KPZ equation, its renormalization and invariant measures

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Plan of the talk

1. KPZ equation (Ill-posedness, Renormalization)
2. Cole-Hopf solution, linear stochastic heat equation (SHE)
3. KPZ approximating equations
   (1) simple approximation
   (2) approximation adapted to finding invariant measures
4. Invariant measures of Cole-Hopf solution and SHE
5. Multi-component KPZ equation

1. KPZ equation

- The KPZ (Kardar-Parisi-Zhang, 1986) equation describes the motion of growing interface with random fluctuation.
- It has the form for height function $h(t,x)$:

$$\partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}(\partial_x h)^2 + \dot{W}(t,x), \quad x \in \mathbb{R} \ \text{(or } \ \mathbb{S}). \quad (1)$$

- We consider in 1D on a whole line $\mathbb{R}$ or on a finite interval $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ under periodic boundary condition.
- The coefficients $\frac{1}{2}$ are not important, since we can change them under space-time scaling.
- $\dot{W}(t,x)$ is a space-time Gaussian white noise with mean 0 and correlation function:

$$E[\dot{W}(t,x)\dot{W}(s,y)] = \delta(t-s)\delta(x-y). \quad (2)$$

Ill-posedness of the KPZ eq (1):

- The nonlinearity and roughness of the noise do not match.
- The linear SPDE:

$$\partial_t h = \tfrac{1}{2}\partial_x^2 h + \dot{W}(t,x),$$

obtained by dropping the nonlinear term has a solution $h \in C^{\frac{1}{4}-,\frac{1}{2}-}([0,\infty)\times\mathbb{R})$ a.s. Therefore, no way to define the nonlinear term $(\partial_x h)^2$ in (1) in a usual sense.

- Actually, the following Renormalized KPZ eq with compensator $\delta_x(x)$ $(=+\infty)$ has the meaning:

$$\partial_t h = \tfrac{1}{2}\partial_x^2 h + \tfrac{1}{2}\{(\partial_x h)^2 - \delta_x(x)\} + \dot{W}(t,x),$$

as we will see later.

2. Cole-Hopf solution to the KPZ equation

- Viscous stochastic Burgers equation: $u := \partial_x h$ satisfies

$$\partial_t u = \frac{1}{2}\partial_x^2 u + \frac{1}{2}\partial_x u^2 + \partial_x \dot{W}(t,x). \tag{3}$$

- Motivated by this, consider the linear stochastic heat equation (SHE) for $Z = Z(t,x)$:

$$\partial_t Z = \frac{1}{2}\partial_x^2 Z + Z\dot{W}(t,x), \tag{4}$$

with a multiplicative noise. This is a well-posed eq.

- The solution $Z(t)$ of (4) is defined in a generalized functions' sense or in a mild form due to Duhamel's principle using heat kernel $p(t,x,y) = \frac{1}{\sqrt{2\pi t}} e^{-(y-x)^2/(2t)}$.
- These two notions are equivalent, and $^{\exists}$unique solution s.t. $Z \in C([0,\infty) \times \mathbb{R})$ and $\sup_{x \in \mathbb{R}} e^{-r|x|}|Z(t,x)| < \infty$ for $^{\forall} r > 0$ a.s.
- (Strong comparison) $Z(0,x) \geq 0$ for $^{\forall} x \in \mathbb{R}$ and $Z(0,x) > 0$ for $^{\exists} x \in \mathbb{R}$
  $\implies \quad Z(t,x) > 0$ for $^{\forall} t > 0, ^{\forall} x \in \mathbb{R}$ a.s.
- Therefore, we can define the Cole-Hopf transformation:

$$h(t,x) := \log Z(t,x). \tag{5}$$

Heuristic derivation of the KPZ eq (with renormalization factor $\delta_x(x)$) from SHE (4) under the Cole-Hopf transformation (5):

- Apply Itô's formula for $h = \log z$:

$$\begin{aligned}\partial_t h &= Z^{-1}\partial_t Z - \tfrac{1}{2}Z^{-2}(\partial_t Z)^2 \\ &= Z^{-1}\left(\tfrac{1}{2}\partial_x^2 Z + Z\dot{W}\right) - \tfrac{1}{2}\delta_x(x) \\ &\qquad \text{by SHE (4) and } (dZ(t,x))^2 = (ZdW(t,x))^2 \\ &\qquad\qquad dW(t,x)dW(t,y) = \delta(x-y)dt \\ &= \tfrac{1}{2}\{\partial_x^2 h + (\partial_x h)^2\} + \dot{W} - \tfrac{1}{2}\delta_x(x)\end{aligned}$$

- This leads to the Renormalized KPZ eq:

$$\partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}\{(\partial_x h)^2 - \delta_x(x)\} + \dot{W}(t,x). \qquad (6)$$

- The function $h(t,x)$ defined by (5) is meaningful and called the Cole-Hopf solution of the KPZ eq, although the equation (1) does not make sense.

- Goal: To introduce approximations for (6), in particular, well adapted to finding invariant measures.
- Hairer (2013, 2014) gave a meaning to (6) without bypassing SHE.

3. KPZ approximating equation-1: Simple

- Symmetric convolution kernel Let $\eta \in C_0^\infty(\mathbb{R})$ s.t. $\eta(x) \geq 0$, $\eta(x) = \eta(-x)$ and $\int_{\mathbb{R}} \eta(x)dx = 1$ be given, and set $\eta^\varepsilon(x) := \eta(x/\varepsilon)/\varepsilon$ for $\varepsilon > 0$.
- Smeared noise The smeared noise is defined by

$$W^\varepsilon(t,x) = \langle W(t), \eta^\varepsilon(x - \cdot)\rangle \ \big(= W(t) * \eta^\varepsilon(x)\big).$$

- Approximating Eq-1:

$$\partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}\left((\partial_x h)^2 - \xi^\varepsilon\right) + \dot{W}^\varepsilon(t,x)$$
$$\partial_t Z = \frac{1}{2}\partial_x^2 Z + Z\dot{W}^\varepsilon(t,x),$$

where $\xi^\varepsilon = \eta_2^\varepsilon(0) \ (:= \eta^\varepsilon * \eta^\varepsilon(0))$.
- It is easy to show that $Z = Z^\varepsilon$ converges to the sol Z of (SHE), and therefore $h = h^\varepsilon$ converges to the Cole-Hopf solution of the KPZ eq.

KPZ approximating equation-2: Suit for inv meas

Goal: We want to introduce an approximation which is suitable to study the invariant measures.

General principle. Consider the SPDE

$$\partial_t h = F(h) + \dot{W},$$

and let A be a certain operator. Then, the structure of the invariant measures essentially does not change for

$$\partial_t h = A^2 F(h) + A\dot{W}.$$

This may not be true in non-reversible situation.

- KPZ approximating equation-2

$$\partial_t h = \tfrac{1}{2}\partial_x^2 h + \tfrac{1}{2}\left((\partial_x h)^2 - \xi^\varepsilon\right) * \eta_2^\varepsilon + \dot{W}^\varepsilon(t,x), \qquad (7)$$

where $\eta_2(x) = \eta * \eta(x)$, $\eta_2^\varepsilon(x) = \eta_2(x/\varepsilon)/\varepsilon$ and $\xi^\varepsilon = \eta_2^\varepsilon(0)$.

- Note that the solution h of (7) is smooth in x, so that we can consider the associated tilt process $\partial_x h$.

Let ν^ε be the distribution of $\partial_x(B * \eta^\varepsilon(x))$, where B is the two-sided Brownian motion. ν^ε is independent of choice of $B(0)$.

Theorem 1

ν^ε is invariant for the tilt process $\partial_x h$ determined by SPDE (7).

- DaPrato-Debussche-Tubaro (2007) studied a similar SPDE to (7) on $\mathbb{S}$.

Sketch of the proof:

- Step 1: Consider on a discrete torus $\mathbb{T}_N = \{1, 2, \ldots, N\}$. The discretization of $(\partial_x h)^2$ should be carefully chosen (cf. Myllys' talk, Krug-Spohn):

$$\frac{1}{3}\Big\{(h_{i+1} - h_i)^2 + (h_i - h_{i-1})^2 + (h_{i+1} - h_i)(h_i - h_{i-1})\Big\}, \ i \in \mathbb{T}_N$$

  Discrete version of ν^ε defined on $\mathbb{T}_N$ is invariant.

- Step 2: Continuum limit as $N \to \infty$ leads to the result on $\mathbb{S}$. This can be easily extended to a torus $\mathbb{S}_M = \mathbb{R}/M\mathbb{Z}$ of size M.
- Step 3: Take an infinite-volume limit as $M \to \infty$ by usual tightness and martingale problem approach.

Remark: Infinitesimal invariance can be directly shown based on Wiener-Itô expansion of tame functions Φ:

$$\int \mathcal{L}^\varepsilon \Phi(h) \nu^\varepsilon(dh) = 0, \tag{8}$$

where $\mathcal{L}^\varepsilon$ is (pre) generator of the SPDE (7).

$$\mathcal{L}^\varepsilon = \mathcal{L}_0^\varepsilon + \mathcal{A}^\varepsilon,$$

$$\mathcal{L}_0^\varepsilon \Phi(h) = \frac{1}{2} \int_{\mathbb{R}^2} D^2\Phi(x_1, x_2; h) \eta_2^\varepsilon(x_1 - x_2) dx_1 dx_2 + \frac{1}{2} \int_{\mathbb{R}} \partial_x^2 h(x) D\Phi(x; h) dx,$$

$$\mathcal{A}^\varepsilon \Phi(h) = \frac{1}{2} \int_{\mathbb{R}} \left((\partial_x h)^2 - \xi^\varepsilon \right) * \eta_2^\varepsilon(x) D\Phi(x; h) dx.$$

Combined with the well-posedness of $\mathcal{L}^\varepsilon$-martingale problem, which can be shown at least on $\mathbb{S}$, it is expected that the infinitesimal invariance implies Thm 1. But this is not clear in infinite-dimensional setting; cf. Echeverria (1982), Bhatt-Karandikar (1993).

Cole-Hopf transform for SPDE (7)

- The goal is to pass to the limit $\varepsilon \downarrow 0$ in the KPZ approximating equation (7):

$$\partial_t h = \tfrac{1}{2}\partial_x^2 h + \tfrac{1}{2}\big((\partial_x h)^2 - \xi^\varepsilon\big) * \eta_2^\varepsilon + \dot{W}^\varepsilon(t,x).$$

- We consider its Cole-Hopf transform: $Z \, (\equiv Z^\varepsilon) := e^h$. Then, by Itô's formula, Z satisfies the SPDE:

$$\partial_t Z = \tfrac{1}{2}\partial_x^2 Z + A^\varepsilon(x,Z) + Z\dot{W}^\varepsilon(t,x), \tag{9}$$

where

$$A^\varepsilon(x,Z) = \frac{1}{2}Z(x)\left\{\left(\frac{\partial_x Z}{Z}\right)^2 * \eta_2^\varepsilon(x) - \left(\frac{\partial_x Z}{Z}\right)^2(x)\right\}.$$

- The complex term $A^\varepsilon(x,Z)$ looks vanishing as $\varepsilon \downarrow 0$.

- But this is not true. Indeed, under the average in time t, $A^\varepsilon(x, Z)$ can be replaced by a linear function $\frac{1}{24}Z$.
- The limit as $\varepsilon \downarrow 0$ (under stationarity of tilt),

$$\partial_t Z = \frac{1}{2}\partial_x^2 Z + \frac{1}{24}Z + Z\dot{W}(t,x).$$

- Or, heuristically at KPZ level,

$$\partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}\{(\partial_x h)^2 - \delta_x(x)\} + \frac{1}{24} + \dot{W}(t,x).$$

Taking the limit $\varepsilon \downarrow 0$ (Similar to Boltzmann-Gibbs principle)

- Asymptotic replacement of $A^\varepsilon(x, Z^\varepsilon(s))$ by $\frac{1}{24} Z^\varepsilon(s,x)$.
- To avoid the complexity arising from the infiniteness of invariant measures, we view $h^\varepsilon(t,\rho) = \int h^e(t,x)\rho(x)dx$ (height averaged by $\rho \in C_0^\infty(\mathbb{R}), \geq 0, \int \rho(x)dx = 1$) in modulo 1 (called wrapped process).

Theorem 2

For every $\varphi \in C_0(\mathbb{R})$ satisfying supp $\varphi \cap$ supp $\rho = \emptyset$, we have that

$$\lim_{\varepsilon \downarrow 0} E^{\pi \otimes \nu^\varepsilon} \left[\left\{ \int_0^t \tilde{A}^\varepsilon(\varphi, Z^\varepsilon(s)) ds \right\}^2 \right] = 0,$$

where π is the uniform measure for $h^\varepsilon(0,\rho) \in [0,1)$,

$$\tilde{A}^\varepsilon(\varphi, Z) = \int_{\mathbb{R}} \tilde{A}^\varepsilon(x, Z)\varphi(x)dx$$

$$\tilde{A}^\varepsilon(x, Z) = A^\varepsilon(x, Z) - \frac{1}{24} Z(x).$$

Proof of Theorem 2

(1) Reduction of equilibrium dynamic problem to static one:

- The expectation is bounded by

$$\leq 20\, t \sup_{\Phi \in L^2(\pi\otimes\nu^\varepsilon)} \left\{ 2E^{\pi\otimes\nu^\varepsilon}\left[\tilde{A}^\varepsilon(\varphi, Z)\Phi\right] - \langle \Phi, (-\mathcal{L}_0^\varepsilon)\Phi\rangle_{\pi\otimes\nu^\varepsilon} \right\},$$
$$\left(= 20t\|A^\varepsilon(\varphi, Z)\|_{-1,\varepsilon}^2\right)$$

  where $\mathcal{L}_0^\varepsilon$ is the symmetric part of $\mathcal{L}^\varepsilon$. This is a generic bound in a stationary situation.

- Here,

$$2E^{\pi\otimes\nu^\varepsilon}\left[\tilde{A}^\varepsilon(\varphi, Z)\Phi\right] = E^\pi\left[Z_\rho E^{\nu^\varepsilon}\left[B^\varepsilon(\varphi, Z)\Phi(h(\rho), \nabla h)\right]\right],$$

  where $Z_\rho = \exp\{\int_{\mathbb{R}} \log Z(x)\rho(x)dx\}$, $B^\varepsilon(x, Z) = \frac{2A^\varepsilon(x,Z)}{Z_\rho}$ and $B^\varepsilon(\varphi, Z) = \int_{\mathbb{R}} B^\varepsilon(x, Z)\varphi(x)dx$.

(2) The key is the following static bound:

Proposition 3

For $\Phi = \Phi(\nabla h) \in L^2(\tilde{\mathcal{C}}, \nu)$ *such that* $\|\Phi\|_{1,\varepsilon}^2 = \langle \Phi, (-\mathcal{L}_0^\varepsilon)\Phi \rangle_{\pi \otimes \nu^\varepsilon} < \infty$ *, and* φ *satisfying the condition of Theorem 2, we have that*

$$\left| E^{\nu^\varepsilon} \left[B^\varepsilon(\varphi, Z)\Phi \right] \right| \le C(\varphi)\sqrt{\varepsilon} \|\Phi\|_{1,\varepsilon}, \qquad (10)$$

with some positive constant $C(\varphi)$*, which depends only on* φ*, for all* ε*:* $0 < \varepsilon \le \frac{\delta}{2} \wedge \frac{1}{6}$.

Once this proposition is shown, the proof of Theorem 2 is concluded, since the sup in the last slide is bounded by

$$\le 20t \sup\{2eC(\varphi)\sqrt{\varepsilon}\|\Phi\|_{1,\varepsilon} - \|\Phi\|_{1,\varepsilon}^2\} = \text{const}(\sqrt{\varepsilon})^2 \to 0.$$

Point of the proof of Proposition 3

- First note that

$$E^{\nu^\varepsilon}\left[B^\varepsilon(\varphi,Z)\Phi\right] = E^{\nu^\varepsilon}\left[\frac{Z(x)}{Z_\rho}\left(\{\Psi^\varepsilon * \eta_2^\varepsilon(x) - \Psi^\varepsilon(x)\} - \frac{1}{12}\right)\Phi\right]$$

- To compute this expectation, since $\{\Psi^\varepsilon * \eta_2^\varepsilon(x) - \Psi^\varepsilon(x)\}$ is 2nd order Wiener functional, we need to pick up the 2nd order and 0th order terms of the products of two Wiener functionals $\frac{Z(x)}{Z_\rho} \times \Phi$. We apply the diagram formula to compute the Winer chaos expansion of products of two functions.

- Note that, under ν,

$$\frac{Z(x)}{Z_\rho} = e^{B(x)-\int_{\mathbb{R}} B(y)\rho(y)dy}$$
$$= e^{a(x)} \left\{ 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\mathbb{R}^n} \phi_x^{\otimes n}(u_1, \dots, u_n) dB(u_1) \cdots dB(u_n) \right\},$$

where,

$$\phi_x(u) = 1_{(-\infty,x]}(u) - \int_u^\infty \rho(y)dy,$$
$$a(x) = \tfrac{1}{2} \int_{\mathbb{R}} \phi_x(u)^2 du.$$

Note that the kernel ϕ_x has jump.

- $\frac{1}{24}$ is the speed of growing interface, and already appears in some previous talks and in many KPZ related papers.
- For general convolution kernel η, this constant is given by $J/2$, where

$$J = P(R_1 + R_3 > 0, R_2 + R_3 > 0) - P(R_1 > 0, R_2 > 0),$$

and $\{R_i\}_{i=1}^3$ are i.i.d. r.v.s distributed under $\eta_2(x)dx$
- If η is symmetric,

$$\begin{aligned} P(R_1 + R_3 > 0, R_2 + R_3 > 0) &= P(R_1 - R_3 > 0, R_2 - R_3 > 0) \\ &= P(R_3 = \min R_i) = \tfrac{1}{3}, \end{aligned}$$

so that $J = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$.
- If the support of $\eta \subset [0, \infty)$ (or $\subset (-\infty, 0]$), then $J = 0$.

- Wrapping can be removed by showing uniform estimate:

$$\sup_{0<\varepsilon<1} E\left[\sup_{0\le t\le T} h^\varepsilon(t,\rho)^2\right] < \infty.$$

Namely, height cannot move very fast. This is shown only on a torus (since we need Poincaré inequality).

- Under the stationary situation of the tilt processes, in the limit, we obtain the SHE:

$$\partial_t Z = \tfrac{1}{2}\partial_x^2 Z + \tfrac{1}{24} Z + Z\dot{W}(t,x). \tag{11}$$

- This looks different from the original SHE (4), but the solution Z_t of (11) gives the solution $\tilde{Z}_t$ of (4) under the simple transformation $\tilde{Z}_t := e^{-\frac{t}{24}} Z_t$.
- This implies the invariance of the distribution of the geometric Brownian motion for the tilt process determined by the SHE (4), and therefore that of BM for Cole-Hopf solution.

4. Invariant measures of Cole-Hopf sol and SHE

As a byproduct, one can give a class of invariant measures for the stochastic heat equation (4) and for the Cole-Hopf solution of the KPZ equation.

- Let $\mu^c, c \in \mathbb{R}$ be the distribution of $e^{B(x)+c\,x}, x \in \mathbb{R}$ on $\mathcal{C}_+$, where $B(x)$ is the two-sided Brownian motion such that $\mu^c(B(0) \in dx) = dx$.
- Let ν^c be the distribution of $B(x) + c\,x$ on $\mathcal{C}$.
- Note that these are not probability measures.

Theorem 4

$\{\mu^c\}_{c\in\mathbb{R}}$ are invariant under SHE (4), i.e.,
$Z(0) \overset{law}{=} \mu^c \Rightarrow Z(t) \overset{law}{=} \mu^c$ for all $t \geq 0$ and $c \in \mathbb{R}$.

Corollary 5

$\{\nu^c\}_{c\in\mathbb{R}}$ are invariant under the Cole-Hopf solution of the KPZ equation.

- c means the average tilt of the interface.
- We have different invariant measures for different average tilts.
- Reversibility does not hold, but a kind of Yaglom reversibility holds.

- (Scale invariance) If $Z(t,x)$ is a solution of (4), then

$$Z^c(t,x) := e^{cx+\frac{1}{2}c^2 t} Z(t, x+ct)$$

is also a solution (with a new white noise). Therefore, once the invariance of μ^0 is shown, μ^c is also invariant for every $c \in \mathbb{R}$.

- One expects μ^c, $c \in \mathbb{R}$ to be all the extremal invariant measures (except constant multipliers), but this remains open; cf. F-Spohn for $\nabla\varphi$-interface model.

- The argument at the end of the last Section combined with Theorem 1 at approximating level shows the invariance of μ for tilt processes.
- To extend this to the height processes Z_t, we introduce the transformation $h^\varepsilon(x, Z) := \log(Z * \eta^\varepsilon(x))$. Then, the evolution of $h^\varepsilon(x, Z_t)$ is governed only by the tilt variables and the initial data $h^\varepsilon(x, Z_0)$.

5. Multi-component KPZ equation

- Ferrari-Sasamoto-Spohn (2013) studied $\mathbb{R}^d$-valued KPZ equation for $h(t,x) = (h^\alpha(t,x))_{\alpha=1}^d$ on $\mathbb{R}$:

$$\partial_t h^\alpha = \tfrac{1}{2}\partial_x^2 h^\alpha + \tfrac{1}{2}\Gamma^\alpha_{\beta\gamma}\partial_x h^\beta \partial_x h^\gamma + \dot{W}^\alpha(t,x), \ x \in \mathbb{R}, \quad (12)$$

where $\dot{W}(t,x) = (\dot{W}^\alpha(t,x))_{\alpha=1}^d$ is an $\mathbb{R}^d$-valued space-time Gaussian white noise. The constants $(\Gamma^\alpha_{\beta\gamma})_{1\le\alpha,\beta,\gamma\le d}$ satisfy the condition:

$$\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta} = \Gamma^\gamma_{\beta\alpha}. \quad (13)$$

- Similar SPDE appears to discuss motion of loops on a manifold, cf. Funaki (1992), Hairer (2013, preprint).

- We introduce the smeared noise:

$$W^\varepsilon(t,x) \equiv (\dot W^{\varepsilon,\alpha}(t,x))_{\alpha=1}^d = \langle W(t), \eta^\varepsilon(x-\cdot)\rangle,$$

  and consider $\mathbb{R}^d$-valued KPZ approximating equation for $h = h^\varepsilon(t,x) \equiv (h^{\varepsilon,\alpha}(t,x))_{\alpha=1}^d$:

$$\partial_t h^\alpha = \frac{1}{2}\partial_x^2 h^\alpha + \frac{1}{2}\Gamma^\alpha_{\beta\gamma}(\partial_x h^\beta \partial_x h^\gamma - \xi^\varepsilon \delta^{\beta\gamma}) * \eta_2^\varepsilon + \dot W^{\varepsilon,\alpha}(t,x), \quad (14)$$

  where $\delta^{\beta\gamma}$ denotes Kronecker's δ.
- Let ν^ε be the distribution of $\partial_x(B * \eta^\varepsilon(x))$ on $\mathcal{C} = C(\mathbb{R};\mathbb{R}^d)$, where B is the $\mathbb{R}^d$-valued two-sided Brownian motion satisfying $B(0)=0$.

Theorem 6

The probability measure ν^ε on $\mathcal{C}$ is infinitesimally invariant for the tilt process $\partial_x h$ of the SPDE (14).

Summary of the talk.

1. KPZ equation:

$$\partial_t h = \tfrac{1}{2}\partial_x^2 h + \tfrac{1}{2}(\partial_x h)^2 + \dot{W}(t,x), \quad x \in \mathbb{R}.$$

2. KPZ approximating equation with $W^\varepsilon(t,x) = \langle W(t), \eta^\varepsilon(x-\cdot)\rangle$:

$$\partial_t h = \tfrac{1}{2}\partial_x^2 h + \tfrac{1}{2}\left((\partial_x h)^2 - \xi^\varepsilon\right) * \eta_2^\varepsilon + \dot{W}^\varepsilon(t,x)$$

   has invariant measure ν^ε (=distribution of $B * \eta^\varepsilon$).

3. Cole-Hopf transform $Z := e^h$ leads to the SPDE:

$$\partial_t Z = \tfrac{1}{2}\partial_x^2 Z + \tfrac{1}{2} Z \left\{ \left(\frac{\partial_x Z}{Z}\right)^2 * \eta_2^\varepsilon - \left(\frac{\partial_x Z}{Z}\right)^2 \right\} + Z\dot{W}^\varepsilon(t,x)$$

4. As $\varepsilon \downarrow 0$, one can replace the middle term by $\frac{1}{24}Z$ under time average and get the SPDE in the limit:

$$\partial_t Z = \tfrac{1}{2}\partial_x^2 Z + \tfrac{1}{24} Z + Z\dot{W}(t,x), \quad x \in \mathbb{R}.$$

5. Multi-component KPZ approximating equation.

Thank you for your attention!