Exact results for the KPZ equation from the replica Bethe ansatz and the sine-Gordon field theory

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Alberto Rosso (LPTMS Orsay)

Thomas Gueudre (LPTENS)

P. Calabrese, P. Le Doussal, A. Rosso EPL 90 20002 (2010)
P. Calabrese, P. Le Doussal, Phys. Rev. Letters 106 250603 (2011) and J. Stat. Mech. P06001 (2012) T. Gueudre, P. Le Doussal, EPL 100 26006 (2012).

- many models in “KPZ class” exhibit universality related to random matrix theory: Tracy Widom distributions: of largest eigenvalue of GUE,GOE..

- provide solution directly continuum KPZ eq./DP (at all times)

KPZ eq. is in KPZ class !

methods of integrable systems (Bethe Ansatz) + disordered systems (replica)

Outline:

- growth of 1D interfaces: KPZ equation, KPZ universality class

- random matrices largest eigenvalues: Tracy Widom universal distributions

- solving KPZ at any time by mapping to directed paths
  then using (imaginary time) quantum mechanics
  attractive bose gas (integrable) => large time TW distrib. for KPZ height

- droplet initial condition

- flat initial condition

- KPZ in half space

- KPZ from sine-Gordon FT

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- not talk about:
  stationary initial condition

T. Inamura, T. Sasamoto
Phys. Rev. Lett. 108, 190603 (2012)

- other works/perspectives:

reviews KPZ: Corwin arXiv 1106.1596, H. Spohn..

also works by: V. Dotsenko, H. Spohn, Sasamoto

(math) Amir, Corwin, Quastel, Borodine,..

also G. Schehr, Reymenik, Ferrari, O'Connell,..

Kardar Parisi Zhang equation

Phys Rev Lett 56 889 (1986)

growth of an interface of height h(x,t)

$$\partial_t h = \nu \partial_x^2 h + \frac{\lambda_0}{2}(\partial_x h)^2 + \eta(x,t)$$

diffusion

noise

$$\overline{\eta(x,t)\eta(x',t')} = D\delta(x-x')\delta(t-t')$$

- 1D scaling exponents $h \sim t^{1/3} \sim x^{1/2}$ $x \sim t^{2/3}$

- P(h=h(x,t)) non gaussian

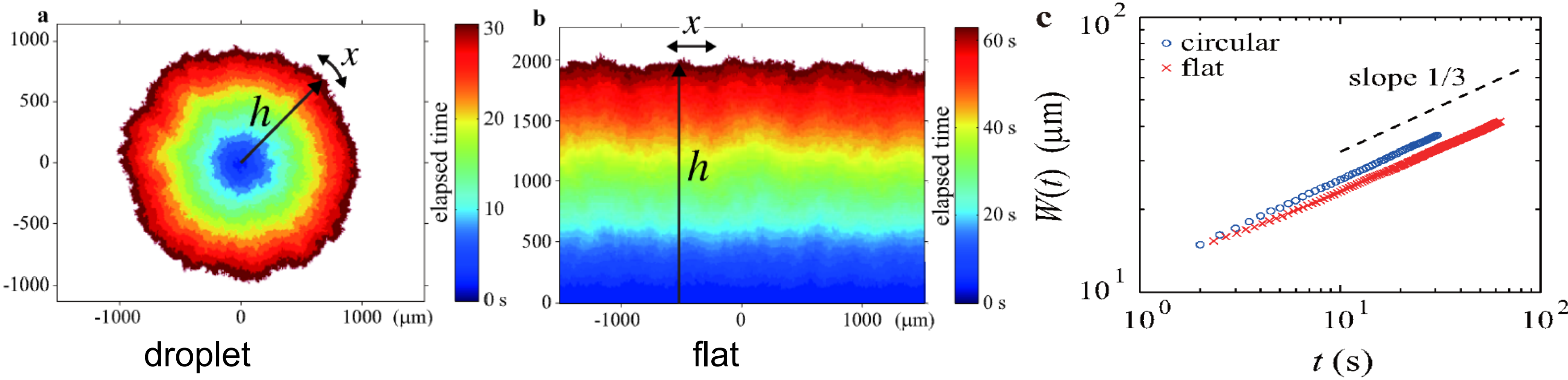
depends on some details of initial condition

flat	h(x,0) =0
wedge (droplet)	h(x,0) = - w |x|

$\lambda_0 = 0$ Edwards Wilkinson P(h) gaussian

- Turbulent liquid crystals

Takeuchi, Sano PRL 104 230601 (2010)



droplet

flat

$$W(t) \equiv \sqrt{\langle [h(x,t) - \langle h \rangle]^2 \rangle}$$

$$h(x,t) \simeq_{t\to+\infty} v_\infty t + \chi\, t^{1/3}$$

χ is a random variable

$$h \sim t^{1/3} \sim x^{1/2}$$

also reported in:

- slow combustion of paper — J. Maunuksela et al. PRL 79 1515 (1997)
- bacterial colony growth — Wakita et al. J. Phys. Soc. Japan. 66, 67 (1996)
- fronts of chemical reactions — S. Atis (2012)
- formation of coffee rings via evaporation — Yunker et al. PRL (2012)

Large N by N random matrices H, with Gaussian independent entries

eigenvalues λ_i $i = 1, ..N$

H is:

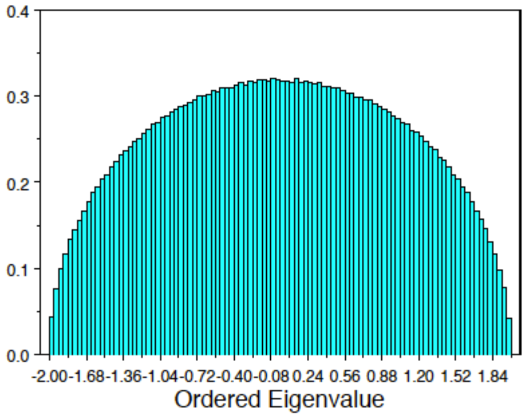
$$P[\lambda] = c_{N,\beta} \prod_{i<j} |\lambda_i - \lambda_j|^\beta e^{-\frac{\beta N}{4} \sum_{k=1}^N \lambda_k^2}$$

$\beta =$

- 1 (GOE) real symmetric
- 2 (GUE) hermitian
- 4 (GSE) symplectic

Universality large N :

- DOS: semi-circle law



histogram of eigenvalues N=25000

- distribution of the largest eigenvalue

$H \to NH$

$$\lambda_{max} = 2N + \chi N^{1/3}$$

$$Prob(\chi < s) = F_\beta(s)$$

Tracy Widom (1994)

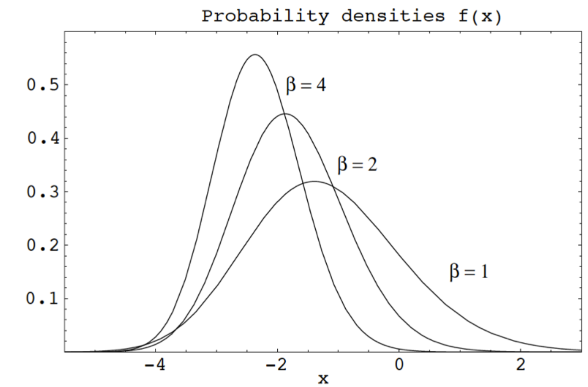
Tracy-Widom distributions (largest eigenvalue of RM)

GOE $$F_1(s) = Det[I - K_1]$$

$$K_1(x,y) = \theta(x) Ai(x+y+s) \theta(y)$$

Fredholm determinants

$$(I-K)\phi(x) = \phi(x) - \int_y K(x,y)\phi(y)$$



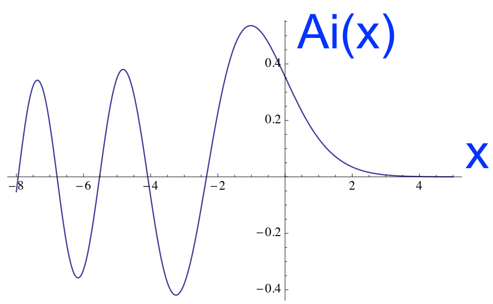
GUE $$F_2(s) = Det[I - K_2]$$

$$K_2(x,y) = K_{Ai}(x+s, y+s)$$

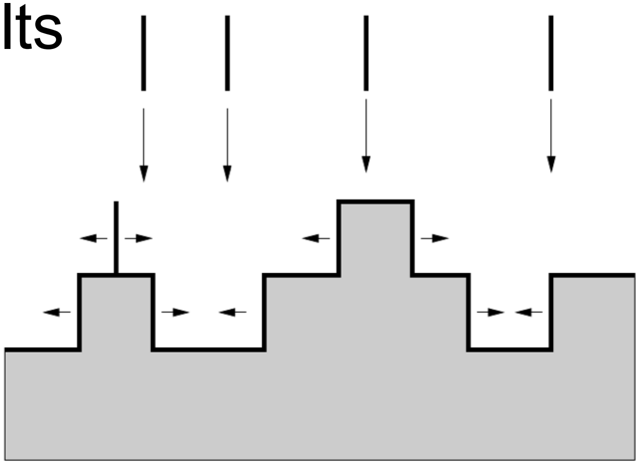
$$K_{Ai}(x,y) = \int_{v>0} Ai(x+v) Ai(y+v)$$

Ai(x-E)

is eigenfunction E
particle linear potential



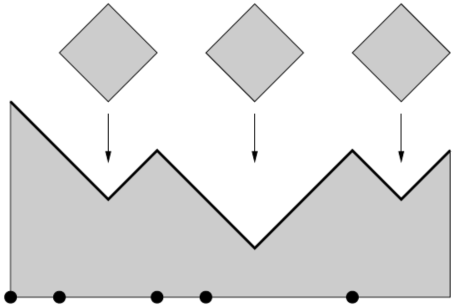
discrete models in KPZ class/exact results

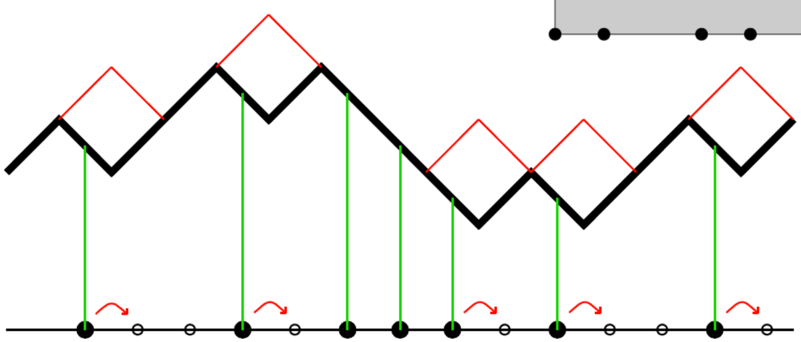


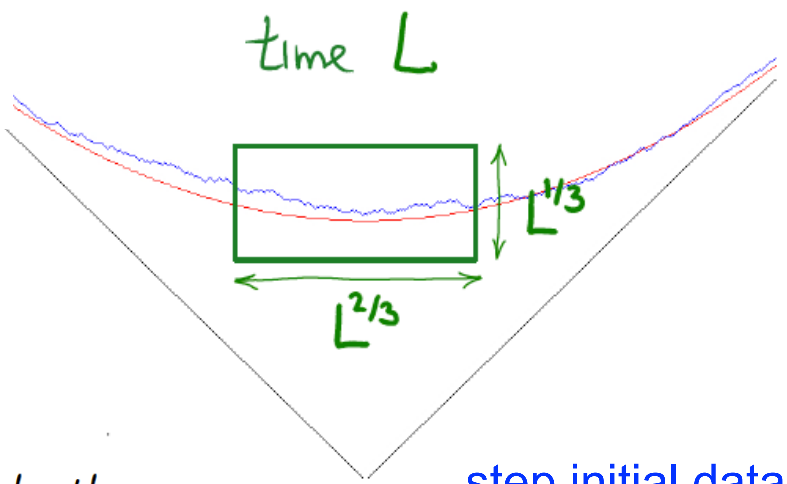
- polynuclear growth model (PNG)

  Prahofer, Spohn, Baik, Rains (2000)

- totally asymmetric exclusion process (TASEP)







Red boxes are added independently at rate 1. Equivalently, particles with no neighbour on the right jump independently with waiting time distributed as $\exp(-x)dx$.

step initial data

Johansson (1999)

Exact results for height distributions for some discrete models in KPZ class

- PNG model

Baik, Deft, Johansson (1999)

$$h(0,t) \simeq_{t\to\infty} 2t + t^{1/3}\chi$$

droplet IC
GUE

Prahofer, Spohn, Ferrari, Sasamoto,..
(2000+)

flat IC $\chi = \chi_1$ GOE

multi-point correlations
Airy processes

$A_2(y)$ GUE

$A_1(y)$ GOE

$$h(yt^{2/3},t) \simeq_{t\to\infty} 2t - \frac{y^2}{2t} + t^{1/3}A_n(y)$$

- similar results for TASEP

Johansson (1999), ...

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Johansson (1999), ...

Question: is KPZ equation in KPZ class ?

Universal distribution of conductance in 2D localized phase

Somoza, Ortuno, Prior (2007)

$$H = \sum_i \epsilon_i c_i^+ c_i - t \sum_{<ij>} c_i^+ c_j + c_j^+ c_i$$

$$\ln g = -\frac{2L}{\xi} + \alpha\left(\frac{L}{\xi}\right)^{1/3} \chi_2$$

ξ localization length

L system size

χ random variable with Tracy Widom distribution

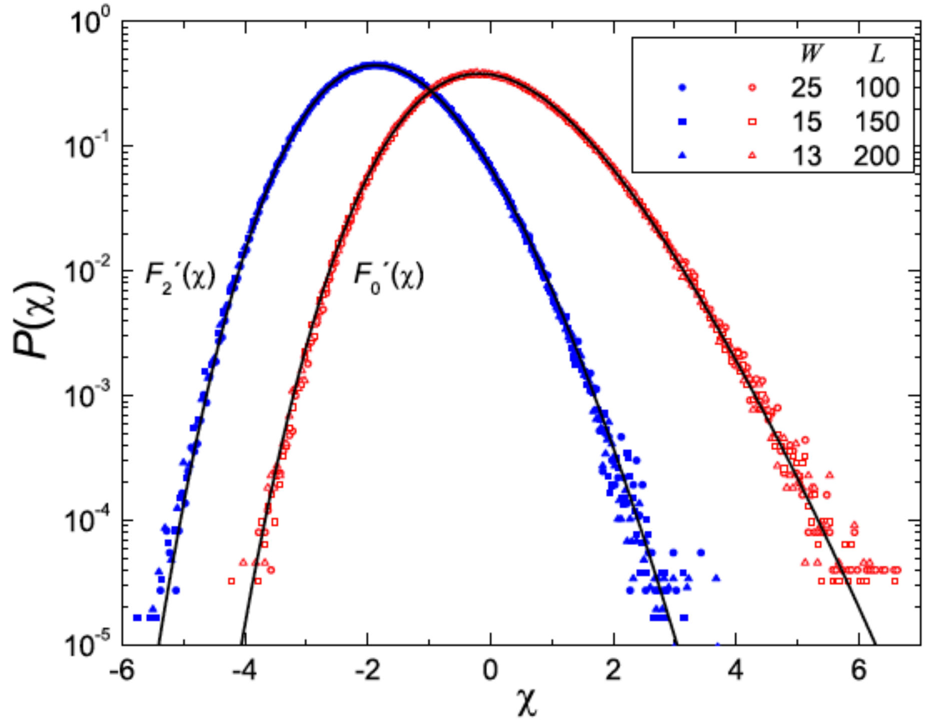

FIG. 1 (color online). Histograms of $\ln g$ versus the scaled variable χ for several sizes and disorders of the Anderson model with narrow (solid symbols) and wide (empty symbols) leads. The continuous lines correspond to $F_2'(\chi)$ and $F_0'(\chi)$.

i

Cole Hopf mapping

KPZ equation

Continuum
Directed paths (polymers)
in a random potential

Quantum mechanics
of bosons
(imaginary time)

Kardar 87

Continuum DP fixed endpoint/KPZ Narrow wedge (droplet)

Replica Bethe Ansatz (RBA)

- P. Calabrese, P. Le Doussal, A. Rosso EPL 90 20002 (2010)
- V. Dotsenko, EPL 90 20003 (2010) J Stat Mech P07010
  Dotsenko Klumov P03022 (2010).

Weakly ASEP

- T Sasamoto and H. Spohn PRL 104 230602 (2010)
  Nucl Phys B 834 523 (2010) J Stat Phys 140 209 (2010).
- G.Amir, I.Corwin, J.Quastel Comm.Pure.Appl.Math. 64 466 (2011)

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- G.Amir, I.Corwin, J.Quastel Comm.Pure.Appl.Math. 64 466 (2011)

Continuum DP one free endpoint/KPZ Flat (RBA)

P. Calabrese, P. Le Doussal, PRL 106 250603 (2011) and J. Stat. Mech. P06001 (2012)

ASEP J. Quastel, J. Ortmann and D. Remenik in preparation

Cole Hopf mapping

KPZ equation:

$$\partial_t h = \nu \partial_x^2 h + \frac{\lambda_0}{2}(\partial_x h)^2 + \eta(x,t)$$

define:

$$Z(x,t) = e^{\frac{\lambda_0}{2\nu} h(x,t)}$$

$$\lambda_0 h(x,t) = T \ln Z(x,t)$$

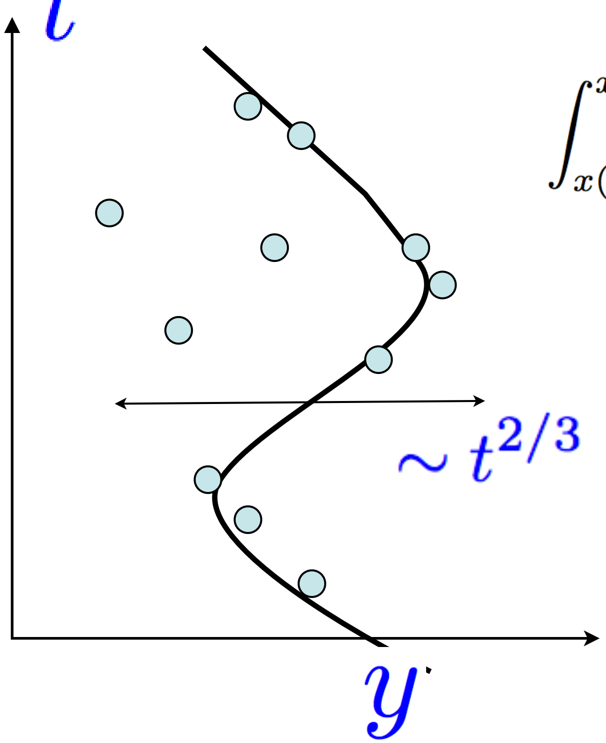
$$T = 2\nu$$

it satisfies:

$$\partial_t Z = \frac{T}{2}\partial_x^2 Z - \frac{V(x,t)}{T} Z$$

$$\lambda_0 \eta(x,t) = -V(x,t)$$

describes directed paths in random potential V(x,t)



$$\sim t^{2/3}$$

$$Z(x,t|y,0) =$$

$$\int_{x(0)=y}^{x(t)=x} Dx(\tau)\, e^{-\frac{1}{T}\int_0^t d\tau \frac{\kappa}{2}\left(\frac{dx(\tau)}{d\tau}\right)^2 + V(x(\tau),\tau)}$$

$$\overline{V(x,t)V(x',t')} = \bar{c}\ \ \delta(t-t')\delta(x-x')$$

Feynman Kac

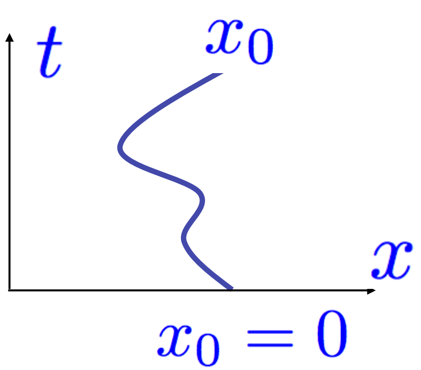
$$Z(x,y,t=0) = \delta(x-y)$$

$$\partial_t Z = \frac{T}{2\kappa}\partial_x^2 Z - \frac{V(x,t)}{T} Z$$

initial conditions

$$e^{\frac{\lambda_0}{2\nu}h(x,t)} = \int dy Z(x,t|y,0) e^{\frac{\lambda_0}{2\nu}h(y,t=0)}$$

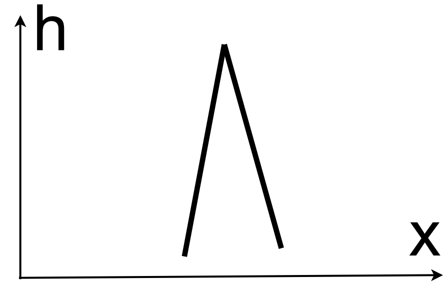
1) DP both fixed endpoints $Z(x_0,t|x_0,0)$



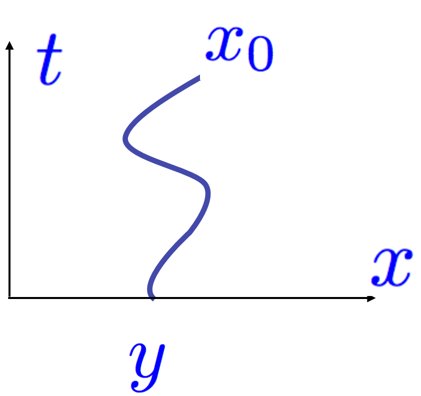
KPZ: narrow wedge <=> droplet initial condition

$$h(x,t=0) = -w|x|$$

$$w \to \infty$$



2) DP one fixed one free endpoint $\int dy Z(x_0,t|y,0)$



KPZ: flat initial condition

$$h(x,t=0) = 0$$

Schematically

$$Z = e^{\frac{\lambda_0 h}{2\nu}}$$

calculate $\overline{Z^n} = \int dZ\, Z^n P(Z)$ $\quad n \in \mathbb{N}$

"guess" the probability distribution from its integer moments:

$$P(Z) \to P(\ln Z) \to P(h)$$

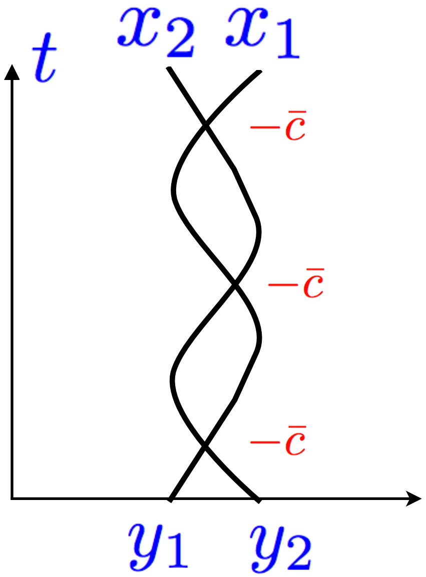
Quantum mechanics and Replica..

$$\mathcal{Z}_n := \overline{Z(x_1,t|y_1,0)..Z(x_n,t|y_n0)} = \langle x_1,..x_n|e^{-tH_n}|y_1,..y_n\rangle$$

$$\partial_t \mathcal{Z}_n = -H_n \mathcal{Z}_n$$

$$x = T^3\kappa^{-1}\tilde{x} \quad , \quad t = 2T^5\kappa^{-1}\tilde{t}$$

drop the tilde..



$$H_n = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} - 2\bar{c} \sum_{1\le i<j\le n} \delta(x_i - x_j)$$

Attractive Lieb-Lineger (LL) model (1963) what do we need from quantum mechanics ?

- KPZ with droplet initial condition

  = fixed endpoint DP partition sum

μ eigenstates

E_μ eigen-energies

$$e^{-tH} = \sum_\mu |\mu > e^{-E_\mu t} < \mu|$$

$$\overline{Z(x_0 t|x_0 0)^n} =< x_0 ... x_0 | e^{-tH_n} | x_0, .. x_0 >$$

symmetric states = bosons

$$= \sum_\mu \Psi^*_\mu(x_0..x_0) \Psi_\mu(x_0..x_0) \frac{1}{||\mu||^2} e^{-E_\mu t}$$

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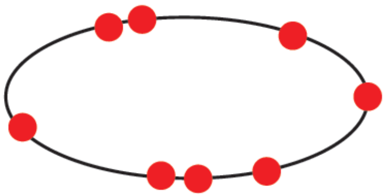
symmetric states = bosons

$$= \sum_\mu \Psi_\mu^*(x_0..x_0)\Psi_\mu(x_0..x_0)\frac{1}{||\mu||^2}e^{-E_\mu t}$$

- flat initial condition

$$\overline{(\int_y Z(x_0 t|y0))^n} = \sum_\mu \Psi_\mu^*(x_0, .x_0) \int_{y_1,.y_n} \Psi_\mu(y_1, .y_n)\frac{1}{||\mu||^2}e^{-E_\mu t}$$

LL model: n bosons on a ring with local delta attraction



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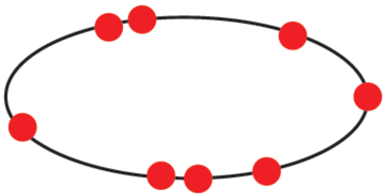

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Bethe Ansatz:

all (un-normalized) eigenstates are of the form (plane waves + sum over permutations)

$$\Psi_\mu = \sum_P A_P \prod_{j=1}^n e^{i\lambda_{P_\ell} x_\ell}$$

$$E_\mu = \sum_{j=1}^{n} \lambda_j^2$$

$$A_P = \prod_{n\geq \ell > k \geq 1} \left(1 - \frac{ic \ \mathrm{sgn}(x_\ell - x_k))}{\lambda_{P_\ell} - \lambda_{P_k}}\right)$$

They are indexed by a set of rapidities $\lambda_1, ..\lambda_n$

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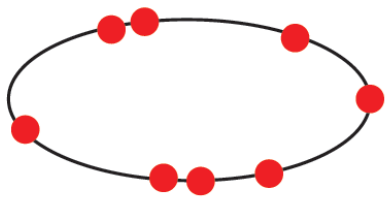

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They are indexed by a set of rapidities $\lambda_1, ..\lambda_n$

which are determined by solving the N coupled Bethe equations (periodic BC)

$$e^{i\lambda_j L} = \prod_{\ell\neq j} \frac{\lambda_j - \lambda_\ell - i\bar{c}}{\lambda_j - \lambda_\ell + i\bar{c}}$$

n bosons+attraction => bound states

Bethe equations + large L => rapidities have imaginary parts

Derrida Brunet 2000

- ground state = a single bound state of n particules Kardar 87

$$\psi_0(x_1,..x_n) \sim \exp(-\frac{\bar{c}}{2}\sum_{i<j}|x_i - x_j|) \qquad E_0(n) = -\frac{\bar{c}^2}{12}n(n^2-1)$$

$$\overline{Z^n} = \overline{e^{n \ln Z}} \quad \sim_{t\to\infty} e^{-tE_0(n)} \sim e^{\frac{\bar{c}^2}{12}n^3 t}$$

exponent 1/3

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exponent 1/3

$$\overline{Z^n} = \overline{e^{n\ln Z}} = e^{\sum_p \frac{1}{p!}n^p \overline{(\ln Z)^p}^c}$$

can it be continued in n ?

NO !

$$F = -\ln Z = \bar{F} + \lambda f \qquad \lambda = (\frac{\bar{c}^2}{4}t)^{1/3}$$

$$P(f) \sim_{f\to-\infty} \exp(-\frac{2}{3}(-f)^{3/2})$$

information about the tail of FE distribution

$$\overline{Z^n} = \int df e^{-n\lambda f - \frac{2}{3}(-f)^{3/2}} \sim e^{\frac{1}{3}\lambda^3 n^3}$$

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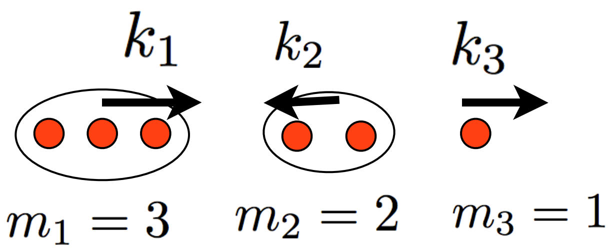
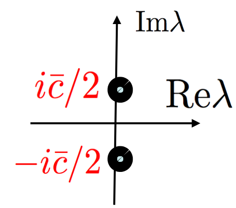
need to sum over all eigenstates !

- all eigenstates are: All possible partitions of n into ns “strings” each with mj particles and momentum kj



$$n = \sum_{j=1}^{n_s} m_j$$



$$E_\mu = \sum_{j=1}^{n_s}(m_j k_j^2 - \frac{\bar{c}^2}{12}m_j(m_j^2-1))$$

Integer moments of partition sum: fixed endpoints (droplet IC)

$$\overline{Z^n} = \sum_{\mu} \frac{|\Psi_\mu(0..0)|^2}{||\mu||^2} e^{-E_\mu t}$$

$$\Psi_\mu(0..0) = n!$$

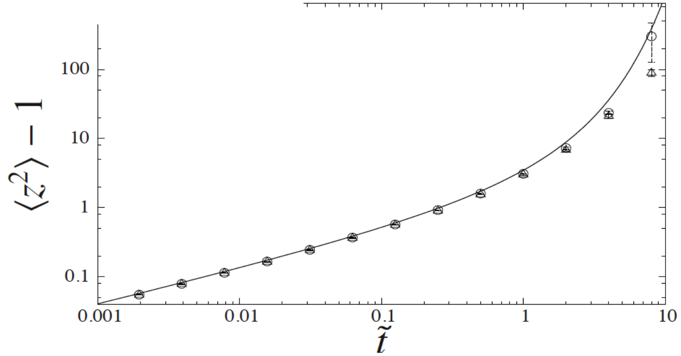
norm of states: Calabrese-Caux (2007)

$$\overline{\hat{Z}^n} = \sum_{n_s=1}^{n} \frac{n!}{n_s!(2\pi\bar{c})^{n_s}} \sum_{(m_1,\dots m_{n_s})_n}$$

$$n = \sum_{j=1}^{n_s} m_j$$

$$\int \prod_{j=1}^{n_s} \frac{dk_j}{m_j} \Phi[k,m] \prod_{j=1}^{n_s} e^{m_j^3 \frac{\bar{c}^2 t}{12} - m_j k_j^2 t},$$

$$\Phi[k,m] = \prod_{1 \le i < j \le n_s} \frac{(k_i - k_j)^2 + (m_i - m_j)^2 c^2/4}{(k_i - k_j)^2 + (m_i + m_j)^2 c^2/4}$$



how to get P(ln Z) i.e. P(h) ?

$$\ln Z = -\lambda f \qquad \lambda = (\frac{\bar{c}^2}{4} t)^{1/3} \qquad f$$

random variable
expected O(1)

introduce generating function of moments g(x):

$$g(x) = 1 + \sum_{n=1}^{\infty} \frac{(-e^{\lambda x})^n}{n!} \overline{Z^n} = \overline{\exp(-e^{\lambda(x-f)})}$$

so that at large time:

$$\lim_{\lambda \to \infty} g(x) = \overline{\theta(f - x)} = Prob(f > x)$$

how to get P(ln Z) i.e. P(h) ?

$$\ln Z = -\lambda f \qquad \lambda = (\frac{\bar{c}^2}{4} t)^{1/3} \qquad f$$

random variable
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introduce generating function of moments g(x):

$$g(x) = 1 + \sum_{n=1}^{\infty} \frac{(-e^{\lambda x})^n}{n!} \overline{Z^n} = \overline{\exp(-e^{\lambda(x-f)})}$$

what we aim to calculate= Laplace transform of P(Z)

what we actually study

so that at large time:

$$\lim_{\lambda \to \infty} g(x) = \overline{\theta(f-x)} = Prob(f > x)$$

reorganize sum over number of strings

$$g(x) = 1 + \sum_{n_s=1}^{\infty} \frac{1}{n_s!} Z(n_s, x)$$

$$Z(n_s, x) = \sum_{m_1, \ldots m_{n_s}=1}^{\infty} \frac{(-1)^{\sum_j m_j}}{(4\pi\lambda^{3/2})^{n_s}}$$

$$\prod_{j=1}^{n_s} \int \frac{dk_j}{m_j} \prod_{1 \le i < j \le n_s} \frac{(k_i - k_j)^2 + (m_i - m_j)^2 \lambda^3}{(k_i - k_j)^2 + (m_i + m_j)^2 \lambda^3} \prod_{j=1}^{n_s} e^{\frac{1}{3}\lambda^3 m_j^3 - m_j k_j^2 + \lambda x m_j}$$

reorganize sum over number of strings

$$g(x) = 1 + \sum_{n_s=1}^{\infty} \frac{1}{n_s!} Z(n_s, x)$$

Airy trick

$$\int_{-\infty}^{\infty} dy Ai(y) e^{yw} = e^{w^3/3}$$

$$Z(n_s, x) = \sum_{m_1,\ldots m_{n_s}=1}^{\infty} \frac{(-1)^{\sum_j m_j}}{(4\pi\lambda^{3/2})^{n_s}}$$

$$\prod_{j=1}^{n_s} \int \frac{dk_j}{m_j} \prod_{1\leq i<j\leq n_s} \frac{(k_i-k_j)^2 + (m_i-m_j)^2\lambda^3}{(k_i-k_j)^2 + (m_i+m_j)^2\lambda^3} \prod_{j=1}^{n_s} e^{\frac{1}{3}\lambda^3 m_j^3 - m_j k_j^2 + \lambda x m_j}$$

double Cauchy formula

$$det[\frac{1}{i(k_i-k_j)\lambda^{-3/2} + (m_i+m_j)}]$$

$$= \prod_{i<j} \frac{(k_i-k_j)^2 + (m_i-m_j)^2\lambda^3}{(k_i-k_j)^2 + (m_i+m_j)^2\lambda^3} \prod_{i=1}^{n_s} \frac{1}{2m_i}$$

$$\frac{1}{X} = \int_0^{\infty} dv e^{-vX}$$

Results: 1) g(x) is a Fredholm determinant at any time t

$$Z(n_s,x) = \prod_{j=1}^{n_s} \int_{v_j>0} dv_j \; det[K(v_j,v_\ell)]$$

$$\lambda = (\frac{\bar{c}^2}{4} t)^{1/3}$$

$$K(v_1,v_2) = -\int \frac{dk}{2\pi} dy Ai(y + k^2 - x + v_1 + v_2) e^{-ik(v_1-v_2)} \frac{e^{\lambda y}}{1+e^{\lambda y}}$$

$$g(x) = 1 + \sum_{n_s=1}^{\infty} \frac{1}{n_s!} Z(n_s,x) \;\; = Det[I+K]$$

by an equivalent definition of a Fredholm determinant

$$K(v_1,v_2) \equiv \theta(v_1) K(v_1,v_2) \theta(v_2)$$

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2) large time limit $\lambda = +\infty$ $\quad \frac{e^{\lambda y}}{1 + e^{\lambda y}} \longrightarrow \theta(y)$

Airy function identity

$$\int dk Ai(k^2 + v + v') e^{ik(v-v')} = 2^{2/3} \pi Ai(2^{1/3} v) Ai(2^{1/3} v')$$

g(x)= $Prob(f > x = -2^{2/3} s) = Det(1 - P_s K_{Ai} P_s) = F_2(s)$

$$K_{Ai}(v, v') = \int_{y>0} Ai(v + y) Ai(v' + y)$$

GUE-Tracy-Widom distribution

An exact solution for the KPZ equation with flat initial conditions

P. Calabrese, P. Le Doussal, (2011)

needed:

$$\int dy_1..dy_n \Psi_\mu(y_1,..y_n)$$

1) g(s=-x) is a Fredholm Pfaffian at any time t

$$Z(n_s) = \sum_{m_i \geq 1} \prod_{j=1}^{n_s} \int_{k_j} \prod_{q=1}^{m_j} \frac{-2}{2ik_j + q} e^{\frac{\lambda^3}{3} m_j^3 - 4 m_j k_j^2 \lambda^3 - \lambda m_j s}$$

$$\times \mathrm{Pf}\left[\begin{pmatrix} \frac{2\pi}{2ik_i} \delta(k_i + k_j)(-1)^{m_i} \delta_{m_i,m_j} + \frac{1}{4}(2\pi)^2 \delta(k_i)\delta(k_j)(-1)^{\min(m_i,m_j)} \mathrm{sgn}(m_i - m_j) & \frac{1}{2}(2\pi)\delta(k_i) \\ -\frac{1}{2}(2\pi)\delta(k_j) & \frac{2ik_i + m_i - 2ik_j - m_j}{2ik_i + m_i + 2ik_j + m_j} \end{pmatrix} \right]$$

$$Z(n_s) = \prod_{j=1}^{n_s} \int_{v_j > 0} \mathrm{Pf}[\mathbf{K}(v_i, v_j)]_{2n_s, 2n_s}$$

$$g_\lambda(s) = \mathrm{Pf}[\mathbf{J} + \mathbf{K}] = \sum_{n_s=0}^{\infty} \frac{1}{n_s!} Z(n_s)$$

$$\mathbf{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

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2) large time limit $\lambda = +\infty$

$$g_\infty(s) = F_1(s) = \det[I - \mathcal{B}_s]$$

GOE Tracy Widom

$$\mathcal{B}_s = \theta(x) Ai(x+y+s)\theta(y)$$

Fredholm Pfaffian Kernel at any time t

$$K_{11} = \int_{y_1,y_2,k} Ai(y_1 + v_i + s + 4k^2) Ai(y_2 + v_j + s + 4k^2) [\frac{e^{-2i(v_i - v_j)k}}{2ik} f_{k/\lambda}(e^{\lambda(y_1+y_2)})$$

$$+ \frac{\pi\delta(k)}{2} F(2e^{\lambda y_1}, 2e^{\lambda y_2})]$$

$$K_{12} = \frac{1}{2} \int_y Ai(y + s + v_i)(e^{-2e^{\lambda y}} - 1) \; \delta(v_j)$$

$$K_{22} = 2\delta'(v_i - v_j) \, ,$$

$$f_k(z) = \frac{-2\pi k z \, {}_1F_2 \left(1; 2 - 2ik, 2 + 2ik; -z\right)}{\sinh\left(2\pi k\right) \Gamma\left(2 - 2ik\right) \Gamma\left(2 + 2ik\right)} \, , \qquad (19)$$

$$F(z_i, z_j) = \sinh(z_2 - z_1) + e^{-z_2} - e^{-z_1} + \int_0^1 du$$

$$\times J_0(2\sqrt{z_1 z_2 (1-u)})[z_1 \sinh(z_1 u) - z_2 \sinh(z_2 u)] \, .$$

$$g_\lambda(s) = \sqrt{Det(1 - 2K_{10})}(1 + \langle \tilde{K} | (1 - 2K_{10})^{-1} | \delta \rangle)$$

$$K_{10}(v_1, v_2) = \partial_{v_1} K_{11}(v_1, v_2) \qquad K_{12}(v_1, v_2) = \tilde{K}(v_1)\delta(v_2)$$

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large time limit

$$\lim_{\lambda \to +\infty} f_{k/\lambda}(e^{\lambda y}) = -\theta(y)$$

$$\lim_{\lambda \to +\infty} F(2e^{\lambda y_1}, 2e^{\lambda y_2}) =$$

$$\theta(y_1 + y_2)(\theta(y_1)\theta(-y_2) - \theta(y_2)\theta(-y_1))$$

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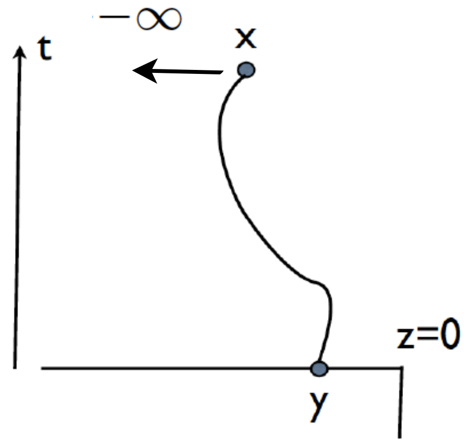
how to calculate $\int dy_1..dy_n \Psi_\mu(y_1,..y_n)$

first method: flat as limit of half-flat (wedge)

$$\lim_{x\to-\infty, w\to 0} Z_{\rm hs}(x,t) \equiv Z_{\rm flat}(x,t)$$

$$Z_{\rm hs,w}(x,t) = \int_{-\infty}^0 dy e^{wy} Z(x,t|y,0)$$

$$Z(x,t=0) = \theta(-x)e^{wx}$$



$$\Psi_\mu = \sum_P A_P \prod_{j=1}^n e^{i\lambda_{P_\ell} x_\ell}$$

$$\Big(\prod_{\alpha=1}^n \int_{-\infty}^0 dy_\alpha e^{wy_\alpha}\Big)\Psi_\mu(y_1\dots y_n) = \sum_P A_P G_{P\lambda}$$

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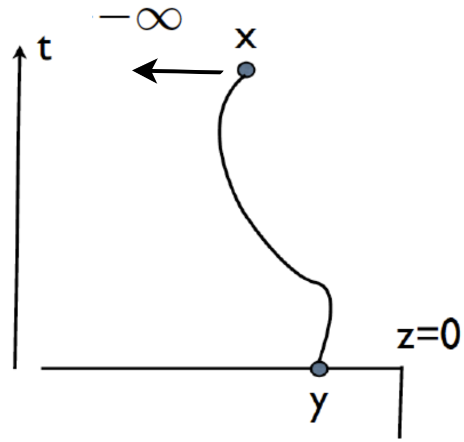

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miracle !

$$= \frac{n!}{\prod_{\alpha=1}^n (w + i\lambda_\alpha)} \prod_{1\le\alpha<\beta\le n} \frac{2w + i\lambda_\alpha + i\lambda_\beta - 1}{2w + i\lambda_\alpha + i\lambda_\beta}$$

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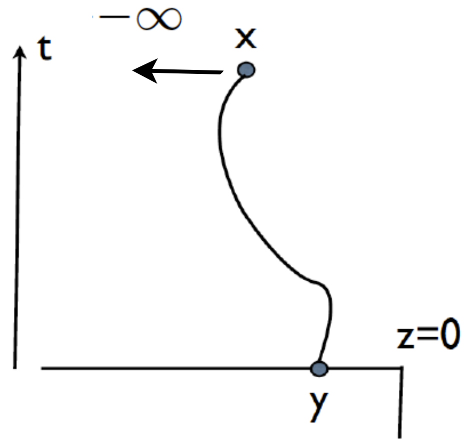

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strings:

$$\lambda^{j,a} = k_j + \frac{i\bar{c}}{2}(j+1-2a)$$

$$a = 1, ..., m_j$$

$$\int^{w} \Psi_\mu = n!(-2)^n \prod_{i=1}^{n_s} S^w_{m_i,k_i} \prod_{1\le i<j\le n_s} D^w_{m_i,k_i,m_j,k_j}$$

$$D^w_{m_1,k_1,m_2,k_2} = \quad (-1)^{m_2} \frac{\Gamma(1-z+\frac{m_1+m_2}{2})\Gamma(z+\frac{m_1-m_2}{2})}{\Gamma(1-z+\frac{m_1-m_2}{2})\Gamma(z+\frac{m_1+m_2}{2})} \qquad z = ik_1 + ik_2 + 2w$$

$$S^w_{m,k} = \frac{(-1)^m \Gamma(z)}{\Gamma(z+m)} \qquad z = 2ik + 2w.$$

in double limit $\lim_{x\to-\infty, w\to 0}$

expand the product $\prod_i S_i \prod_{i<j} D_{ij}$

each momentum k_ℓ appears only in exactly one pole

$$S^w_{m_i,k_i} \to \frac{(-1)^{m_i}}{2\Gamma(m_i)} 2\pi\delta(k_i) + s^0_{m_i,k_i}$$

$$D^w_{m_i,k_i,m_j,k_j} \to (-1)^{m_i} m_i \delta_{m_i,m_j}\ 2\pi\delta(k_i+k_j) + d^w_{m_i,k_i,m_j,k_j}$$

pairing of string momenta and pfaffian structure emerges second method:

calculate: $$\prod_{\alpha=1}^{n} \int_0^L dy_\alpha \Psi_\mu(y_1, .., y_n)$$

use Bethe equations: $$e^{i\lambda_j L} = \prod_{\ell \neq j} \frac{\lambda_j - \lambda_\ell - i\bar{c}}{\lambda_j - \lambda_\ell + i\bar{c}}$$

=> integral vanishes for generic state

oberve: requires pairs opposite rapidities second method:

calculate: $\prod_{\alpha=1}^{n} \int_0^L dy_\alpha \Psi_\mu(y_1,..,y_n) = \langle \Phi_0 | \mu \rangle$

is the overlap with uniform state

$\Phi_0(x_1,..x_n) = 1$

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Can be seen as interaction quench in Lieb-Liniger model with initial state BEC (c=0)

de Nardis et al., arXiv 1308.4310

overlap is non zero only for parity invariant states $\{\lambda_1, -\lambda_1, .., \lambda_{n/2}, -\lambda_{n/2}\}$

infinity of conserved charges $Q_p = \sum_{\alpha=1}^{n} \lambda_\alpha^p$

second method: calculate: $\prod_{\alpha=1}^{n}\int_0^L dy_\alpha \Psi_\mu(y_1,..,y_n) = \langle \Phi_0|\mu\rangle$

use Bethe equations: $e^{i\lambda_j L} = \prod_{\ell \neq j} \frac{\lambda_j - \lambda_\ell - i\bar{c}}{\lambda_j - \lambda_\ell + i\bar{c}}$

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$$\langle \Phi_0|\mu\rangle = n! c^{n/2} \prod_{\alpha=1}^{n/2} \frac{1}{\lambda_\alpha^2} \prod_{1\leq\alpha<\beta\leq n/2} \frac{(\lambda_\alpha-\lambda_\beta)^2+c^2}{(\lambda_\alpha-\lambda_\beta)^2} \frac{(\lambda_\alpha+\lambda_\beta)^2+c^2}{(\lambda_\alpha+\lambda_\beta)^2} \times \det G^Q.$$

$$G^Q_{\alpha\beta} = \delta_{\alpha\beta}(L + \sum_{\gamma=1}^{n/2} K^Q(\lambda_\alpha, \lambda_\gamma)) - K^Q(\lambda_\alpha, \lambda_\beta)$$

$$K^Q(x,y) = K(x-y) + K(x+y),$$

$$K(x) = \frac{2c}{x^2+c^2}.$$

Brockmann, arXiv1402.1471.

P. Calabrese, P. Le Doussal, arXiv 1402.1278

large L limit, overlap for strings

partially recovers the moments Z^n for flat

Summary: we found

for droplet initial conditions

$$\frac{\lambda_0 h}{2\nu} \equiv \ln Z = v_\infty t + 2^{2/3} (\frac{t}{t^*})^{1/3} \chi$$

χ at large time has the same distribution as the largest eigenvalue of the GUE

for flat initial conditions

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similar (more involved)

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$$t^* = \frac{8(2\nu)^5}{D^2 \lambda_0^4}$$

in addition: g(x) for all times
=> P(h) at all t (inverse LT)

decribes full crossover from Edwards Wilkinson to KPZ

t^* is crossover time scale

large for weak noise, large diffusivity

GSE ?

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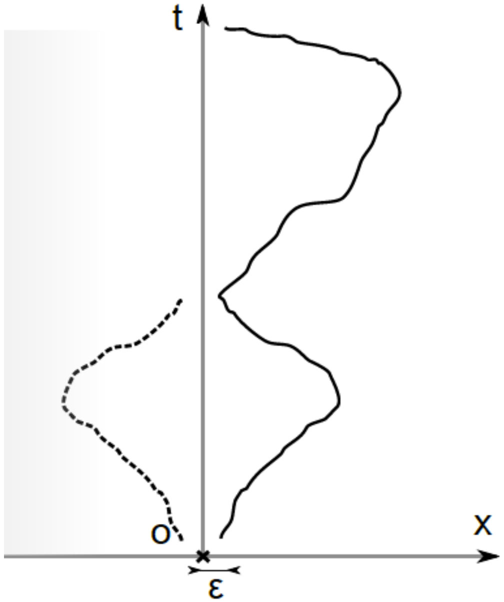
GSE ? KPZ in half-space

DP near a wall = KPZ equation in half space

T. Gueudre, P. Le Doussal, EPL 100 26006 (2012)



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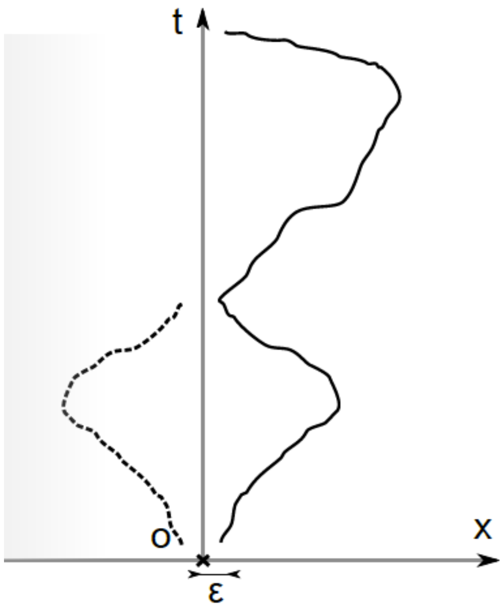
$$- {}_1F_2\,(1; 1 - 2ik, 1 + 2ik; -1/z)$$

$Z(x, 0, t) = Z(0, y, t) = 0$

$\nabla h(0, t)$ fixed

DP near a wall = KPZ equation in half space

T. Gueudre, P. Le Doussal, EPL 100 26006 (2012)



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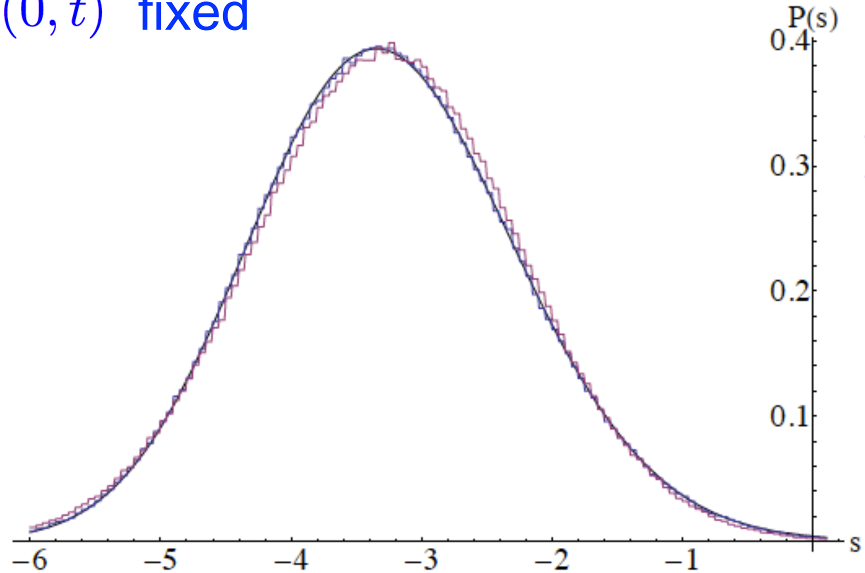
$$f_k[z] = \frac{2\pi k}{\sinh(4\pi k)} \left(J_{-4ik}(\frac{2}{\sqrt{z}}) + J_{4ik}(\frac{2}{\sqrt{z}}) \right) - {}_1F_2\,(1; 1 - 2ik, 1 + 2ik; -1/z)$$

$$\lim_{\lambda \to \infty} f_{k/\lambda}[e^{\lambda y}] = -\theta(y)(1 - \cos(2ky))$$

$Z(x, 0, t) = Z(0, y, t) = 0$

$\nabla h(0, t)$ fixed

$$\lambda = (\frac{\bar{c}^2 t}{8T^5})^{1/3} = (\frac{D\lambda_0^2 t}{8(2\nu)^5})^{1/3}$$

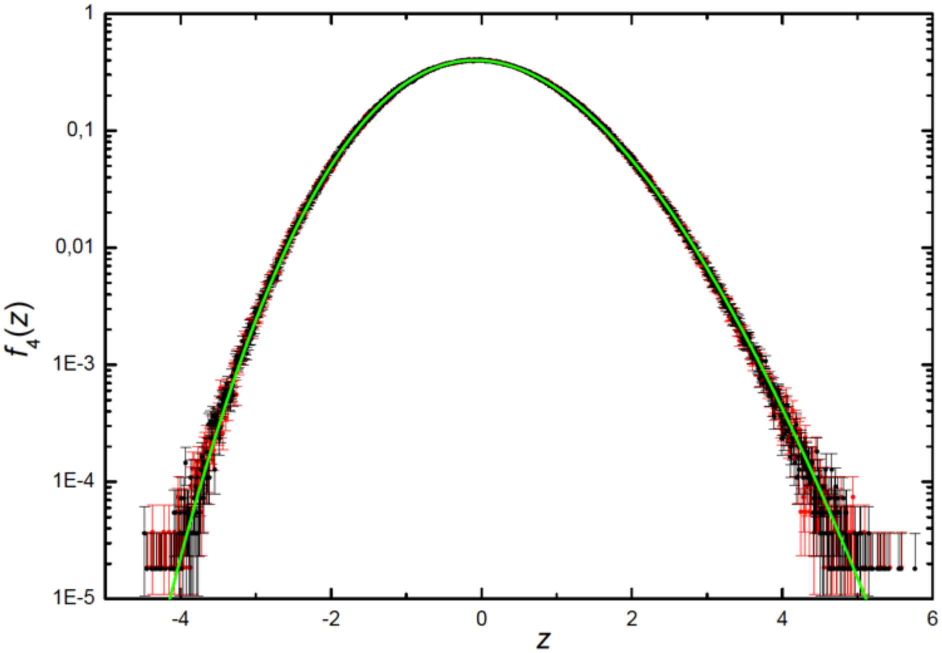


$$\ln Z = \frac{\lambda_0}{2\nu} \tilde{h}(0, t) = v_\infty t + 2^{2/3} \lambda \chi_4$$

χ_4 distributed as $F_4(s)$

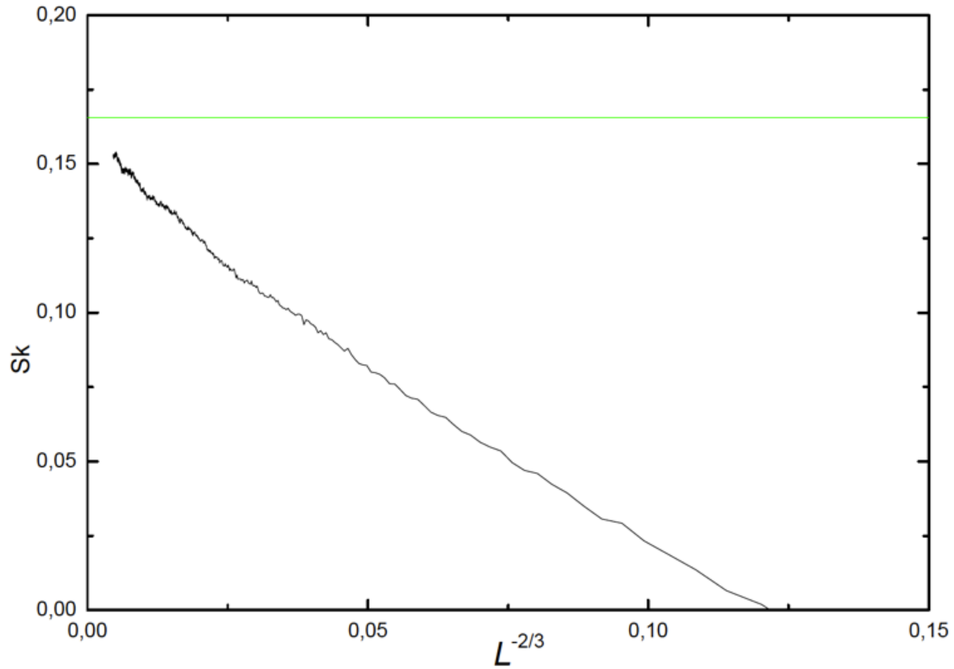
Gaussian Symplectic Ensemble

Ortuno, Somoza, PLD (2014)



1- half plane sample:
log of conductance point to point
near sample edges

box distribution W=10
L=100-3200



2- edge hopping t larger than bulk:
"unbinding transition"
crossover from F4 to F1 to log-normal

From the sine Gordon field theory to KPZ

P. Calabrese, M. Kormos, PLD
arXiv/1405.2582, EPL (2014)

1. integrable quantum field theory $\phi(x,t)$

imaginary time
$$\int D\phi \, e^{-\int dx dt \mathcal{L}_{\rm sG}[\phi]}$$

$$\mathcal{L}_{\rm sG}[\phi] = \frac{1}{2c_l^2}(\partial_t \phi)^2 + \frac{1}{2}(\partial_x \phi)^2 - \frac{m_0^2 c_l^2}{\beta^2}(\cos(\beta\phi) - 1)$$

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2. excitation spectrum

- solitons mass -> infinity decouple
- breathers B_1 "particle" sinh-Gordon

B_m m-breather

"bound state"

$$H = \int dx \Pi(x)^2 + \frac{1}{2}(\partial_x \phi)^2 + ..$$

$$E_m(\theta) = M_m c_l^2 \cosh\theta$$

$$P_m(\theta) = M_m c_l \sinh(\theta)$$

$$M_m = M \frac{\sin m\pi\alpha/2}{\sin \pi\alpha/2}$$

$$m_{max} = [1/\alpha]$$

$$\alpha = c_l \beta^2 / (8\pi - c_l \beta^2)$$

$$\mathcal{L}_{\rm sG}[\phi] = \frac{1}{2c_l^2}(\partial_t\phi)^2 + \frac{1}{2}(\partial_x\phi)^2 - \frac{m_0^2 c_l^2}{\beta^2}(\cos(\beta\phi) - 1)$$

3. non-relativistic limit (NRL): $\begin{cases} c_l \to +\infty \\ \beta \to 0 \end{cases}$ $\beta c_l = 4\sqrt{\bar{c}}$

$\longrightarrow$ non-linear Schrodinger

$$\phi(x,t) = e^{-m_0 c_l^2 t}\Psi(x,t) + e^{m_0 c_l^2 t}\Psi^+(x,t)$$

$$\alpha \approx c_l \beta^2/(8\pi) \to 0$$

$$m_{max} = [1/\alpha] \to \infty$$

ShG $\longrightarrow$ repulsive Lieb-Liniger

Mussardo, Kormos et al. (2014)

SG $\longrightarrow$ attractive LL

B_m $\longrightarrow$ m-string

$$E_m(\theta) = Mmc_l^2 + \frac{\bar{c}^2}{24M}(m - m^3) + m\frac{p^2}{2M}$$

$$M \approx m_0$$

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$M \approx m_0$

4. SG is integrable QFT

Form factors known explicitly

satisfy functional recursion relations, analyticity,..

Smirnov, Mussardo, ..

$$\langle 0|e^{ik\beta\phi(0,0)}|\theta_1, .., \theta_n\rangle = F_n^k(\theta_1, ..\theta_n)$$

$$\downarrow$$

$$|B_{m_1}(\theta_1), .., B_{m_{n_s}}(\theta_{n_s})\rangle$$

$$e^{ik\beta\phi(0,0)} = e^{i\tilde{k}\phi(0,0)}$$

$$k = \tilde{k}/\beta$$

2-point correlation function in SG

$$G(\tilde{k},t) = \langle 0 | e^{i\tilde{k}\phi(0,t)} e^{-i\tilde{k}\phi(0,0)} | 0 \rangle$$

Lehman formula:

$$G(\tilde{k},t) \simeq \sum_{n_s=0}^{\infty} \frac{1}{n_s!} \prod_{j=1}^{n_s} \sum_{m_j=1}^{m_{max}} \int \frac{\mathrm{d}\theta_1}{2\pi} \dots \frac{\mathrm{d}\theta_{n_s}}{2\pi} |\langle 0 | e^{i\tilde{k}\phi(0,0)} | B_{m_1}(\theta_1) \dots B_{m_{n_s}}(\theta_{n_s}) \rangle|^2 e^{-\sum_{j=1}^{n_s} E_{m_j}(\theta_j)|t|}$$

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↓ non-relativistic limit

$$G(\tilde{k},t) \simeq |\langle e^{i\tilde{k}\phi}\rangle|^2 \sum_{n_s=0}^{\infty} \frac{\bar{c}^{n-n_s}}{n_s!} \prod_{j=1}^{n_s} \sum_{m_j=1}^{+\infty} \left[\frac{2}{\sqrt{\bar{c}}} \sin\left(\frac{\sqrt{\bar{c}}}{2}\tilde{k}\right)\right]^{2m_j}$$

$$\times \prod_{j=1}^{n_s} \int \frac{\mathrm{d}p_j}{2\pi m_j} e^{-m_j M c_l^2 t - \frac{\bar{c}^2}{12}(m_j - m_j^3)t - m_j p_j^2 t} \Phi[p,m]$$

$$\Phi[p,m] = \prod_{1\leq j<l\leq n_s} \frac{4(p_i-p_j)^2 + \bar{c}^2(m_i-m_j)^2}{4(p_i-p_j)^2 + \bar{c}^2(m_i+m_j)^2}$$

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$$g(u) = \sum_{n=0}^{+\infty} \frac{(-u)^n}{n!} \overline{Z(t)^n}|_{KPZ}$$

$$\Phi[p,m] = \prod_{1\le j<l\le n_s} \frac{4(p_i - p_j)^2 + \bar{c}^2(m_i - m_j)^2}{4(p_i - p_j)^2 + \bar{c}^2(m_i + m_j)^2}$$

$$\langle e^{i\tilde{k}(\phi(0,0)-\phi(0,t))}\rangle / \langle e^{i\tilde{k}\phi(0,0)}\rangle^2 \to_{NRL} g(u) \qquad u = -(\frac{2}{\sqrt{\bar{c}}}\sin(\frac{\sqrt{\bar{c}}}{2}))^2 e^{-Mc_l^2 t}$$

$$\langle e^{\tilde{k}(\phi(0,0)-\phi(0,t))}\rangle / \langle e^{\tilde{k}\phi(0,0)}\rangle^2 \to_{NRL} g(u) \qquad u = (\frac{2}{\sqrt{\bar{c}}}\sinh(\frac{\sqrt{\bar{c}}}{2}))^2 e^{-Mc_l^2 t}$$

2-point correlation in SG ⟶ point to point (droplet) KPZ moments

Perspectives/other works

- replica BA method

Airy process

stationary KPZ — Sasamoto Inamura — $t \to \infty$ — $A_2(y)$

2 space points — $Prob(h(x_1,t), h(x_2,t))$ — Prohlac-Spohn (2011), Dotsenko (2013)

2 times — $Prob(h(0,t), h(0,t'))$ — Dotsenko (2013)

endpoint distribution of DP — Dotsenko (2012) — Schehr, Quastel et al (2011)

- rigorous replica..

Borodin, Corwin, Quastel, O Neil, ..

q-TASEP — $q \to 1$ Bose gas

WASEP

avoids moment problem $\overline{Z^n} \sim e^{cn^3}$

moments as nested contour integrals

- sine-Gordon FT