tilde{S}(\tilde{\Phi}^p)$ due to some nontrivial combinatorial cancellations (see subsection 5.3 of [53]).

As stated previously, the homotopy types of cyclic A_∞-algebras are classified by their minimal cyclic A_∞-algebras. For a given cyclic A_∞-algebra, its minimal one is unique up to cyclic A_∞-isomorphisms. Namely, we have at least a decomposed cyclic A_∞-algebra whose minimal part is the one given explicitly in this subsection. Note however that a cyclic A_∞-isomorphism from the decomposed one to the original one is not given explicitly. It might be interesting to explore the relation between the explicit construction of a minimal model and the way of the proof of the decomposition theorem in subsection 5.2.

6 The minimal model theorem in the BV-formalism

In this section we shall apply the homotopy algebraic structures discussed in the previous section to field theory equipped with classical BV-structures. In subsection 6.1, it is shown that any cyclic field theory equipped with a classical BV-structure has a cyclic A_∞-structure (Theorem 6.1). Subsection 6.2 is devoted to a brief review of perturbative expansion in the BV-formalism and to translating it into our language. The perturbative expansion is necessary for computing correlation functions in the subsequent subsections. Also, we shall show that the propagator in the BV-formalism, which is defined in subsection 6.2, is a homotopy operator Q^+ in the previous section (Proposition 6.5). In subsection 6.3 it is then shown that the tree on-shell correlation functions of a cyclic field theory equipped with a classical BV-structure define just the minimal cyclic A_∞-algebra defined in subsection 5.5 (Corollary 6.14 (cf.[53])). Moreover in subsection 6.4 the arguments in section 5 are applied to the classification of classical open string field theories, and it is shown that all classical string field theories on a fixed conformal background are cyclic A_∞-isomorphic to each other (Theorem 6.18). The theorem means all classical string field theories on a fixed conformal background are related by field redefinitions and so physically equivalent to each other.

6.1 Cyclic A_∞-structures in the BV-formalism

The BV-formalism is formulated on formal supermanifolds equipped with odd symplectic forms, where the coordinates of supermanifolds are just the fields. Since we discuss field theories related to open string theory, we let the fields noncommutative as explained in subsection 1.4. We shall first explain that our noncommutative symplectic supergeometry just fits the BV-formalism.

For any odd symplectic form one can take a Darboux coordinate due to Theorem 4.15. We denote the odd symplectic form on $\mathbb{Z}$-graded vector space $\mathcal{H}$ in a Darboux coordinate by

$$\{\omega_{ij}\} = \begin{pmatrix} \mathbf{0} & -\mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} \ . \tag{6.1}$$

In this coordinate, let us decompose the basis $\{\mathbf{e}_i\}$ into $\{\mathbf{e}_a\}$ and $\{\mathbf{e}^*_a\}$ such that $-\omega(\mathbf{e}_a, \mathbf{e}^*_b) = \omega(\mathbf{e}^*_b, \mathbf{e}_a) = \delta_{ab}$ and $\omega(\mathbf{e}_a, \mathbf{e}_b) = \omega(\mathbf{e}^*_a, \mathbf{e}^*_b) = 0$. Then we have $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, where $\mathcal{H}_+$ and $\mathcal{H}_-$ are the $\mathbb{Z}$-graded vector spaces spanned by $\{\mathbf{e}_a\}$ and $\{\mathbf{e}^*_a\}$, respectively. As above, we use indices $a, b, \cdots$ for the basis of $\mathcal{H}_+$ or $\mathcal{H}_-$. For bases $\mathbf{e}_a$ and $\mathbf{e}^*_a$, we denote the associated dual fields by ϕ^a and $\phi^{a,*}$, respectively. Also, it is more convenient to prepare the notation $\phi^*_a := \omega_{ab}\phi^{b,*}$ where ω_{ab} is the one in eq.(6.1). The odd Poisson bracket associated to eq.(6.1) is then written as

$$(\ ,\ ) = \frac{\overleftarrow{\partial}}{\partial\phi^i}\omega^{ij}\frac{\overrightarrow{\partial}}{\partial\phi^j} = \frac{\overleftarrow{\partial}}{\partial\phi^a}\frac{\overrightarrow{\partial}}{\partial\phi^*_a} - \frac{\overleftarrow{\partial}}{\partial\phi^*_a}\frac{\overrightarrow{\partial}}{\partial\phi^a} \ . \tag{6.2}$$

This is just the situation in the BV-formalism [10, 11, 44, 33]. In the context of the BV-formalism, $\{\phi^a\}$ consists of usual fields of degree zero, ghost fields of degree one, and ghosts of ghosts with degree two, ... and also so-called antighosts whose degree is defined negative (so that the gauge fixing (Definition 6.3) can be performed in the BV-formalism). For each ϕ^a, its *antifield* ϕ^*_a is then introduced, where its degree is defined as

$$\deg(\phi^*_a) = -1 - \deg(\phi^a) \ .$$

This is equivalent to $\deg(\mathbf{e}_a) + \deg(\mathbf{e}^*_a) = 1$. For a constant symplectic form ω, this fact determines the degree of ω to be minus one. Thus, our definition for the degree of ω (Definition 2.10) is natural also from the viewpoint of the BV-formalism.

The BV-formalism is applied mainly to two cases. Originally [10, 11], it is a general method to quantize gauge-invariant actions consistently. In such a usage one begins with the gauge invariant action which does not include antifields, one adds the terms including antifields to the original action so that the action satisfies the master equation and is proper (Definition 6.2). The BV-quantization of Poisson-σ model in [18] is a good example. On the other hand, it is used to determine higher terms of actions. Namely, starting from an action which consists only a kinetic term, when one includes higher interaction terms as deformation of the action preserving its symmetry, the BV-master equation becomes constraints for the determination of the higher interaction terms. String field theory as reviewed in subsection 1.2 is just the latter case. A similar application to topological field theories is given by [8] and developed for example in [3, 48, 87, 19, 9].

In any case the action in the BV-formalism is, by power series of fields, written as

$$S = \frac{1}{2}\mathcal{V}_{i_1 i_2}\phi^{i_2}\phi^{i_1} + \sum_{k\geq 3}\frac{1}{k}\mathcal{V}_{i_1\cdots i_k}\phi^{i_k}\cdots\phi^{i_1} \ , \qquad \mathcal{V}_{i_1\cdots i_k} \in \mathbb{C} \ , \tag{6.3}$$

where $\{\phi^i\}$ consists of both fields and antifields. We consider the case the action is cyclic $S \in C(\phi)_c$. We call such a field theory *a cyclic field theory.* Moreover suppose that S satisfies the classical BV-master equation

$$(S, S) = 0 \ , \tag{6.4}$$

where (,) is the odd Poisson bracket in eq.(6.2). In this situation we say that the action is equipped with a *classical BV-structure*.

The BV-BRST transformation is then defined by the Hamiltonian vector field of S

$$\delta = (\ , S) \ . \tag{6.5}$$

With this δ, the classical master equation (6.4) is written as $\delta S = 0$. Moreover, by using the Jacobi identity of the BV-bracket and the classical BV-master equation (6.4), $(\delta)^2 = 0$ holds. Namely, the following three statements are equivalent; the action S satisfies the BV-master equation (6.4), the action S is invariant under the BV-BRST transformation (6.5), and the BV-BRST transformation δ is nilpotent.

As stated in subsection 4.5, this δ is nothing but the A_∞-odd vector field. It is written in the form

$$\delta = (\ , S) = \sum_{k=1}^{\infty} \frac{\overleftarrow{\partial}}{\partial \phi^j} c^j_{i_1 \cdots i_k} \phi^{i_k} \cdots \phi^{i_1} \ , \tag{6.6}$$

where $c^j_{i_1 \cdots i_k} = (-1)^{\mathbf{e}_l} \omega^{jl} \mathcal{V}_{l i_1 \cdots i_k}$ (eq.(4.9)). Note that the $\{c^j_{i_1 \cdots i_k}\}$ not only defines an A_∞-structure but also has a cyclic structure since the set $\{\mathcal{V}_{i i_1 \cdots i_k}\}$ defines the cyclic field theory. The algebraic structures of these field theories are the cyclic A_∞-structure in Definition 2.11.

Generally the following fact holds.

Theorem 6.1 *Any field theory has a cyclic A_∞-structure if the action is cyclic, $S \in C(\phi)_c$, and satisfies a classical BV-master equation.*

In Theorem 6.1 we assumed the fields $\{\phi^i\}$ are associative. An example of field theory with a cyclic action is nonabelian single trace gauge theory. Here nonabelian means each field is $N \times N$ matrix for some $N \in \mathbb{N}$ (see subsection 1.4). Single trace then corresponds to $S \in C(\phi)_c$, not $TC(\phi_c)$. If we assume the fields (graded) commutative, the A_∞-structure reduces to an L_∞-structure. Namely, Theorem 6.1 implies that any field theory which consists of usual graded commutative fields has a (cyclic) L_∞-structure if it is equipped with a classical BV-structure (such as in [8, 3, 48, 87, 19, 9]). Of course this fits also the case when the matrix fields are decomposed into graded commutative component fields as mentioned in subsection 1.4.

Usually field theory deals with fields $\{\phi^i\}$, the dual side, where $\mathbb{Z}$-graded vector space $\mathcal{H}$ is implicit. However we can now express the A_∞-structure explicitly by employing the arguments in subsection 3.3. For each field ϕ^i one can define its dual base $\mathbf{e}_i \in \mathcal{H}$ whose degree is minus the degree of ϕ^i. Then one can associate to $\mathcal{V}_{i_1 \cdots i_k}$ for $k \geq 2$ a cyclic multilinear map $\mathcal{V}_k : \mathcal{H} \otimes \cdots \otimes \mathcal{H} \to \mathbb{C}$ as

$$\mathcal{V}_k(\mathbf{e}_{i_1}, \cdots, \mathbf{e}_{i_k}) = \mathcal{V}_{i_1 \cdots i_k} \ .$$

It can also be written as $\mathcal{V}(\mathbf{e}_{i_1}, \cdots, \mathbf{e}_{i_k}) = (-1)^{\mathbf{e}_{i_1}} \omega(\mathbf{e}_{i_1}, m_{k-1}(\mathbf{e}_{i_2}, \cdots, \mathbf{e}_{i_k}))$ (Remark 2.12) since ω is nondegenerate. Also, for the operation $\omega(\ ,\ ) : \mathcal{H} \otimes \mathcal{H} \to \mathbb{C}$ we count the degree minus one by not ‘,′ but ω, and extend naturally the operation to the one over $C(\phi)$. Each term of the action (6.3) is then represented as

$$\begin{aligned}\mathcal{V}_{i_1\cdots i_{k+1}}\phi^{i_{k+1}}\cdots\phi^{i_1} &= (-1)^{\mathbf{e}_{i_1}}\omega_{i_1 j}c^{j}_{i_2\cdots i_{k+1}}\phi^{i_{k+1}}\cdots\phi^{i_2}\cdot\phi^{i_1}\\ &= \phi^{i_1}\omega(\mathbf{e}_{i_1},\mathbf{e}_j c^{j}_{i_2\cdots i_{k+1}})\phi^{i_{k+1}}\cdots\phi^{i_2}\\ &= \omega(\mathbf{e}_{i_1}\phi^{i_1}, m_k(\mathbf{e}_{i_2}\phi^{i_2},\cdots,\mathbf{e}_{i_{k+1}}\phi^{i_{k+1}}))\\ &= \omega(\Phi, m_k(\Phi,\cdots,\Phi))\ ,\end{aligned}$$

and the action is given by eq.(5.27),

$$S=\frac{1}{2}\omega(\Phi,Q\Phi)+\sum_{k\geq 2}\frac{1}{k+1}\omega(\Phi,m_k(\Phi))\ . \tag{6.7}$$

From the form of the kinetic term one can see that the degrees of Q, ω and Φ are assigned consistently. Of course classical open string field theory constructed as in subsection 1.2 also takes this form. In this case, $\Phi=\mathbf{e}_i\phi^i$ is just the string field where $\{\mathbf{e}_i\}$ is the basis of the string Hilbert space $\mathcal{H}$ and ϕ^i are its coordinate whose degree is minus the degree of $\mathbf{e}_i$. The inner product ω is known as BPZ-inner product.

6.2 Gauge fixing in the BV-formalism

To proceed the path-integral in the BV-formalism, it is necessary to fix a gauge. Roughly speaking, the gauge fixing corresponds to killing the degree of freedom of gauge transformations. When choose a gauge fixing, one can construct a propagator canonically. The aim of this subsection is to explain these facts and to show that the propagator is just a homotopy operator Q^+ as in the previous section (Proposition 6.5). The statement of the path-integral (perturbative expansion) given in this subsection is formal. We shall write down the definition explicitly in the next subsection.

Formally, given an action $S\in C(\phi)_c$ as in eq.(6.7), the starting point of the path-integral is the partition function of the field theory,

$$Z=\int\mathcal{D}\Phi\ e^{-S}\ . \tag{6.8}$$

Let us separate the action into the quadratic term and others. We denote the quadratic term by $S_2=\frac{1}{2}\omega(\Phi,Q\Phi)$ and the rest terms by $S_{int}:=s_3+s_4+\cdots$. Then we have $e^{-S}=e^{-S_{int}}e^{-S_2}$ in eq.(6.8). In perturbation theory, the partition function (6.8) is computed by perturbative expansion, which is essentially the Gaussian integral of e^{-S_2} where $e^{-S_{int}}$ is Taylor expanded. Here we should define the integration $\int\mathcal{D}\Phi$ in some sense. In fact, the Hessian of the kinetic term S_2 is degenerate and one cannot integrate over the whole space Φ by perturbative expansion.

For this purpose, the properties of the kinetic term should be examined. The degeneracy is directly related to the gauge transformation. For the BV-BRST transformation of the string field, $\delta\Phi=\sum_{k\geq 1}m_k(\Phi)=\mathfrak{m}_*(e^{\Phi})$, the gauge transformation is defined correspondingly as a degree zero transformation δ_α given by

$$\begin{aligned}\delta_\alpha\Phi &= \mathfrak{m}_*(e^\Phi\alpha e^\Phi)\\ &:= Q\alpha + m_2(\alpha,\Phi) + m_2(\Phi,\alpha) + m_3(\alpha,\Phi,\Phi) + m_3(\Phi,\alpha,\Phi) + m_3(\Phi,\Phi,\alpha) + \cdots\ ,\end{aligned} \tag{6.9}$$

where $\alpha = \mathbf{e}_i\alpha^i$ is a gauge parameter of degree minus one. More precisely, α^i is treated as a graded parameter that belongs to $\mathcal{H}^*[1]$ and can be identified with $\alpha^i = \phi^i[1]$. The degree of α^i is then minus the degree of $\mathbf{e}_i$ minus one. One may notice that this is the gauge transformation for A_∞-algebras discussed in subsection 7.2. We shall discuss more on the properties of gauge transformations there. The gauge transformation is written as $\delta_\alpha\Phi = m_*(e^\Phi)\frac{\overleftarrow{\partial}}{\partial\phi^i}\alpha^i$, and in the language of the component fields, it is

$$\delta_\alpha = \sum_{k=1}^{\infty}\frac{\overleftarrow{\partial}}{\partial\phi^j}c^j_{i_1\cdots i_k}\left(\phi^{i_k}\cdots\phi^{i_1}\frac{\overleftarrow{\partial}}{\partial\phi^i}\alpha^i\right)\ . \tag{6.10}$$

By standard arguments in the BV-formalism [11, 44], one sees the following facts. First, the action is invariant under this δ_α because $0 = (S,S)\frac{\overleftarrow{\partial}}{\partial\phi^i}\alpha^i$ implies $\delta_\alpha S = 0$. Moreover,

$$0 = \omega_{kj}\frac{\overrightarrow{\partial}}{\partial\phi^j}(S,S)\frac{\overleftarrow{\partial}}{\partial\phi^i}\alpha^i\Bigg|_{\frac{\partial S}{\partial\phi}=0} = 2\left(\omega_{kj}\frac{\overrightarrow{\partial}}{\partial\phi^j}S\frac{\overleftarrow{\partial}}{\partial\phi^l}\right)\left(\omega^{lm}\frac{\overrightarrow{\partial}}{\partial\phi^m}S\frac{\overleftarrow{\partial}}{\partial\phi^i}\right)\alpha^i\Bigg|_{\frac{\partial S}{\partial\phi}=0} \tag{6.11}$$

indicates that the generator of the gauge transformation is degenerate and the rank of the Hessian for the quadratic part of the action S_2 is less than half of the number of the basis $\{\mathbf{e}_i\}$ on the space $\{\phi|\frac{\partial S}{\partial\phi} = 0\}$, though the number of the basis is infinity. The origin $\phi = 0$ is also the solution for $\{\phi|\frac{\partial S}{\partial\phi} = 0\}$, and eq.(6.11) at the origin is nothing but the condition $(Q)^2 = 0$. Hereafter for simplicity we choose the origin for the solution for $\{\phi|\frac{\partial S}{\partial\phi} = 0\}$ (without loss of generality).

Definition 6.2 (Proper) At the origin $\phi = 0$ if the ratio of the rank of the Hessian over the number of the basis is just half, the action is called *proper* (see [10, 11, 44, 33]). (This is a traditional definition, but when we say an action is proper, we assume an additional condition stated later in eq.(6.15).)

The Hessian at the origin is $\mathcal{V}_{i_1i_2}$ in eq.(6.3), which is determined by Q. Let us consider the decomposition (5.2) $\mathcal{H} = \mathcal{H}^t \oplus \mathcal{H}^u \oplus \mathcal{H}^p$. The rank of the Hessian is equal to the rank of unphysical states $\mathcal{H}^u$ which generate the gauge transformations. rank($\mathcal{H}^u$) is equal to rank($\mathcal{H}^t$), where $\mathcal{H}^t$ is Q-trivial states. The condition of the proper is then equivalent to the condition that rank($\mathcal{H}^u$)/rank($\mathcal{H}$) $= \frac{1}{2}$. Note that it does not imply that rank($\mathcal{H}^p$) $= 0$. When an action is proper, $\mathcal{H}^p$ corresponds to so-called the Green kernel and rank($\mathcal{H}^p$)/rank($\mathcal{H}^u$) $= 0$ holds. String field theory is also proper where Q is the BRST-operator [58] of conformal field theory on a fixed background. (The reducibility of the gauge group of string field theory action then comes from the Virasoro symmetry of Q.)

Given a proper action, the gauge fixing and the path-integral measure $\mathcal{D}\Phi$ are defined as follows.

Definition 6.3 (Gauge fixing) Let us consider a degree minus one element Ψ in $C(\phi)_c$, which is called *the gauge fixing fermion.* The power of fields for Ψ is assumed to be greater than or equal to two. A *BV-gauge fixing* is defined as restriction of antifields to the lagrangian submanifold $\phi_a^* = \frac{\partial\Psi}{\partial\phi^a}$. The path-integral measure in eq.(6.8) is then the integration over the space of fields $\{\phi^a\}$, the dual of the graded vector space $\mathcal{H}_+$. We denote it by $\mathcal{D}\Phi_{gf}$.

Thus, choosing Ψ determines the gauge fixing. In the original context of the BV-formalism, restricting the antifields to zero ($\Psi = 0$) recovers the original action that consists of only degree zero fields, where the rank of the Hessian is possibly less than the rank of the fields. We call it the trivial gauge. The gauge fixing is then performed by shifting the trivial gauge $\phi_a^* = 0$ to $\phi_a^* = \frac{\partial\Psi}{\partial\phi^a}$ so that the rank of the Hessian is equal to the rank of the fields, *i.e.* half of the rank of the total space $\mathcal{H}$. In case of string field theories, however, the antifields are originally included in the quadratic term S_2, and BV-master equation is used in order to determine the form of higher vertices. Therefore the trivial gauge fixing can also be a candidate for consistent gauge fixing. This trivial gauge is called *Siegel gauge* in string field theory.

Let us now examine some properties of the kinetic term of a proper action. We write $Q\mathbf{e}_j = \mathbf{e}_k c_j^k$ and in matrix expression

$$c := \{c_j^k\} = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} .$$

The kinetic term $\mathcal{V}_{ij}$ is then written as

$$\omega c = \begin{pmatrix} -c_3 & -c_4 \\ c_1 & c_2 \end{pmatrix} . \tag{6.12}$$

$\mathcal{V}_{ij}$ is graded symmetric since vertices are defined to be cyclic. This implies that c satisfies

$$(\omega c)_{ji} = (-1)^{\mathbf{e}_i}(\omega c)_{ij}$$

where ω is the one in eq.(6.1). When we write † for the transpose with the sign factor, the above equation becomes

$$\omega c = (\omega c)^\dagger .$$

One then obtains that $c_2 = c_2^\dagger$, $c_3 = c_3^\dagger$ and $c_4 = -c_1^\dagger$.

Alternatively, when fixing a gauge, one can bring the gauge fixing condition $\phi_a^* = \frac{\partial\Psi}{\partial\phi^a}$ to the form $\phi_a^* = 0$ by a coordinate transformation of the form

$$\Phi' = \Phi + \Phi\frac{\overleftarrow{\partial}}{\partial\phi_a^*}\frac{\overrightarrow{\partial}\Psi}{\partial\phi^a} .$$

The second term in the right hand side is written as $-(\Phi, \Psi)$. This transformation preserves the symplectic form, since it is a special case of the transformation eq.(4.5), that is,

$$\Phi' = e^{(\ ,\Psi)}\Phi . \tag{6.13}$$

For the kinetic term ωc, only the linear part of the coordinate transformation is relevant. When we represent the gauge fixing fermion as $\Psi_a = \phi^a \psi_{ab} \phi^b + \cdots$, the linear part of the transformation is as follows

$$\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \psi & \mathbf{1} \end{pmatrix} .$$

In this coordinate the kinetic term is just $-c_3$ since the gauge fixing is $\phi_a^* = 0$. The rank of c_3 in eq.(6.12) is then half of the rank of total space.

Though c_3 is degenerate because of the Green kernel generally, let us first consider the case c_3 is nondegenerate, where the cohomology with respect to $Q = m_1$ is trivial. In this situation, the condition $c^2 = 0$ implies that c can be written as

$$\omega c = T^\dagger \begin{pmatrix} -c_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} T \ , \qquad T := \begin{pmatrix} \mathbf{1} & (c_3)^{-1} c_1 \\ \mathbf{0} & \mathbf{1} \end{pmatrix} .$$

Note that T preserves the BV-symplectic form in the Darboux coordinate

$$T^\dagger \begin{pmatrix} \mathbf{0} & -\mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} T = \begin{pmatrix} \mathbf{0} & -\mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} . \tag{6.14}$$

By the definition of proper (Definition 6.2) it is natural that the properties of the kinetic term above holds also in general situations. Namely, when we say an action is proper, we assume that, in the coordinate where the gauge fixing is $\phi_a^* = 0$, there exists a linear transformation

$$T = \begin{pmatrix} \mathbf{1} & t \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \ , \qquad t - t^\dagger = 0$$

which preserves the symplectic form (6.14) and transforms the kinetic term $\{c_j^k\}$ of the form

$$\begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} = T^{-1} \begin{pmatrix} 0 & 0 \\ c_3 & 0 \end{pmatrix} T \ . \tag{6.15}$$

One can see that $\mathcal{H}^p$ is just linearly isomorphic to two copies of the kernel of c_3. $t = (c_3)^{-1} c_1$ is a solution of eq.(6.15). On the other hand, $c^2 = 0$ leads to $P_L c_4 = c_4$ and $c_1 P_L = c_1$, and in addition $P_L c_1 = c_1$ and $c_4 P_L = c_4$ hold if eq.(6.15) is satisfied.

Given a gauge fixing Ψ, a propagator is constructed canonically as follows.

Definition 6.4 (Propagator in BV-formalism) Let $\Psi \in C(\phi)_c$ be a gauge fixing fermion. By this gauge fixing, the quadratic term of the action is written in terms of fields only (not antifields) as $\frac{1}{2}\phi^a (\mathcal{V}_{gf})_{ab} \phi^b$ for some graded symmetric (*i.e.* cyclic) matrix $\mathcal{V}_{gf}$. It is also regarded as a bilinear map $\mathcal{V}_{gf} : \mathcal{H}_+ \otimes \mathcal{H}_+ \to \mathbb{C}$ such that $\mathcal{V}_{gf}(\mathbf{e}_a, \mathbf{e}_b) = (\mathcal{V}_{gf})_{ab}$. The gauge fixing fermion is taken so that the rank of $\mathcal{V}_{gf}$ is maximal. Note that $\mathcal{V}_{gf}$ is degenerate only for on-shell states. The degeneracy corresponding to gauge orbits (orbits of the gauge transformations) is killed by the gauge fixing. Let $P_{gf} : \mathcal{H}_+ \to \mathcal{H}_+$ be the projection onto the kernel of $\mathcal{V}_{gf}$. The *BV-propagator* $\mathcal{V}_{gf}^+$ is then given by the inverse of $\mathcal{V}_{gf}$ such that

$$\mathcal{V}_{gf}^+ \mathcal{V}_{gf} = \mathcal{V}_{gf} \mathcal{V}_{gf}^+ = \mathbf{1} - P_{gf} \tag{6.16}$$

on $\mathcal{H}_+$ in the matrix expression.

Proposition 6.5 *A propagator in the BV-formalism is a homotopy operator of* $(\mathcal{H}, Q)$*.*

proof. As stated previously, in the coordinate where the gauge fixing is $\phi^* = 0$, the gauge fixed kinetic term is $\mathcal{V}_{gf} = -c_3$. The propagator $\mathcal{V}_{gf}^+ \in \mathcal{H}_+ \otimes \mathcal{H}_+$ given in Definition 6.4 is naturally extended to the one in $\mathcal{H} \otimes \mathcal{H}$. We denote it by $\mathcal{V}_L^+$. Let us define a degree one operator $Q^+ : \mathcal{H} \to \mathcal{H}$, $Q^+(\mathbf{e}_i) = \mathbf{e}_j \bar{c}_i^j$ in matrix expression by

$$\bar{c} := \mathcal{V}_L^+ \omega = \begin{pmatrix} \mathcal{V}_{gf}^+ & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \omega = \begin{pmatrix} \mathbf{0} & \mathcal{V}_{gf}^+ \\ \mathbf{0} & \mathbf{0} \end{pmatrix} . \tag{6.17}$$

Q^+ then satisfies the Hodge-Kodaira decomposition (5.1)

$$QQ^+ + Q^+Q + P = \mathbf{1} , \tag{6.18}$$

where $P : \mathcal{H} \to \mathcal{H}$ is the projection onto the on-shell state. That is, Q^+ is a homotopy operator. This is because the action is proper. More precisely, one can make a coordinate transformation T in eq.(6.15) where the kinetic term ωc is transformed to be of the form

$$\begin{pmatrix} -c_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} .$$

Note that T preserves the matrix (6.17) corresponding to the homotopy operator Q^+. In this coordinate it is clear that eq.(6.18) holds. ■

6.3 Path integral, Feynman diagram and the minimal model theorem

In subsection 5.4, we obtained an explicit form of the minimal model $(\mathcal{H}^p, \tilde{\omega}^p, \tilde{\mathfrak{m}}^p)$ of cyclic A_∞-algebra $(\mathcal{H}, \omega, \mathfrak{m})$. There, a cyclic A_∞-morphism $\tilde{\mathcal{F}}^p$ from minimal cyclic A_∞-algebra $(\mathcal{H}^p, \tilde{\omega}^p, \tilde{\mathfrak{m}}^p)$ to $(\mathcal{H}, \omega, \mathfrak{m})$ was also constructed. Here let $(\mathcal{H}, \omega, \mathfrak{m})$ be the cyclic A_∞-algebra of a field theory whose action is proper (Definition 6.2). Then we can see that the n-point vertex defined by A_∞-structure $\tilde{\mathfrak{m}}^p$ is nothing but the tree level n-point correlation function of the field theory(Lemma 6.13).

This subsection is devoted to show that the scattering amplitudes of the field theory computed by the Feynman rule coincides with $\pm\frac{1}{n}\omega(\mathbf{e}_{i_1}^p, \tilde{m}_{n-1}^p(\mathbf{e}_{i_2}^p, \cdots, \mathbf{e}_{i_n}^p))$ where $\mathbf{e}_{i_1}^p, \cdots, \mathbf{e}_{i_n}^p \in \mathcal{H}^p$ are the external states of the amplitude.

Let us first write down the explicit definition of the perturbative expansion. It is obtained by fixing the gauge, constructing the propagator and eq.(6.8). Let $\mathcal{O}(\Phi) \in TC(\phi)_c$ be any operators. Then its path-integral is formally given by

$$\langle \mathcal{O}(\Phi) \rangle \sim \int \mathcal{D}\Phi \ \mathcal{O}(\Phi) e^{-S}\Big|_{gf} = \int \mathcal{D}\Phi_{gf} \ \mathcal{O}(\Phi_{gf}) e^{-\sum_{k\geq 3} \frac{1}{k}\mathcal{V}_k(\Phi_{gf}, \cdots, \Phi_{gf})} e^{-S_2|_{gf}} , \tag{6.19}$$

where $|_{gf}$ denotes a gauge fixing (Definition 6.3) in the previous subsection, Φ_{gf} denotes the gauge fixed Φ, and $S_2|_{gf} = \frac{1}{2}\omega(\Phi_{gf}, Q\Phi_{gf})$. The path integral (6.19) is not well-defined precisely since we do not define the integral $\mathcal{D}\Phi_{gf}$ on a formal noncommutative supermanifold. Instead, we shall define the path integral precisely at the level of the perturbative expansion below. Since

$S_2|_{gf}$ is quadratic with respect to Φ_{gf}, $e^{-S_2|_{gf}}$ is a gaussian. Then the perturbative expansion is essentially the gaussian integral where $e^{-\sum_{k\geq 3}\frac{1}{k}\mathcal{V}_k(\Phi_{gf},\cdots,\Phi_{gf})}$ is Taylor expanded. In field theory it is calculated by so-called Wick contraction. One can rewrite it in a purely algebraic manner as follows. We take it as the starting point of our definition.

Definition 6.6 (Perturbative expansion) For $\mathcal{O}(\Phi)\in TC(\phi)_c$, The perturbative expansion of $\mathcal{O}(\Phi)$ is defined by

$$\langle\mathcal{O}(\Phi)\rangle=\left(\mathcal{O}(\Phi)\cdot e^{-\sum_{k\geq 3}\frac{1}{k}\mathcal{V}_k(\Phi,\cdots,\Phi)}\right)e^{\left(\frac{1}{2}\mathcal{V}_L^{+,ij}\frac{\overleftarrow{\partial}}{\partial\phi^i}\frac{\overleftarrow{\partial}}{\partial\phi^j}\right)}\Bigg|_{\Phi=0}\ , \tag{6.20}$$

where $\mathcal{V}_L^+$ is the propagator constructed in the gauge $|_{gf}$ as in Definition 6.4 and the below. Here the derivation $\frac{\overleftarrow{\partial}}{\partial\phi^i}$ acts on $\left(\mathcal{O}(\Phi)\cdot e^{-\sum_{k\geq 3}\frac{1}{k}\mathcal{V}_k(\Phi,\cdots,\Phi)}\right)$ from right with an appropriate Kostul sign.

The above expression is the one derived directly from eq.(6.19) only if the gauge fixing fermion is quadratic $\Psi=\phi^a\psi_{ab}\phi^b$ (since otherwise $\mathcal{V}_k(\Phi_{gf},\cdots,\Phi_{gf})$ in eq.(6.19) includes terms higher than k powers of fields $\{\phi^a\}$). However, as stated in the previous subsection, a gauge fixing is equivalent to a particular field transformation preserving the symplectic form; perturbative expansion with gauge fixing Ψ is performed by trivial gauge $\Psi=0$ after field redefinition (6.13). Thus, we indicate by $\mathcal{V}_k(\Phi,\cdots,\Phi)$ in eq.(6.20) the term of k powers after the such a field redefinition associated to Ψ. The dependence or independence of the choice of the gauge fixing will be stated in the end of this subsection.

The perturbative expansion (6.20) gives a well-defined linear map $\langle\cdots\rangle : TC(\phi)_c\to\mathbb{C}$. Especially the path integral above reduces to the ordinary one if the fields are (graded) commutative. [18] The value (6.20) is calculated by *Feynman rules* as follows. Let us consider eq.(6.20) in the case when $\mathcal{O}(\Phi)=a_1(\phi)\bullet\cdots\bullet a_n(\phi)$ where $a_r(\phi):=\frac{1}{k_r}a_{i_1^r\cdots i_{k_r}^r}\phi^{i^r_{k_r}}\cdots\phi^{i^r_1}\in C(\phi)_c$ for each $1\leq r\leq n$. We call each $a_r(\phi)$ an *observable vertex* and assign o_r to it. Moreover we associate o^r_a, $1\leq a\leq k_r$ to each field in $a_r(\Phi)$. On the other hand, $e^{-\sum_{k\geq 3}\frac{1}{k}\mathcal{V}_k(\Phi,\cdots,\Phi)}$ is Taylor expanded. Each term is characterized by a multiindex $\Lambda:=\{\lambda_3\in\mathbb{Z}_{\geq 0},\lambda_4\in\mathbb{Z}_{\geq 0},\cdots\}$ and given by

$$\Pi_{l\geq 3}\frac{1}{\lambda_l!}\left(-\frac{1}{l}\mathcal{V}_l(\Phi)\right)^{\lambda_l}=\frac{1}{\lambda_3!}\left(-\frac{1}{3}\mathcal{V}_3(\Phi)\right)^{\lambda_3}\bullet\frac{1}{\lambda_4!}\left(-\frac{1}{4}\mathcal{V}_4(\Phi)\right)^{\lambda_4}\bullet\cdots\ .$$

It is a product of $m:=(\sum_{l\geq 3}\lambda_l)$ numbers of vertices. Namely, the equation above is

$$\mathcal{V}_{e_1}(\Phi)\bullet\cdots\bullet\mathcal{V}_{e_m}(\Phi)$$

up to an appropriate coefficient, where $3\leq e_1\leq\cdots\leq e_m$. To each $\mathcal{V}_{e_q}(\Phi)$, $1\leq q\leq m$, we assign a *vertex* v_q. Moreover we associate v^q_a, $1\leq a\leq e_q$ to each field in $\mathcal{V}_{e_q}(\Phi)$. We indicate

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[18] For the Feynman rule in operator language, see [53]. Comparing it from the argument here in component field theory picture, one can see the equivalence between them. The reference [117] also provides us with useful informations.

both observable vertices and vertices by (observable) vertices. The set of fields of (observable) vertices is denoted by

$$Vert(\mathcal{O},\Lambda) := \{o^1_1, \cdot\cdot, o^1_{k_1}, \cdots, o^n_1, \cdot\cdot, o^n_{k_n}, v^1_1, \cdot\cdot, v^1_{e_1}, \cdots, v^m_1, \cdot\cdot, v^m_{e_m}\} \ .$$

In this situation, consider the value

$$\left(\mathcal{O}(\Phi) \bullet \Pi_{l\geq 3} \frac{1}{\lambda_l!}\left(-\frac{1}{l}\mathcal{V}_l(\Phi)\right)^{\lambda_l}\right) e^{\left(\frac{1}{2}\mathcal{V}_L^{+,ij}\frac{\overleftarrow{\partial}}{\partial\phi^i}\frac{\overleftarrow{\partial}}{\partial\phi^j}\right)}\Bigg|_{\Phi=0} \ .$$

The value vanishes if $(k_1 + \cdots + k_n) + (e_1 + \cdots + e_m)$ is odd. When it is even, the equation above is equal to

$$\left(\mathcal{O}(\Phi) \bullet \Pi_{l\geq 3} \frac{1}{\lambda_l!}\left(-\frac{1}{l}\mathcal{V}_l(\Phi)\right)^{\lambda_l}\right) \frac{1}{2^{I'}I'!}\left(\mathcal{V}_L^{+,ij}\frac{\overleftarrow{\partial}}{\partial\phi^i}\frac{\overleftarrow{\partial}}{\partial\phi^j}\right)^{I'} \tag{6.21}$$

where $2I' = (k_1 + \cdots + k_n) + (e_1 + \cdots + e_m)$. We have $2I'$ differentials that act on (observable) vertices. We assign ed^p_a, $1 \leq p \leq I'$, $a = 1, 2$ to each differential and denote the set by

$$Edge(I') := \{ed^1_1, ed^1_2, ed^2_1, ed^2_2, \cdots, ed^{I'}_1, ed^{I'}_2\} \ .$$

Consider isomorphisms from the set $Edge(I')$ to the set $Vert(\mathcal{O},\Lambda)$. We have $(2I')!$ such isomorphisms. Denote the set of the isomorphisms by $\tilde{F}(\mathcal{O},\Lambda)$.

Then eq.(6.20) is represented in the following form

$$\Pi_{l\geq 3}\frac{1}{\lambda_l!}\frac{1}{l^{\lambda_l}}\frac{1}{2^{I'}I'!} \cdot \sum_{\tilde{\Upsilon}(\mathcal{O},\Lambda)\in\tilde{F}(\mathcal{O},\Lambda)} N_{\tilde{\Upsilon}(\mathcal{O},\Lambda)} \ , \tag{6.22}$$

where $N_{\tilde{\Upsilon}(\mathcal{O},\Lambda)} \in \mathbb{C}$ is given by the product of $a_{i^r_1\cdots i^r_{k_r}}$, $\mathcal{V}_{i^q_1\cdots i^q_{e_q}}$, $\mathcal{V}_L^{+,ij}$ and the Kostul sign factor. One can consider two actions on the set $Edge(I')$; the exchange between ed^p_1 and ed^p_2 for each p, and the exchange between the pair (ed^p_1, ed^p_2) and $(ed^{p'}_1, ed^{p'}_2)$ for any $p \neq p'$. One can also consider an action on $Vert(\mathcal{O},\Lambda)$, the cyclic permutations in $o^r_1, \cdots, o^r_{k_r}$ for each r or in $v^q_1, \cdots, v^q_{e_q}$ for each q. These actions induce automorphisms on $\tilde{F}(\mathcal{O},\Lambda)$ and $N_{\tilde{\Upsilon}(\mathcal{O},\Lambda)}$ is independent of the automorphisms. Let us denote by $F(\mathcal{O},\Lambda)$ the set of isomorphisms $\tilde{F}(\mathcal{O},\Lambda)$ over these automorphisms. Moreover we introduce the direct sum, $F(\mathcal{O}) := \bigoplus_{\Lambda} F(\mathcal{O},\Lambda)$.

The perturbative expansion eq.(6.20) is the sum of eq.(6.22) with respect to Λ. It is rewritten as

$$\langle\mathcal{O}(\Phi)\rangle = \sum_{\Upsilon(\mathcal{O})\in F(\mathcal{O})} \frac{1}{\Lambda!} N_{\Upsilon(\mathcal{O})} \ , \tag{6.23}$$

where $\Lambda! := \Pi_{l\geq 3}\frac{1}{\lambda_l!}$. Note that $\frac{1}{l^{\lambda_l}}\frac{1}{2^{I'}I'!}$ in eq.(6.22) is canceled by the automorphisms. Associated to each $\Upsilon(\mathcal{O}) \in F(\mathcal{O},\Lambda)$, we define Feynman graphs for a cyclic field theory $(\mathcal{H},\omega,S)$ as follows.

Definition 6.7 (Feynman graph) In the situation above, let us arrange the observable vertices o_r and vertices v_q from left to right on a plane such as

$$\bullet_{o_1} \quad \bullet_{o_2} \quad \cdots \quad \bullet_{o_n} \quad \bullet_{v_1} \quad \cdots \quad \bullet_{v_{m-1}} \quad \bullet_{v_m} \quad .$$

Moreover, connect any two (observable) vertices to each other by edges so that the number of incident edges is k_r at observable vertex o_r and e_q at vertex v_q. We call such graphs the *Feynman graphs* for a cyclic field theory $(\mathcal{H}, \omega, S)$. Hereafter we identify an element $\Upsilon(\mathcal{O}) \in F(\mathcal{O})$ with a Feynman graph. The cyclic order of the edges around each (observable) vertex is distinguished. Two edges that intersect on the plane can pass through each other and two graphs before and after this process are not distinguished.

Connecting two (observable) vertices by an edge is called the *Wick contraction.* The contraction indicates that two differentials in $\frac{1}{2}\mathcal{V}_L^{+,ij}\frac{\overleftarrow{\partial}}{\partial\phi^i}\frac{\overleftarrow{\partial}}{\partial\phi^j}$ act on the fields in the two (observable) vertices.

Now we are interested in tree on-shell amputated amplitudes.

Definition 6.8 (Feynman graphs for n-point amplitudes) An n-point amplitude is calculated by eq.(6.23) with $a_r(\Phi) = \phi^{j_r}$, that is, $\mathcal{O}(\Phi) = \phi^{j_1} \bullet \cdots \bullet \phi^{j_n}$. Each ϕ^{j_r} is called an *external field.* We call the corresponding vertex a *external vertex.* A Feynman graph $\Upsilon(\mathcal{O}) \in F(\mathcal{O})$ is called *connected* if any two (observable) vertices of $\Upsilon(\mathcal{O})$ is connected to each other by the edges. When $\Upsilon(\mathcal{O})$ includes a circle S^1 consisting of the edges, the S^1 is called a *loop.* The set of connected Feynman graphs is denoted by $F^{conn}(\mathcal{O}) \subset F(\mathcal{O})$. Moreover we denote the set of every connected tree Feynman graphs by $\mathcal{T}(\mathcal{O}) \subset F^{conn}(\mathcal{O})$, where a *tree* Feynman graph means a Feynman graph without loops. The value

$$\langle \mathcal{O}(\Phi) \rangle|_{conn} := \sum_{\Upsilon(\mathcal{O}) \in F^{conn}(\mathcal{O})} \frac{1}{\Lambda!} N_{\Upsilon(\mathcal{O})}$$

is then the *n-point amplitude.* Moreover, restricting the Feynman graphs in $F^{conn}(\mathcal{O})$ to those in $\mathcal{T}(\mathcal{O})$ one gets the *n-point tree amplitude*,

$$\langle \mathcal{O}(\Phi) \rangle|_{\substack{conn \\ tree}} := \sum_{\Upsilon(\mathcal{O}) \in \mathcal{T}(\mathcal{O})} \frac{1}{\Lambda!} N_{\Upsilon(\mathcal{O})} \ .$$

For a connected tree Feynman graph, the Wick contraction by the edges can be divided into two processes; the contractions between n external fields $\phi^{j_1} \bullet \cdots \bullet \phi^{j_n}$ and the vertices, and the contractions between the vertices. As explained below, the latter process produces some function of n powers of Φ that associates to planar tree graphs (Definition 5.19 below). We shall define cyclic functions associated to the planar tree graphs in Definition 6.9 as we defined the multilinear map $\tilde{m}_{\Gamma_k}$ associated to rooted planar k-tree. After the latter process, the former one, the contractions of the cyclic function with n external fields, finishes the calculation for the value associated to the Feynman graph.

Definition 6.9 (Cyclic function associated to planar graphs) Let G_n^{cyc} be the set of planar tree graphs with n-leaves. An element $\Gamma_n^{cyc} \in G_n^{cyc}$ is a rooted planar $(n-1)$-tree without the distinction of the root edge and the leaves, that is, the root edge is regarded as a leaf. [19] Denote the natural surjection by $\check{r} : G_{n-1} \to G_n^{cyc}$. Now we regard G_{n-1} and G_n^{cyc} as vector spaces and denote them also by themselves. Namely, for the set G_{n-1} (resp. G_n^{cyc}), their elements are regarded as the bases of the vector space G_{n-1} (resp. G_n^{cyc}). $\check{r} : G_{n-1} \to G_n^{cyc}$ is then regarded as a vector bundle. For an element $\Gamma_n^{cyc} \in G_n^{cyc}$, there exist n choices to pick up one of the leaves as the root edges. Summing over these n numbers of $(n-1)$-trees and dividing by n defines a section $s : G_n^{cyc} \to G_{n-1}$.

First we define a $(\mathcal{H})^{\otimes k} \to \mathcal{H}$ valued linear function $\tilde{m}_k^{cyc}$ on vector space G_k in a similar way as in Definition 5.20. To an elementary k-tree we associate $m_k : \mathcal{H}^{\otimes k} \to \mathcal{H}$. For any k-tree Γ_k denote the associated endomorphisms by $\tilde{m}_{\Gamma_k}^{cyc} : (\mathcal{H})^{\otimes k} \to \mathcal{H}$. For a grafting $\Gamma_k \circ_i \Gamma_l$ we associate

$$\tilde{m}_{\Gamma_k \circ_i \Gamma_l}^{cyc} = \tilde{m}_{\Gamma_k}^{cyc} \circ \left(\mathbf{1}^{\otimes(i-1)} \otimes \tilde{f}_{\Gamma_l} \otimes \mathbf{1}^{\otimes(k-i)} \right) : \mathcal{H}^{\otimes(k+l-1)} \to \mathcal{H} \ ,$$

where $\tilde{f}_{\Gamma_l} : (\mathcal{H})^{\otimes l} \to \mathcal{H}$ are those in Definition 5.20. $\tilde{m}_{\Gamma_k}^{cyc}$ are then defined so that they are compatible with this grafting. Note that $\tilde{m}_{\Gamma_k} = \pi \circ \tilde{m}_{\Gamma_k}^{cyc}$, that is, removing π on the root edge in Definition 5.20 leads to $\tilde{m}^{cyc}$.

$(\mathcal{H})^{\otimes n} \to \mathbb{C}$ valued cyclic function on vector space G_n^{cyc} is then defined by

$$\frac{1}{n} \tilde{\mathcal{V}}_{\Gamma_n^{cyc}} = (s^* \tilde{m}^{cyc})(\Gamma_n^{cyc}) := \omega \circ (\mathbf{1} \otimes (\tilde{m}^{cyc}(s(\Gamma_n^{cyc})))) \ .$$

Definition 6.10 ((Amputated) tree on-shell scattering amplitude) For each element in $\mathcal{T}(\mathcal{O})$, $\mathcal{O}(\Phi) = \phi^{j_1} \bullet \cdots \bullet \phi^{j_n}$, remove the external vertices together with the edges whose one end is the external vertices. One gets a tree graph with n free ends. Denote by $\mathcal{T}_n$ the set of the graphs obtained in such a way. We denote the surjection by $Ampu : \mathcal{T}(\mathcal{O}) \to \mathcal{T}_n$ and call it the *amputation map.* It is an $n!$-to-one map. We regard $\mathcal{T}(\mathcal{O})$ and $\mathcal{T}_n$ also as vector spaces. Any element in $\mathcal{T}_n$ is isomorphic to an element in G_n^{cyc} as a planar tree graph. Thus we have a surjection $t : \mathcal{T}_n \to G_n^{cyc}$.

$$\begin{tikzcd}
 & & G_{n-1} \arrow[d, shift left, "\check{r}"] \\
F(\mathcal{O}) \supset \mathcal{T}(\mathcal{O}) \arrow[r, "Ampu"] & \mathcal{T}_n \arrow[r, "t"] & G_n^{cyc} \arrow[u, shift left, "s"]
\end{tikzcd}$$

The *(amputated) tree correlation functions* for a cyclic field theory $(\mathcal{H}, \omega, S)$ is the collection of the following cyclic functions $\{\tilde{\mathcal{V}}_v\}_{n \geq 3}$,

$$\tilde{\mathcal{V}}_n := \sum_{\Upsilon_n \in \mathcal{T}_n} \frac{1}{\Lambda!} \tilde{\mathcal{V}}_{t(\Upsilon_n)} \ . \tag{6.24}$$

The *tree on-shell correlation functions* is given by

$$\tilde{\mathcal{V}}_n^p := \tilde{\mathcal{V}}_n \circ (\iota)^{\otimes n} \ : (\mathcal{H}^p)^{\otimes n} \to \mathbb{C} \ .$$

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[19] Here a notation is changed compared to that in [53]. Γ_n^{cyc} here is denoted by Γ_{n-1}^{cyc} in [53].

It is called also the *tree on-shell scattering amplitudes* or the *tree on-shell S(cattering)-matrices.*

Remark 6.11 The scattering amplitudes are usually defined on the gauge fixed subspace of $\mathcal{H}$. In this sense the scattering amplitude in the Definition above is an extended one that is defined on the whole graded vector space $\mathcal{H}$.

Lemma 6.12 *For $\mathcal{O}(\Phi)=\phi^{j_1}\bullet\cdots\bullet\phi^{j_n}$ and for a fixed $\Upsilon_n\in\mathcal{T}_n$, we have*

$$\mathcal{V}_L^{+,j_1j_1'}\frac{\overrightarrow{\partial}}{\partial\phi^{j_1'}}\cdots\mathcal{V}_L^{+,j_nj_n'}\frac{\overrightarrow{\partial}}{\partial\phi^{j_n'}}\cdot\left(-\frac{1}{n}\tilde{\mathcal{V}}_{t(\Upsilon_n)}(\Phi,\cdots,\Phi)\right)=\sum_{\Upsilon(\mathcal{O})\in Ampu^{-1}(\Upsilon_n)}N_{\Upsilon(\mathcal{O})}\ . \qquad (6.25)$$

Consequently,

$$\sum_{\Upsilon(\mathcal{O})\in\mathcal{T}(\mathcal{O})}\frac{1}{\Lambda!}N_{\Upsilon(\mathcal{O})}=\mathcal{V}_L^{+,j_1j_1'}\frac{\overrightarrow{\partial}}{\partial\phi^{j_1'}}\cdots\mathcal{V}_L^{+,j_nj_n'}\frac{\overrightarrow{\partial}}{\partial\phi^{j_n'}}\cdot\left(-\frac{1}{n}\tilde{\mathcal{V}}_n(\Phi,\cdots,\Phi)\right)$$

holds [53]. ■

This implies that the Definition 6.10 actually gives the amputated n point tree amplitudes in a usual sense in field theory.

Lemma 6.13 ([53]) $\{\tilde{\mathcal{V}}_n\}_{n\geq 3}$ *is given by*

$$\tilde{\mathcal{V}}_n=\sum_{\Gamma_{n-1}\in G_{n-1}}\omega\circ\left(\mathbf{1}\otimes\tilde{m}^{cyc}_{\Gamma_{n-1}}\right)=\sum_{\Gamma^{cyc}_n\in G^{cyc}_n}\frac{1}{\sharp\left(\check{r}^{-1}(\Gamma^{cyc}_n)\right)}\tilde{\mathcal{V}}_{\Gamma^{cyc}_n}\ . \qquad (6.26)$$

Here $\sharp\left(\check{r}^{-1}(\Gamma^{cyc}_n)\right)$ indicates the number of the elements of the set $\check{r}^{-1}(\Gamma^{cyc}_n)$. ■

In the terminology of Feynman graphs, $1/\sharp\left(\check{r}^{-1}(\Gamma^{cyc}_n)\right)$ is the *symmetric factor* of graph Γ^{cyc}_n. [20]

Corollary 6.14 ([53]) *For a given cyclic field theory $(\mathcal{H},\omega,S)$, the tree on-shell correlation functions are given by*

$$\tilde{\mathcal{V}}^p_n=\omega\circ\left(\mathbf{1}\otimes\tilde{m}^p_{n-1}\right)$$

and therefore they form the minimal cyclic A_∞-algebra $(\mathcal{H}^p,\tilde{\omega}^p,\tilde{\mathfrak{m}}^p)$ in eq.(5.28).

For the proof of Lemma 6.12, one may calculate each $\Upsilon(\mathcal{O})\in\mathcal{T}(\mathcal{O})$ using the correspondence

$$\frac{\overleftarrow{\partial}}{\partial\phi^i}\mathcal{V}_L^{+,ij}\frac{\overrightarrow{\partial}}{\partial\phi^j}\left(\frac{1}{k+1}\mathcal{V}_{k+1}(\Phi)\right)=\frac{\overleftarrow{\partial}}{\partial\phi^i}\vec{c}^i_l c^l(\phi)\ ,$$

that comes from $\mathcal{V}_L^{+,ii_1}\omega_{i_1l}=\vec{c}^i_l$ where $Q^+(\mathbf{e}_l)=\mathbf{e}_i\vec{c}^i_l$. Lemma 6.13, that is, the equivalence of eq.(6.24) and (6.26) follows from concentrating the inverse of t for each element in G^{cyc}_n. ■

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[20] The number $\sharp\left(\check{r}^{-1}(\Gamma^{cyc}_n)\right)$ coincides with the number of the automorphisms acting on Γ^{cyc}_n which we denoted by $\sharp\mathrm{Aut}(\Gamma^{cyc}_n)$ in eq.(1.6) in the Introduction.

Remark 6.15 Corollary 6.14 indicates that the collection of on-shell correlation functions in Definition 6.10 forms the 'minimal action' $\tilde{S}(\tilde{\Phi}^p)$ in eq.(5.28),

$$\tilde{S}(\tilde{\Phi}^p) = \sum_{k\geq 2} \frac{1}{k+1}\omega(\tilde{\Phi}^p, \tilde{m}^p_k(\tilde{\Phi}^p)) \ .$$

The action $\tilde{S}(\tilde{\Phi}^p)$ is in fact an effective action in the following sense. $\tilde{S}(\tilde{\Phi}^p)$ is obtained by substituting $\Phi|_{gf} = \tilde{\mathcal{F}}^p(\tilde{\Phi}^p)$ into $S(\Phi)$ as explained above. When we express $\Phi = \Phi^t + \Phi|_{gf} = \Phi^t + \Phi^p + \Phi^u$ where Φ^t, Φ^p and Φ^u denotes the trivial, physical and unphysical modes of Φ, respectively, the substitution means $\Phi^p = \tilde{\Phi}^p$ and $\Phi^u = f(\tilde{\Phi}^p)$. As seen from eq.(5.15), the latter is nothing but the equation of motion for Φ^u. Moreover, $\tilde{S}(\tilde{\Phi}^p)$ is related to $S(\Phi)$ by integrating Φ^u at tree level through a gauge fixing as

$$\int \mathcal{D}\Phi^u \ e^{-S(\Phi)}|_{gf} = e^{-\tilde{S}(\tilde{\Phi}^p)} \ , \tag{6.27}$$

under an appropriate definition of the integration. In this sense the action $\tilde{S}(\tilde{\Phi}^p)$ is an effective action. The pair of gauge fixing $|_{gf}$, which extract Φ^t, and the extraction of Φ^u by substituting is an analogue of symplectic reduction, where the extraction of Φ^u can also be regarded as the restriction $\Phi|_{S(\Phi)=0}$ (see also a comment in Remark 6.19). Also, it is clear that our arguments are applicable to constructions of any other tree level effective actions.

Remark 6.16 (Independence of the choice of gauge fixing) We encoded the dependence of gauge fixing Ψ in the following two; a cyclic A_∞-isomorphism from the original cyclic A_∞-algebra $(\mathcal{H}, \omega, S)$, and propagator $\mathcal{V}_L^+$. The higher terms than the quadratic term of Ψ are absorbed into the cyclic A_∞-isomorphism. Whereas, a propagator $\mathcal{V}_L^+$ is derived by a well-defined gauge fixing. Note that $\mathcal{V}_L^+$ determines the decomposition $\mathcal{H} = \mathcal{H}^t \oplus \mathcal{H}^p \oplus \mathcal{H}^u$. Any $\mathcal{V}_L^+$ derived by a well-defined gauge fixing picks up the same (or isomorphic) minimal part $\mathcal{H}^p$.

Suppose that we have two minimal cyclic A_∞-algebra $(\mathcal{H}^p, \tilde{\omega}^p, \tilde{\mathfrak{m}}^p)$ and $(\mathcal{H}^{p'}, \tilde{\omega}^{p'}, \tilde{\mathfrak{m}}^{p'})$ that are obtained by two gauge fixing conditions. The uniqueness of minimal model (Corollary 5.10) implies that minimal cyclic A_∞-algebras obtained by perturbative expansion are independent of the choice of the gauge fixing at least up to isomorphisms on them.

In general in field theory, it is known that S-matrices are 'invariant' under (certain class of) field redefinition, which is called an *equivalence theorem* [57]. Physically, the fact stated above might be thought of as a version of this theorem.

6.4 Equivalence of classical open string field theories

In this subsection we shall apply the relation between the minimal model theorem and Feynman graph in the previous subsection to classical open string field theories constructed as in subsection 1.2. For the family of the well-defined string field theories, Lemma 6.13 gives us a classification of string field theories (Theorem 6.18) described below. The field transformations induce one-to-one correspondence of moduli spaces of classical solutions between such string field theories in the context of deformation theory. We shall explain that the classical solutions are regarded as those corresponding to marginal deformations.

A classical open string field theory is defined on a fixed *conformal background* (see the beginning of subsection 1.2). For a fixed conformal background, a $\mathbb{Z}$-graded vector space $\mathcal{H}$ is given canonically. $\mathcal{H}$ is called an open string Hilbert space, where each base $\mathbf{e}_i$ is called an open string state and the associated dual coordinate ϕ^i is a field in the sense of field theory. The superfield $\Phi := \mathbf{e}_i\phi^i \in \mathcal{H}\otimes\mathcal{H}^*$ is then called the *string field*. Moreover, a degree one coboundary operator $Q : \mathcal{H} \to \mathcal{H}$ and an odd constant symplectic structure $\omega(\ ,\ ) : \mathcal{H}\otimes\mathcal{H} \to \mathbb{C}$ are defined canonically. Q and ω are called the BRST-operator [58] [21] and the BPZ-inner product [13], respectively, on the fixed conformal background. The information with which open string theory provides us is the collection of *on-shell open string scattering amplitudes*. As a subset of the collection of the on-shell open string scattering amplitudes, we have *on-shell tree open string scattering amplitudes* which are cyclic multilinear maps $(\mathcal{H}^p)^{\otimes k} \to \mathbb{C}$ for $k \geq 3$.

Though string field theories are usually constructed by decomposing the moduli spaces of Riemann surfaces into cell as explained in subsection 1.2, we shall give a purely algebraic definition for classical open string field theories as follows.

Definition 6.17 (Classical open string field theory : axiom) A *classical open string field theory* is an action $S(\Phi) \in C(\phi)_c$ satisfying the following properties:

(a) Φ is the string field, $\omega : \mathcal{H}\otimes\mathcal{H} \to \mathbb{C}$ is the BPZ-inner product and $Q : \mathcal{H} \to \mathcal{H}$ is the BRST operator on a fixed conformal background ,

(b) The action $S(\Phi)$ is of the form $S(\Phi) = \frac{1}{2}\omega(\Phi, Q\Phi) + \sum_{k\geq 2}\frac{1}{k+1}\omega(\Phi, m_k(\Phi,\cdots,\Phi))$ where $S(\Phi) \in C(\phi)_c$,

(c) the action $S(\Phi)$ satisfies the classical BV-master equation $(S,S) = 0$ and is proper ,

(d) the on-shell scattering amplitudes (Definition 6.10) of the action $S(\Phi)$ by perturbative expansion reproduces the tree on-shell open string scattering amplitudes on the fixed conformal background.

The fact that $(\mathcal{H},\omega,S)$ is a cyclic A_∞-algebra follows from the condition (b) and (c).

Theorem 6.18 (cf. [53]) *All the well-defined classical open string field theories which are constructed on a fixed conformal background are cyclic A_∞-isomorphic to each other.*

proof. Let $(\mathcal{H},\omega,S)$ be a cyclic A_∞-algebra describing a classical open string field theory on a fixed conformal background. Lemma 6.13 states that the collection of the on-shell scattering amplitudes for the action S forms a minimal cyclic A_∞-algebra just in the way given in subsection 5.5. We denote it by $(\mathcal{H}^p,\omega^p,\tilde{S}^p)$. As stated in the end of subsection 5.5, Theorem 5.15 further implies that $(\mathcal{H},\omega,S)$ are cyclic A_∞-isomorphic to the decomposed cyclic A_∞-algebra

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[21] Here BRST-operator indicates the operator that induces the BRST transformation in the sense of string world sheet theory. Do not confuse it with the (BV-)BRST transformation in string field theory explained in subsection 6.2.

$(\mathcal{H}^p, \omega^p, \tilde{S}^p) \oplus (\mathcal{H}^t \oplus \mathcal{H}^p, Q)$, where $(\mathcal{H}^t \oplus \mathcal{H}^u, Q)$ is the linear contractible cyclic A_∞-algebra such that $\mathcal{H} = \mathcal{H}^p \oplus (\mathcal{H}^t \oplus \mathcal{H}^u)$ and Q is the restriction of the original differential $Q : \mathcal{H} \to \mathcal{H}$ in $(\mathcal{H}, \omega, S)$ onto $\mathcal{H}^t \oplus \mathcal{H}^u$. Suppose $(\mathcal{H}, \omega, S')$ be another cyclic A_∞-algebra of a classical open string field theory on the same conformal background. Then, it is cyclic A_∞-isomorphic to the decomposed cyclic A_∞-algebra $(\mathcal{H}^p, \omega^p, \tilde{S'}^p) \oplus (\mathcal{H}^t \oplus \mathcal{H}^p, Q)$. On the other hand, by Definition 6.17 (d), both of these two minimal cyclic A_∞-algebras $(\mathcal{H}^p, \omega^p, \tilde{S}^p)$ and $(\mathcal{H}^p, \omega^p, \tilde{S'}^p)$ give the same minimal cyclic A_∞-algebra consisting of the on-shell *open string* scattering amplitudes. This implies that two decomposed cyclic A_∞-algebras $(\mathcal{H}^p, \omega^p, \tilde{S}^p) \oplus (\mathcal{H}^t \oplus \mathcal{H}^p, Q)$ and $(\mathcal{H}^p, \omega^p, \tilde{S'}^p) \oplus (\mathcal{H}^t \oplus \mathcal{H}^p, Q)$ are cyclic A_∞-isomorphic to each other. Since the composition of any two cyclic A_∞-isomorphisms forms a cyclic A_∞-isomorphism, one can conclude that $(\mathcal{H}, \omega, S)$ and $(\mathcal{H}, \omega, S')$ are cyclic A_∞-isomorphic to each other. ■

In physical terms, this theorem means any two classical string field theories on a conformal background are transformed to each other by a field redefinition (preserving classical BV-structures). This may also be thought of as a converse statement of the equivalence theorem ('S-matrices are invariant under field redefinitions') [57] for field theories equipped with classical BV-structures. In contrast two cyclic A_∞-algebras are not connected by a field definition if their minimal cyclic A_∞-algebras are different from each other. This fact is also clear and follows from Theorem 5.15. Usually a string field theory is constructed by decomposing Riemann surfaces as in subsection 1.2. However, the actions which are transformed to each other by field redefinitions with preserving BV-symplectic forms are regarded to be equivalent [100, 26]. Thus, Theorem 6.18 implies also the sufficiency of the axiom of string field theory in Definition 6.17.

Remark 6.19 Since there exists a cyclic A_∞-isomorphism between any two classical open string field theories on a fixed conformal background, one may expect that it transforms all the equations of motions in one side to those in another side. However, the problem depends on convergence problem of the transformation, that depends on the models we consider, and the answer may be 'no' in general. On the other hand, in the context of deformation theory, the solutions for equations of motions, *i.e.* the Maurer-Cartan equations, are assumed to be of the form $\Phi = \hbar\tilde{\Phi}^p + \mathcal{O}(\hbar^2)$ for $\hbar$ a 'small' deformation parameter. The argument in subsection 5.4 is just the case where the small parameter is thought to be included in $\tilde{\Phi}^p$. Such solutions are expressed as $\Phi = \tilde{\mathcal{F}}^p_*(\tilde{\Phi}^p)$ where $\tilde{\Phi}^p$ is a solution for Maurer-Cartan equation $\tilde{\mathfrak{m}}^p_*(e^{\tilde{\Phi}^p}) = 0$. Namely, they are the solutions which corresponds to the continuous deformations from the origin $\Phi = 0$. They are regarded as the marginal deformation in the terminology of conformal field theory. The solutions satisfy not only the equation of motion but $S(\Phi) = 0$. Now the problem of the convergence still remains, though we expect in string field theories they are valid at least in some neighborhood of the origin. However, if we treat $\hbar$ as a formal deformation parameter, then all the solutions are valid. Namely, all classical open string field theories on a fixed conformal background have isomorphic moduli spaces of 'formal marginal deformations'. It follows from Theorem 6.18 and Theorem 7.16 in subsection 7.2.

The fact that the collection of open string tree scattering amplitudes possesses a minimal cyclic A_∞-structure has essentially appeared in some literatures. It is described in [127] that the S^2 tree amplitudes for closed strings has a L_∞-structure, where the external states are restricted to physical states and therefore it has vanishing Q. This implies that the tree level closed string free energy satisfies the classical BV-master equation. The result is extended to quantum closed string, and it is shown that the free energy which consists of the closed string loop amplitudes satisfies the quantum BV-master equation [121]. These structures are derived in the context of 2D-string theory, *i.e.* the dimension of the target space is two. However they are in fact the general structures of the string world sheet, and reinterpreted in [128] from a world sheet viewpoint explained below. The open string version of this L_∞-structure is nothing but the A_∞-structure $\tilde{\mathfrak{m}}^p$ above.

From string world sheet point of view, a string field theory action is usually constructed by decomposing $\{\mathcal{M}_n\}_{n\geq 3}$, moduli spaces of disk with n punctures on the boundary, into cell (see subsection 1.2). The vertex map $\mathcal{V} := \omega\circ(\mathbf{1}\otimes m_{n-1}) : \mathcal{H}^{\otimes n}\to\mathbb{C}$ is determined by a certain integral over the cell $\mathcal{M}_n^0\subset\mathcal{M}_n$. $\{\mathcal{M}_k^0\}_{k\geq 3}$ satisfy some consistency condition and then forms a topological operad. The string field theory constructed at the limit $\{\mathcal{M}_k^0\}_{k\geq 3}\to\{\mathcal{M}_k\}_{k\geq 3}$ then turns out to be the on-shell open string scattering amplitudes. This also implies that the on-shell open string scattering amplitudes form a minimal cyclic A_∞-algebra. This interpretation is just the one given in [128] for closed string case. Note that, for a $\{\mathcal{M}_k^0\}_{k\geq 3}$ constructed consistently, one can in fact take a continuous deformation from $\{\mathcal{M}_k^0\}_{k\geq 3}$ to $\{\mathcal{M}_k\}_{k\geq 3}$ with preserving the cyclicity. This implies that all $\{\mathcal{M}_k^0\}_{k\geq 3}$ that are constructed consistently are homotopic to each other as topological operads [54].

The continuous deformation from $\{\mathcal{M}_k^0\}_{k\geq 3}$ to $\{\mathcal{M}_k\}_{k\geq 3}$ is related to renormalization group flow [88] of string world sheet theory in the sense of [6, 47], etc. Note that, for a given conformal background, the field ϕ^i is the coupling constant for open string state $\mathbf{e}_i$ and also thought of as 'source' for $\mathbf{e}_i$ from world sheet theory viewpoints. Let $(\mathcal{H},\omega,\tilde{S})$ be the decomposed cyclic A_∞-algebra whose minimal part is $(\mathcal{H}^p,\omega^p,\tilde{S}^p)$. It is an algebra over operad $\{\mathcal{M}_k\}_{k\geq 3}$ and is essentially the generating function (or free energy) of tree open string world sheet theory. The action $S(\Phi)$ over $\{\mathcal{M}_k^0\}_{k\geq 3}$ then defines some kind of deformed free energy of world sheet theory. The renormalization group flow is the flow (solution of differential equations) of the coupling constants such that the free energy $S(\Phi)$ over $\{\mathcal{M}_k^0\}_{k\geq 3}$ is preserved under the continuous deformation. Now, by Theorem 6.18, there exists continuous field redefinition that preserves the free energy $S(\Phi)$ under the continuous deformation of $\{\mathcal{M}_k^0\}_{k\geq 3}$. Namely, the field redefinition of string field theory is just the renormalization group flow of string world sheet. On the other hand, string field theory is a field theory on the target space where strings are mapped. One can also consider the renormalization group as a target space field theory [16]. The correspondence between the world sheet renormalization group and the target space one is then discussed in [17, 85]. Namely, the world sheet renormalization group, which is target space field redefinition, can also be regarded as the target space renormalization group, though our theory is classical as a target space field theory.

Though the (abstract) way of the construction is given, physically string field theory has had mainly two general puzzles that should be resolved. One is the relation between different string field theory actions on the same conformal background. The another one is then the background independence of the action; namely, if a string field theory action does give an nonperturbative definition of string theory, any classical solution should correspond to a vacuum (conformal background) of the string theory and the action expanded around the classical solution should describe a string field theory on the corresponding conformal background. Both problems have ever resolved only at infinitesimal level. In [43] (for quantum closed string field theory) it is shown that on a fixed conformal background any infinitesimal variation of the way of decomposition of moduli space is absorbed by an infinitesimal field redefinition. The background independence for (infinitesimal) marginal deformation is discussed and shown in [96, 97, 98, 90, 100, 101, 102, 71]. Theorem 6.18 then gives the answer for the first problem for classical string field theory. We believe that similar result holds for quantum case [22] and it could be the starting point for the problem of the background independence.

7 Homotopy equivalence, gauge equivalence and moduli spaces

In this section we shall discuss homotopy theoretical aspects of A_∞-algebras. In subsection 7.1 we shall define homotopy between an A_∞-morphism, and show that two A_∞-algebras are homotopy equivalent to each other iff there exists an A_∞-quasi-isomorphism between them (Theorem 7.5). Similar results are obtained in the framework of twisting cochains [51, 52], in terms of operads [79], in a purely algebraic way [25], more recently in [72] in terms of closed model categories [72], and so on. In subsection 7.2 we shall define the notion of gauge equivalence, and define the moduli space of an A_∞-algebra as the solution space of the Maurer-Cartan equation for the A_∞-algebra modulo gauge equivalence. The homotopy invariance of the moduli spaces are then shown in Theorem 7.16. A characteristic of our approach is to apply the decomposition theorem (Theorem 5.4), which enables us to show both Theorem 7.5 and Theorem 7.16. See also [56].

7.1 Homotopy equivalence

Definition 7.1 (Homotopy of A_∞-morphisms) Given two A_∞-algebras $(\mathcal{H},\mathfrak{m})$ and $(\mathcal{H}',\mathfrak{m}')$ and two (weak) A_∞-morphisms $\mathcal{F},\mathcal{G}:(\mathcal{H},\mathfrak{m})\to(\mathcal{H}',\mathfrak{m}')$, we say that $\mathcal{G}$ is *homotopic* to $\mathcal{F}$ when there exists degree minus one operator $H:C(\mathcal{H})\to C(\mathcal{H}')$ such that

$$\mathcal{G}-\mathcal{F}=\mathfrak{m}'H+H\mathfrak{m}\ . \tag{7.1}$$

We call H the homotopy operator between A_∞-morphisms $\mathcal{G}$ and $\mathcal{F}$.

Note that if we forget the cohomomorphism structures of $\mathcal{F}$, G and regard $(C(\mathcal{H}),\mathfrak{m})$, $(C(\mathcal{H}'),\mathfrak{m}')$ as complexes of groups such as deRham complex, then the definition is the homotopy operator H is just the usual one.

Remark 7.2 (A condition for H) Note that $\mathcal{F}$ and $\mathcal{G}$ are cohomomorphisms. Therefore H has a condition defined by $(\mathcal{F}\otimes\mathcal{F})\triangle=\triangle\mathcal{F}$, $(\mathcal{G}\otimes\mathcal{G})\triangle=\triangle\mathcal{G}$ and eq.(7.1). One of the most simplest solutions for the condition is given by [39, 104]

$$\triangle H=(\mathcal{F}\otimes H+H\otimes\mathcal{G})\triangle\ . \tag{7.2}$$

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[22] Some (homotopy) algebraic structures for quantum closed SFT are discussed in [101, 102, 78].

Under this condition, H is determined if the collection of degree minus one map $h_n : \mathcal{H}^{\otimes n} \to \mathcal{H}'$ for $n \geq 0$ is given as follows:

$$H = \sum_{n\geq 0} \mathcal{F} \otimes h_n \otimes \mathcal{G} \ . \tag{7.3}$$

One can see that if the image of H in $C(\mathcal{H}')$ is restricted to $\mathcal{H}'$, H just reduces to $\sum_n h_n$. General solutions are then written of the form

$$\triangle H = ((\mathcal{F} \otimes H + H \otimes \mathcal{G}) + A) \triangle$$

for a degree minus one element $A \in C(\mathcal{H}) \otimes C(\mathcal{H})$. In this paper we treat all such H as homotopies without fixing a condition for $\triangle H$.

Lemma 7.3 *The homotopy in Definition 7.1 actually defines an equivalence relation.*

proof. It is easy to confirm the fact.

Lemma 7.4 *Let* $(\mathcal{H}_{dc}, \mathfrak{m}_{dc})$ *be a direct sum of a minimal* A_∞*-algebra and a linear contractible* A_∞*-algebra, and* $(\mathcal{H}^p_{dc}, \mathfrak{m}^p_{dc})$ *the corresponding minimal* A_∞*-algebra. We have* A_∞*-quasi-isomorphisms* $\pi : (\mathcal{H}_{dc}, \mathfrak{m}_{dc}) \to (\mathcal{H}^p_{dc}, \mathfrak{m}^p_{dc})$ *and* $\iota : (\mathcal{H}^p_{dc}, \mathfrak{m}^p_{dc}) \to (\mathcal{H}_{dc}, \mathfrak{m}_{dc})$ *and the composition* $P = \iota \circ \pi : (\mathcal{H}_{dc}, \mathfrak{m}_{dc}) \to (\mathcal{H}_{dc}, \mathfrak{m}_{dc})$ *is also an* A_∞*-quasi-isomorphism. In this situation,* A_∞*-quasi-isomorphisms* Id $: (\mathcal{H}_{dc}, \mathfrak{m}_{dc}) \to (\mathcal{H}_{dc}, \mathfrak{m}_{dc})$ *and* $P : (\mathcal{H}_{dc}, \mathfrak{m}_{dc}) \to (\mathcal{H}_{dc}, \mathfrak{m}_{dc})$ *are homotopic to each other.*

proof. The homotopy H satisfying $\mathrm{Id} - P = \mathfrak{m}H + H\mathfrak{m}$ is obtained explicitly by setting $\mathcal{F} = \mathrm{Id}$, $\mathcal{G} = P$, $h_1 = Q^+$ and $h_n = 0$ for $n \neq 1$ in the condition (7.2), that is,

$$H = \mathrm{Id} \otimes Q^+ \otimes P \ .$$

Note that here we denote Id and P as cohomomorphisms on $C(\mathcal{H}_{dc})$. ■

The following theorem is known, which follows from the results in [51, 52] in the framework of *twisting cochains*, where homotopy is defined as in Definition 7.1 but with the additional condition (7.2). Now we can show this theorem by applying the decomposition theorem (Theorem 5.4).

Theorem 7.5 *For two* A_∞*-algebras* $(\mathcal{H}, \mathfrak{m})$ *and* $(\mathcal{H}', \mathfrak{m}')$*, suppose that there exists an* A_∞*-quasi-isomorphism* $\mathcal{F} : (\mathcal{H}, \mathfrak{m}) \to (\mathcal{H}', \mathfrak{m}')$*. Then there exists an inverse* A_∞*-quasi-isomorphism* $\mathcal{F}^{-1} : (\mathcal{H}', \mathfrak{m}') \to (\mathcal{H}, \mathfrak{m})$ *such that* $\mathcal{F}^{-1} \circ \mathcal{F}$ *is homotopic to the identity* Id *on* $(\mathcal{H}, \mathfrak{m})$ *and* $\mathcal{F} \circ \mathcal{F}^{-1}$ *is homotopic to the identity* Id *on* $(\mathcal{H}', \mathfrak{m}')$*.*

proof. The inverse A_∞-quasi-isomorphism is just given by Theorem 5.17 combined with Corollary 5.9. It is of the form

$$\mathcal{F}^{-1} = \mathcal{F}_{dc} \circ \iota \circ (\mathcal{F}^p)^{-1} \circ \pi' \circ ({\mathcal{F}'}_{dc})^{-1} \ ,$$

where we have the following commutative diagram

$$\begin{tikzcd}
(\mathcal{H},\mathfrak{m}) \arrow[rr, "\mathcal{F}"] \arrow[d, shift left, "(\mathcal{F}_{dc})^{-1}"] & & (\mathcal{H}',\mathfrak{m}') \arrow[d, shift left, "(\mathcal{F}'_{dc})^{-1}"] \\
(\mathcal{H}_{dc},\mathfrak{m}_{dc}) \arrow[u, shift left, "\mathcal{F}_{dc}"] \arrow[d, shift left, "\pi"] & & (\mathcal{H}'_{dc},\mathfrak{m}'_{dc}) \arrow[u, shift left, "\mathcal{F}'{}_{dc}"] \arrow[d, shift left, "\pi'"] \\
(\mathcal{H}^{p}_{dc},\mathfrak{m}^{p}_{dc}) \arrow[u, shift left, "\iota"] \arrow[rr, shift left, "\mathcal{F}^{p}"] & & (\mathcal{H}'^{p}_{dc},\mathfrak{m}'^{p}_{dc}) \arrow[u, shift left, "\iota'"] \arrow[ll, shift left, "(\mathcal{F}^{p})^{-1}"]
\end{tikzcd} \quad .$$

Since one already knows the existence of $\mathcal{F}^{-1}$, one can see that

$$(\mathcal{F}_{dc})^{-1}\circ(\mathcal{F}^{-1}\circ\mathcal{F})\circ\mathcal{F}_{dc}=P\ .$$

From Lemma 7.4, there exists a homotopy operator H_o such as

$$\mathrm{Id}-P=\mathfrak{m}_{dc}H_o+H_o\mathfrak{m}_{dc}$$

on $(\mathcal{H}_{dc},\mathfrak{m}_{dc})$. Acting $\mathcal{F}_{dc}$ from left and $(\mathcal{F}_{dc})^{-1}$ from right in both sides of the equation above, one can obtain

$$\mathrm{Id}-(\mathcal{F})^{-1}\circ\mathcal{F}=\mathfrak{m}H+H\mathfrak{m}$$

where $H:=\mathcal{F}_{dc}\circ H_o\circ(\mathcal{F}_{dc})^{-1}$ and we used the fact that $\mathfrak{m}=\mathcal{F}_{dc}\circ\mathfrak{m}_{dc}\circ(\mathcal{F}_{dc})^{-1}$. Thus it is shown that Id and $\mathcal{F}^{-1}\circ\mathcal{F}$ are homotopic to each other with homotopy operator H. The fact that $\mathcal{F}\circ\mathcal{F}^{-1}$ is homotopic to Id can also be shown in a similar way. ■

Note that an A_∞-morphism $(\mathcal{H},\mathfrak{m})\to(\mathcal{H},\mathfrak{m})$ that is not an A_∞-quasi-isomorphism cannot be homotopic to the identity. However, the converse is not true. Namely, an A_∞-quasi-isomorphism is not necessarily homotopic to the identity. For instance, in general there exist A_∞-isomorphisms associated to discrete A_∞-automorphisms on $(\mathcal{H},\mathfrak{m})$.

Given two A_∞-algebras $(\mathcal{H},\mathfrak{m})$, $(\mathcal{H}',\mathfrak{m}')$, an A_∞-morphism $\mathcal{F}:(\mathcal{H},\mathfrak{m})\to(\mathcal{H}',\mathfrak{m}')$ is called a homotopy equivalence between the two A_∞-algebras iff there exists an A_∞-morphism $\mathcal{F}^{-1}:(\mathcal{H}',\mathfrak{m}')\to(\mathcal{H},\mathfrak{m})$ such that $\mathcal{F}^{-1}\circ\mathcal{F}$ and $\mathcal{F}\circ\mathcal{F}^{-1}$ are homotopic to the identity (see [24]). Then Theorem 7.5 implies that the notion of the homotopy equivalence of A_∞-algebras is equivalent to the existence of A_∞-quasi-isomorphisms between the A_∞-algebras.

We end this subsection with some byproducts from the decomposition theorem (Theorem 5.4) and the notion of homotopy equivalence of A_∞-algebras. There is the notion of homotopy invariant algebraic structures by Boardman and Vogt [14] and the notion is translated into homotopy algebraic language in [79](see also [80]). Namely, homotopy algebras should have the following three properties (Corollary 7.6, Corollary 7.7, Corollary 7.8) so that they actually define homotopy invariant algebraic structures. They can be shown due to our arguments until now.

Corollary 7.6 *For an A_∞-algebra $(\mathcal{H},\mathfrak{m})$, a chain complex $(\mathcal{H}',m'_1)$ and a chain complex $f_1:(\mathcal{H},m_1)\to(\mathcal{H}',m'_1)$, there exists an A_∞-structure on $\mathcal{H}'$ and an A_∞-morphism $\mathcal{F}$ whose leading term is f_1.*

Corollary 7.7 *Suppose two* A_∞*-algebras* $(\mathcal{H},\mathfrak{m})$, $(\mathcal{H}',\mathfrak{m}')$ *and an* A_∞*-morphism* $\mathcal{F}:(\mathcal{H},\mathfrak{m})\to(\mathcal{H}',\mathfrak{m}')$ *are given. Moreover suppose there exists a chain map* $g_1:(\mathcal{H},m_1)\to(\mathcal{H}',m'_1)$ *that is chain homotopic to* f_1*. Then there exists an* A_∞*-morphism* $\mathcal{G}:(\mathcal{H},\mathfrak{m})\to(\mathcal{H}',\mathfrak{m}')$ *whose leading term is* g_1*.*

Corollary 7.8 *Given an* A_∞*-quasi-isomorphism* $\mathcal{F}:(\mathcal{H},\mathfrak{m})\to(\mathcal{H}',\mathfrak{m}')$ *and* $g_1:(\mathcal{H}',m'_1)\to(\mathcal{H},m_1)$*, a chain homotopy inverse of* f_1*, there exists an* A_∞*-quasi-isomorphism* $\mathcal{G}:(\mathcal{H}',\mathfrak{m}')\to(\mathcal{H},\mathfrak{m})$*.*

proof. Corollary 7.6 can be shown due to the decomposition theorem. For an A_∞-algebra $(\mathcal{H},\mathfrak{m})$ one can consider a decomposed A_∞-algebra $(\mathcal{H}_{dc},\mathfrak{m}_{dc})$ and it is clear that f_1 is naturally extended to an A_∞-quasi-isomorphism and it induces a decomposed A_∞-structure on $\mathcal{H}'$. $\mathcal{F}$ is then obtained by the composition of the A_∞-quasi-isomorphism with $(\mathcal{F}_{dc})^{-1}:(\mathcal{H},\mathfrak{m})\to(\mathcal{H}_{dc},\mathfrak{m}_{dc})$.

Corollary 7.7 follows from the notion of homotopy of A_∞-morphisms. That is, $\mathcal{G}:(\mathcal{H},\mathfrak{m})\to(\mathcal{H}',\mathfrak{m}')$ is obtained by

$$\mathcal{G}=\mathcal{F}+\mathfrak{m}'H+H\mathfrak{m} \tag{7.4}$$

if a homotopy operator H exists. Here the problem is that whether there exists the H such that the restriction of eq.(7.4) to the term $\mathcal{H}\to\mathcal{H}'$ is $g_1=f_1+m'_1h_1+h_1m_1$ for h_1 the chain homotopy. Such H does exist. One such is constructed by eq.(7.3) with $h_0=h_2=h_3=\cdots=0$.

Corollary 7.8 is almost the same as Theorem 7.5 and also follows from the decomposition theorem. ■

7.2 Gauge equivalence and the moduli space of an A_∞-algebra

Roughly speaking, the moduli space of an A_∞-algebra is degree zero solution space of the Maurer-Cartan equation over gauge equivalence. Therefore we should define the gauge equivalence. Two elements in $\mathcal{H}$ are called gauge equivalent if they are transformed to each other by a gauge transformation. Then the gauge transformation is usually defined as integrals of some infinitesimal transformations along paths in $\mathcal{H}$. There exist mainly two candidates of the infinitesimal transformation. In terms of formal noncommutative supermanifolds, they are the followings.

(a) The infinitesimal gauge transformation is defined by

$$\delta_\alpha=\frac{\overleftarrow{\partial}}{\partial\phi^i}\left(c^i(\phi)\frac{\overleftarrow{\partial}}{\partial\phi^j}\alpha^j\right)$$

where $\alpha^j\in\mathbb{C}$ for $\deg(\mathbf{e}_j)=0$ and $\alpha^j=0$ for $\deg(\mathbf{e}_j)\neq 0$.

(b) The infinitesimal gauge transformation is defined by

$$\delta_\alpha=[\delta,\alpha(\phi)]\ ,\qquad \alpha(\phi):=\frac{\overleftarrow{\partial}}{\partial\phi^i}\alpha^i(\phi)=\frac{\overleftarrow{\partial}}{\partial\phi^i}\sum_{k\geq 0}\alpha^i_{j_1\cdots j_k}\phi^{j_k}\cdots\phi^{j_1}\ ,\quad \alpha^i_{j_1\cdots j_k}\in\mathbb{C}$$

where δ is the A_∞-odd vector field and $\alpha(\phi)$ is a degree minus one vector field (see [65]).

The infinitesimal transformation (a) is just the gauge transformation in cyclic field theory (6.10) explained in subsection 6.1. More precisely, restricting the graded gauge parameter α^j, $\deg(\mathbf{e}_j) \neq 0$, to zero leads to the definition (a) above. Note that the gauge transformation (6.10) just reduces to transformation (a) in the degree-zero subvector space of $\mathcal{H}$. The infinitesimal transformation (a) forms a Lie algebra only *on-shell*, where on-shell means the solution space of the Maurer-Cartan equation. Namely, the Lie bracket closes modulo the Maurer-Cartan equation. This implies the infinitesimal transformations are integrable on-shell and so the definition (a) is sufficient to define the moduli space of A_∞-algebras. The integral is given by the usual exponential map (see below).

On the other hand, infinitesimal transformation (b) forms a Lie algebra on the whole space $\mathcal{H}$. This fact can also be confirmed by a direct calculation. Note that the transformation (b) reduces to transformation (a) if $\alpha^i_{j_1\cdots j_k} = 0$ for $k \geq 1$. One can see that extending α to ϕ-dependent one leads to a Lie algebra which closes even off-shell.

A natural extension of gauge transformation in differential graded Lie algebra case [32, 91] to A_∞-algebras leads to the choice (a) (see [24]). Thus, (a) is also natural and in fact sufficient for our purpose. However, in this subsection we shall use (b) as the definition of the gauge transformation. By employing the arguments in section 3, we translate the above into coalgebra language and define precisely the gauge transformation.

Definition 7.9 (Gauge transformation) Given an A_∞-algebra $(\mathcal{H}, \mathfrak{m})$, let us consider a piecewise smooth path of weak A_∞-automorphisms $U_{\alpha[0,t]} : (\mathcal{H}, \mathfrak{m}) \to (\mathcal{H}, \mathfrak{m})$, $0 \leq t \leq 1$ defined as a coalgebra homomorphism $U_{\alpha[0,t]} : C(\mathcal{H}) \to C(\mathcal{H})$ satisfying the following differential equation

$$\frac{d}{dt} U_{\alpha[0,t]} = \left([\mathfrak{m}, \alpha(t)] U_{\alpha[0,t]}\right) \ , \qquad U_{\alpha[0,0]} = \mathrm{Id} \ .$$

Here $\alpha(t) : C(\mathcal{H}) \to C(\mathcal{H})$ is a degree minus one coderivation which consists of $\alpha(t) = \alpha_0(t) + \alpha_1(t) + \alpha_2(t) + \cdots$, where each coderivation $\alpha_k(t)$ is defined by the lift of a multilinear map $\mathcal{H}^{\otimes k} \to \mathcal{H}$ like as $\mathfrak{m} = \mathfrak{m}_0 + \mathfrak{m}_1 + \mathfrak{m}_2 + \cdots$, and then $[\mathfrak{m}, \alpha(t)] := \mathfrak{m}\alpha(t) + \alpha(t)\mathfrak{m} : C(\mathcal{H}) \to C(\mathcal{H})$. We call a weak A_∞-automorphism $U_{\alpha[0,1]} : (\mathcal{H}, \mathfrak{m}) \to (\mathcal{H}, \mathfrak{m})$ a *gauge transformation.*

The integral along the path $[0, s]$ is represented explicitly by an iterated integral,

$$U_{\alpha[0,s]} := \mathcal{P}e^{\int_0^s dt[\mathfrak{m},\alpha(t)]} = \mathbf{1} + \int_0^s dt[\mathfrak{m}, \alpha(t)] + \int_{s>t>t'>0} dtdt'[\mathfrak{m}, \alpha(t)][\mathfrak{m}, \alpha(t')] + \cdots \ , \qquad (7.5)$$

where $\mathcal{P}$ is the one which is what is called the path-ordering. Note that if $\alpha(t)$ is constant with respect to t the equation above reduces to the ordinary exponential map (in terms of coalgebra side).

$$U_{\alpha[0,1]} = \mathcal{P}e^{\int_0^1 dt[\mathfrak{m},\alpha]} = e^{[\mathfrak{m},\alpha]} := \mathbf{1} + [\mathfrak{m}, \alpha] + \frac{1}{2!}([\mathfrak{m}, \alpha])^2 + \frac{1}{3!}([\mathfrak{m}, \alpha])^3 + \cdots \ .$$

Lemma 7.10 *$U_{\alpha[0,s]}$ in eq.(7.5) is actually a weak A_∞-automorphism on $(\mathcal{H}, \mathfrak{m})$. Namely, the gauge transformations are weak A_∞-automorphisms.*

proof. The fact that $U_{\alpha[0,s]}\mathfrak{m} = \mathfrak{m}U_{\alpha[0,s]}$ can be shown directly. In this calculation, one may use $[\mathfrak{m},[\mathfrak{m},\alpha]] = 0$, which is just the infinitesimal description of the statement of this Lemma and follows from the Jacobi identity. ■

Lemma 7.11 *The gauge transformation $U_{\alpha[0,1]}$ is homotopic to the identity on A_∞-algebra $(\mathcal{H},\mathfrak{m})$.*

proof. First we have

$$U_{\alpha[0,1]} - \mathrm{Id} = \int_0^1 ds \frac{d}{ds} U_{\alpha[0,s]} \ ,$$

where by definition $\frac{d}{ds}U_{\alpha[0,s]} = [\mathfrak{m},\alpha(s)]U_{\alpha[0,s]}$. Moreover Lemma 7.10 and the fact that $\mathcal{F}$ is an A_∞-morphism imply that the equation above is rewritten as

$$U_{\alpha[0,1]} - \mathrm{Id} = \mathfrak{m}H + H\mathfrak{m} \ , \qquad H = \int_0^1 ds\, \alpha(s) U_{\alpha[0,s]} \ .$$

The arguments above are similar to those of Weinstein-Darboux theorem [123], though the usage is different. ■

Note that the converse of Lemma 7.11 is not true. As seen in Lemma 7.4 the retraction for the linear contractible direction is homotopic to the identity, which is not a gauge transformation.

Remark 7.12 The composition of gauge transformations is a gauge transformation. In fact, for given two gauge transformation $U_{\alpha[0,1]}$ and $U_{\alpha'[0,1]}$, the composition is given by

$$U_{\alpha'[0,1]} \circ U_{\alpha[0,1]} = U_{\alpha''[0,1]} \ , \qquad \alpha''(t) = \begin{cases} \alpha(2t) & 0 \le t \le \frac{1}{2} \\ \alpha'(2t-1) & \frac{1}{2} \le t \le 1 \end{cases} \ .$$

Of course it is clear that one can take other α'' that come from a reparametrization of t.

We shall below consider the moduli space of an A_∞-algebra in $\mathcal{H}^0$, the degree zero part of $\mathcal{H}$, though our formalism may be appropriate to extend it to the whole degree (cf. subsection 3.3, subsection 5.4 and [7]). In order to avoid the problem of convergence, one often considers in general the tensor product of maximal ideal of an Artin algebra with $\mathcal{H}$ (see [24]). For simplicity, we shall introduce a formal parameter $\hbar$ and deal with $\mathcal{H} \otimes \hbar\mathbb{C}[[\hbar]]$ from now on.

Definition 7.13 (Moduli space of an A_∞-algebra) For an A_∞-algebra $(\mathcal{H},\mathfrak{m})$, let Φ be a degree zero element in $\hbar\mathcal{H}[[\hbar]] := \mathcal{H} \otimes \mathbb{C}\hbar[[\hbar]]$, where $\hbar$ is a formal deformation parameter. We define the solution space of Maurer-Cartan equation for A_∞-algebra $(\mathcal{H},\mathfrak{m})$ by

$$\mathcal{MC}(\mathcal{H},\mathfrak{m}) = \{\Phi \in \hbar\mathcal{H}[[\hbar]] \mid \mathfrak{m}_*(e^\Phi) = 0,\ \deg(\Phi) = 0\} \ .$$

By definition, gauge transformations (Definition 7.9) preserve the space $\mathcal{MC}(\mathcal{H},\mathfrak{m})$. We call $\Phi_1, \Phi_2 \in \mathcal{MC}(\mathcal{H},\mathfrak{m})$ are *gauge equivalent* to each other when they are transformed to each other by a gauge transformation, that is, $e^{\Phi_2} = U_{\alpha[0,1]}e^{\Phi_1}$ or equivalently $\Phi_2 = (U_{\alpha[0,1]})_*(\Phi_1)$ for some $\alpha(t)$, $0 \le t \le 1$. Remark 7.12 guarantees that this actually defines an equivalence relation. The *moduli space* of A_∞-algebra $(\mathcal{H},\mathfrak{m})$ is then defined by $\mathcal{MC}(\mathcal{H},\mathfrak{m})$ modulo the gauge equivalence $\sim$,

$$\mathcal{M}(\mathcal{H},\mathfrak{m}) := \mathcal{MC}(\mathcal{H},\mathfrak{m})/\sim \ .$$

Lemma 7.14 *Given two A_∞-algebras $(\mathcal{H},\mathfrak{m})$, $(\mathcal{H}',\mathfrak{m}')$ and an A_∞-morphism $\mathcal{F}:(\mathcal{H},\mathfrak{m})\to(\mathcal{H}',\mathfrak{m}')$, one can define the gauge transformation on the image of $\mathcal{F}$ so that it is compatible with the gauge transformation in $(\mathcal{H},\mathfrak{m})$ when restricted to the solution spaces of the Maurer-Cartan equations.*

proof. It is sufficient to show it infinitesimally. What should be shown is then that there exists a degree minus one coderivation α' on $\mathcal{H}'$ such that

$$[\mathfrak{m}',\alpha']\mathcal{F}\left(e^\Phi\right) = \mathcal{F}[\mathfrak{m},\alpha]\left(e^\Phi\right) \qquad (7.6)$$

holds if $\mathfrak{m}\left(e^\Phi\right)=0$.

Let us define a degree minus one coderivation α' on $\mathcal{H}'$ by

$$\alpha'(\Phi') = \mathcal{F}(\alpha(e^\Phi))\big|_{(\mathcal{H}')^{\otimes 1}} = f_1(\alpha(\Phi)) + f_2(\alpha(\Phi),\Phi) + f_2(\Phi,\alpha(\Phi)) + \cdots$$

where $\alpha(\Phi)=\alpha_0+\alpha_1(\Phi)+\alpha_2(\Phi,\Phi)+\cdots$. This α' actually satisfies eq.(7.6). One can see that it follows from the condition of A_∞-morphisms (Definition 2.7)

$$(\mathfrak{m}'\mathcal{F}-\mathcal{F}\mathfrak{m})\alpha\left(e^\Phi\right)=0\ .$$

■

Note that this Lemma, combined with the fact that an A_∞-morphism preserves the solution spaces of the Maurer-Cartan equations as shown in subsection 2.4, implies that an A_∞-morphism $\mathcal{F}:(\mathcal{H},\mathfrak{m})\to(\mathcal{H}',\mathfrak{m}')$ actually induces a well-defined map from $\mathcal{M}(\mathcal{H},\mathfrak{m})$ to $\mathcal{M}(\mathcal{H}',\mathfrak{m}')$. In particular, if $\mathcal{F}$ is an A_∞-isomorphism, it induces an isomorphism on the moduli spaces.

Lemma 7.15 *For a direct sum $(\mathcal{H}_{dc},\mathfrak{m}_{dc})$ of a minimal A_∞-algebra $(\mathcal{H}^p_{dc},\mathfrak{m}^p_{dc})$ and a linear contractible A_∞-algebra, the projection $P:\mathcal{H}_{dc}\to\mathcal{H}_{dc}$ induces the identity map* Id *on $\mathcal{M}(\mathcal{H}_{dc},\mathfrak{m}_{dc})$.*

proof. For the Maurer-Cartan equation of the decomposed A_∞-algebra $(\mathcal{H}_{dc},\mathfrak{m}_{dc})$,

$$m_{dc,1}\Phi + \sum_{k\ge 2} m_{dc,k}(\Phi) = 0\ ,$$

the first term and the second term should be zero independently, and the second term is nothing but the Maurer-Cartan equation of the minimal part $(\mathcal{H}^p_{dc},\mathfrak{m}^p_{dc})$. Here, for the solutions of $m_{dc,1}(\Phi)=0$, the $m_{dc,1}$-exact part is obviously gauge equivalent to zero; for instance, one may take $\alpha(t)=\alpha:\mathbb{C}\to\mathcal{H}^{-1}_{dc}$ and then $m_{dc,1}(\alpha)\in\mathcal{H}^0_{dc}$ is generated by a gauge transformation. Thus, any gauge equivalent class in $\mathcal{MC}(\mathcal{H}_{dc},\mathfrak{m}_{dc})$ can be represented by an element in $\mathcal{H}^p$ and from which the statement of this Lemma follows. ■

Then we obtain the following, the A_∞ version of the theorem in differential graded Lie algebra case [32, 91].

Theorem 7.16 *Given two A_∞-algebras $(\mathcal{H},\mathfrak{m})$, $(\mathcal{H}',\mathfrak{m}')$ and suppose that there exists an A_∞-quasi-isomorphism $\mathcal{F}:(\mathcal{H},\mathfrak{m})\to(\mathcal{H}',\mathfrak{m}')$. Then the moduli spaces of these two A_∞-algebras are isomorphic to each other:*

$$\mathcal{M}(\mathcal{H},\mathfrak{m})\simeq\mathcal{M}(\mathcal{H}',\mathfrak{m}')\ .$$

proof. By Theorem 7.5, we have an inverse A_∞-quasi-isomorphism $\mathcal{F}^{-1}:(\mathcal{H}',\mathfrak{m}')\to(\mathcal{H},\mathfrak{m})$ such that $\mathcal{F}^{-1}\circ\mathcal{F}$ and $\mathcal{F}\circ\mathcal{F}^{-1}$ are homotopic to the identity Id. Denote by $\mathcal{F}_\sim:\mathcal{M}(\mathcal{H},\mathfrak{m})\to\mathcal{M}(\mathcal{H}',\mathfrak{m}')$ and $(\mathcal{F})^{-1}_\sim:\mathcal{M}(\mathcal{H}',\mathfrak{m}')\to\mathcal{M}(\mathcal{H},\mathfrak{m})$ the corresponding maps between the moduli spaces. One may show the following two equations

$$(\mathcal{F})^{-1}_\sim\circ\mathcal{F}_\sim=\mathrm{Id}\ ,\qquad \mathcal{F}_\sim\circ(\mathcal{F})^{-1}_\sim=\mathrm{Id}\ . \tag{7.7}$$

On the other hand, in the proof of Theorem 7.5 we know that

$$(\mathcal{F}_{dc})^{-1}\circ(\mathcal{F}^{-1}\circ\mathcal{F})\circ\mathcal{F}_{dc}=P\ ,\qquad (\mathcal{F}'_{dc})^{-1}\circ((\mathcal{F}')^{-1}\circ\mathcal{F}')\circ\mathcal{F}'_{dc}=P'$$

hold, and furthermore Lemma 7.15 implies that $P:\mathcal{H}_{dc}\to\mathcal{H}_{dc}$ and $P':\mathcal{H}'_{dc}\to\mathcal{H}'_{dc}$ induce the identities Id on $\mathcal{M}(\mathcal{H}_{dc},\mathfrak{m}_{dc})$ and $\mathcal{M}(\mathcal{H}'_{dc},\mathfrak{m}'_{dc})$, respectively. This implies eq.(7.7). ■

The L_∞-version of this theorem is used in the proof of Kontsevich's deformation quantization [65].

We ends with leaving some comments. Though we defined homotopy between A_∞-morphisms as in Definition 7.1, one can consider another definition of homotopy based on interpolating two A_∞-morphisms with one parameter family of A_∞-morphisms. Such one is used in [24, 56]. This homotopy is included in the homotopy in Definition 7.1 in the sense that integrating the homotopy along the one parameter path yields a homotopy operator H in Definition 7.1. However the converse is not true. Actually, the homotopy obtained in such a way satisfies a condition in Remark 7.2 but the condition is different from, for instance, eq.(7.2). This another definition of homotopy is more compatible with the gauge transformation, where the gauge transformation (a) is in fact more natural than (b) used in this paper. Moreover, all of the theorems and corollaries in this section, for instance, remain true even if we replace Definition 7.1 to this another one [56].

All the arguments in this section hold in a similar way for cyclic A_∞-algebras. Note that, in the cyclic case, the gauge transformation preserves the form of the action S but does not preserve the form of the odd constant symplectic form. Moreover, all the arguments above hold true if an A_∞-algebra is replaced to an L_∞-algebra, too. In the cyclic case, these properties are closely related to the BV-formalism [10, 11, 33, 44]. To exploring these relations should be very interesting (see for instance [113, 112]).

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