Martingales for Determinantal Log-Gases

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Contents

1. Introduction

First I would like to explain the following two things.

1.1 Martingale: a property of Markov processes;

background, definition, expressions

1.2 Krattenthaler's Determinantal Identity;

a useful equality and its generalizations

Then I will give

1.3 A Combination of the Above Two

1.1 Martingale is a betting strategy

① You bet ＄100, the banker bets ＄100.

→ You win a bet, you get ＄200.

→ You stop the game, then your gain is −＄100＋＄200＝＄100

↓ You lose a bet. You continue the game.

② You double your bet. You bet ＄200, the banker bets ＄200.

→ You win a bet, you get ＄400.

→ You stop the game, then your gain is −＄(100＋200)＋＄400＝＄100

↓ You lose a bet. You continue the game.

③ You double your bet again. You bet ＄400, the banker bets ＄400.

You stop the game when you win: Your gain is always＋＄100

- Let $B(t), t \geq 0$ be a **one-dimensional standard Brownian motion (BM)**. Its expectation and conditional expectation are denoted by $\mathrm{E}[\cdots]$ and $\mathrm{E}[\cdots|\mathrm{C}]$.

- The **filtration** is the smallest σ-field (the collection of events which is closed with respect to 'summation' $\cup$) generated by the BM up to time t,

$$\mathcal{F}_t = \sigma(B(s) : 0 \leq s \leq t).$$

- If a process $f(t, B(t)), t \geq 0$ satisfies the following, it is called the continuous-time **martingale**;

$$\mathrm{E}[f(t, B(t))|\mathcal{F}_s] = f(s, B(s)) \quad \text{a.s.} \quad \text{for all } 0 \leq s \leq t.$$

- Martingale is the property of Markov processes such that **the expectation is conserved in time**.

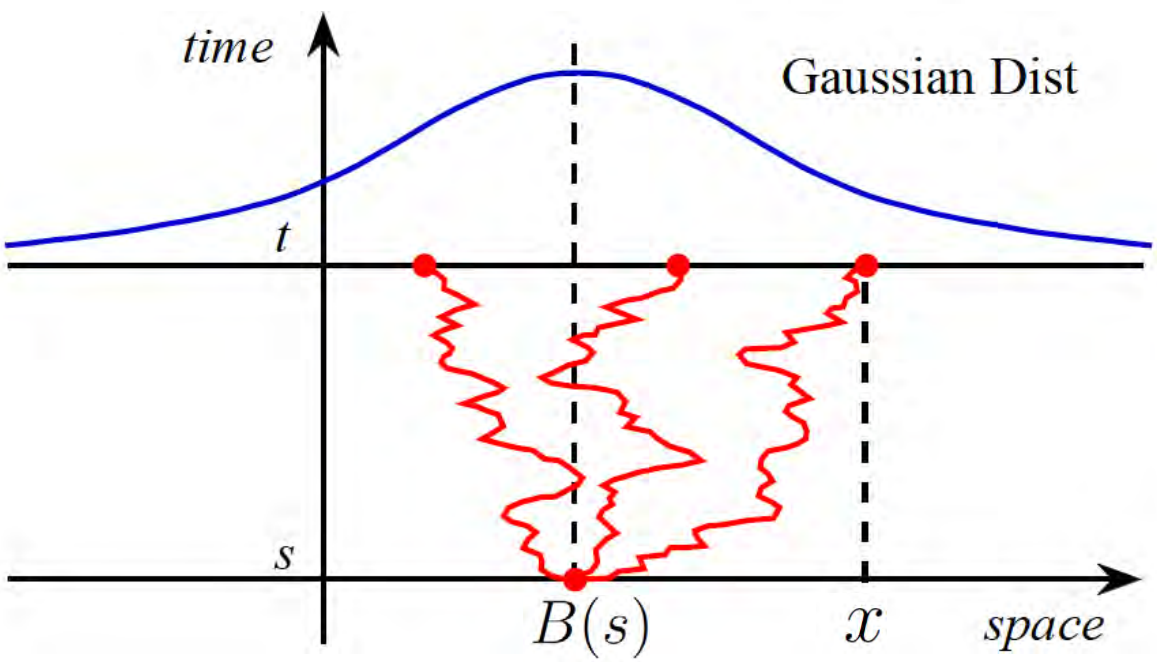
- Martingale is a Markov process representing a **fluctuation**.

BM is a martingale;

$$\mathrm{E}[B(t)|\mathcal{F}_s] = B(s) \quad \text{a.s.} \quad \text{for all } 0 \le s \le t.$$

Proof. Let $p(t, y|x)$ be the transition probability density of BM.

$$\begin{aligned}\mathrm{E}[B(t)|\mathcal{F}_s] &= \int_{-\infty}^{\infty} x p(t-s, x|B(s)) dx \\ &= \int_{-\infty}^{\infty} x \frac{e^{-(x-B(s))^2/2(t-s)}}{\sqrt{2\pi(t-s)}} dx = B(s).\end{aligned}$$



$B(t)$ is a martingale, but $B(t)^2$ is not.
For $0 \leq s < t$,

$$\begin{aligned}
\mathrm{E}[B(t)^2|\mathcal{F}_s] &= \int_{-\infty}^{\infty} x^2 p(t-s, x|B(s))dx \\
&= \int_{-\infty}^{\infty} x^2 \frac{e^{-(x-B(s))^2/2(t-s)}}{\sqrt{2\pi(t-s)}}dx \\
&= \int_{-\infty}^{\infty} \{(x-B(s))^2 + 2B(s)(x-B(s)) + B(s)^2\} \frac{e^{-(x-B(s))^2/2(t-s)}}{\sqrt{2\pi(t-s)}}dx \\
&= (t-s) + 0 + B(s)^2 = B(s)^2 + (t-s) \neq B(s)^2.
\end{aligned}$$

⬇

$$\mathrm{E}[B(t)^2 - t|\mathcal{F}_s] = B(s)^2 - s \quad \text{a.s.} \quad \text{for all } 0 \leq s \leq t.$$

$m_2(t, B(t)) \equiv B(t)^2 - t$ is a martingale.

- Let $B(t)$ and $\widetilde{B}(t)$ are independent BMs. The **complex BM** is defined by

$$Z(t) \equiv B(t) + i\widetilde{B}(t), \quad i = \sqrt{-1}, \quad t \geq 0.$$

Let $\mathbf{E}[\cdots] \equiv \mathrm{E}[\widetilde{\mathrm{E}}[\cdots]]$.

- $Z(t)$ is a martingale, since both of the real and imaginary parts are martingales;

$$\begin{aligned} \mathbf{E}[Z(t)|\mathcal{F}_s] &= \mathbf{E}[B(t) + i\widetilde{B}(t)|\mathcal{F}_s] = \mathrm{E}[B(t)|\mathcal{F}_s] + i\widetilde{\mathrm{E}}[\widetilde{B}(t)|\mathcal{F}_s] = B(s) + i\widetilde{B}(s) \\ &= Z(s) \quad \text{a.s.} \quad \text{for all } 0 \leq s \leq t. \end{aligned}$$

Consider $Z(t)^2 = (B(t) + i\widetilde{B}(t))^2 = B(t)^2 + 2iB(t)\widetilde{B}(t) - \widetilde{B}(t)^2$.

$$
\begin{aligned}
\mathbf{E}[Z(t)^2|\mathcal{F}_s] &= \mathrm{E}[B(t)^2|\mathcal{F}_s] + 2i\mathrm{E}[B(t)|\mathcal{F}_s]\widetilde{\mathrm{E}}[\widetilde{B}(t)|\mathcal{F}_s] - \widetilde{\mathrm{E}}[\widetilde{B}(t)^2|\mathcal{F}_s] \\
&= \mathrm{E}[(B(t)^2 - t)|\mathcal{F}_s] + 2i\mathrm{E}[B(t)|\mathcal{F}_s]\widetilde{\mathrm{E}}[\widetilde{B}(t)|\mathcal{F}_s] - \widetilde{\mathrm{E}}[(\widetilde{B}(t)^2 - t)|\mathcal{F}_s] \\
&= (B(s)^2 - s) + 2iB(s)\widetilde{B}(s) - (\widetilde{B}(s)^2 - s) \\
&= B(s)^2 + 2iB(s)\widetilde{B}(s) - \widetilde{B}(s)^2 = Z(s)^2.
\end{aligned}
$$

$Z(t)^2$ is also a martingale.

Note that, if $\widetilde{B}(t)$ starts at 0 $\Rightarrow \widetilde{\mathrm{E}}[\widetilde{B}(t)] = 0, \widetilde{\mathrm{E}}[\widetilde{B}(t)^2] = t$,

$$
\begin{aligned}
\widetilde{\mathrm{E}}[Z(t)^2] &= \widetilde{\mathrm{E}}[B(t)^2 + 2iB(t)\widetilde{B}(t) - \widetilde{B}(t)^2] \\
&= B(t)^2 + 2iB(t)\widetilde{\mathrm{E}}[\widetilde{B}(t)] - \widetilde{\mathrm{E}}[\widetilde{B}(t)^2] \\
&= B(t)^2 + 0 - t \\
&= B(t)^2 - t = m_2(t, B(t)).
\end{aligned}
$$

- $B(t)^3$ is not a martingale, but

$$m_3(t, B(t)) \equiv B(t)^3 - 3tB(t) = \widetilde{\mathrm{E}}[Z(t)^3]$$

is a martingale.

- For $n \in \mathbb{N}_0 \equiv \{0, 1, 2, \dots\}$

$$m_n(t, B(t)) \equiv \left(\frac{t}{2}\right)^{n/2} H_n\left(\frac{B(t)}{\sqrt{2t}}\right) = \widetilde{\mathrm{E}}[Z(t)^n]$$

are martingales, where

$$H_n(x) = \sum_{j=0}^{[n/2]} (-1)^j \frac{n!}{j!(n-2j)!}(2x)^{n-2j}, \quad n \in \mathbb{N}_0, \quad \textbf{Hermite polynomials}.$$

Integral representation of `martingale-polynomials'

$$m_n(t, B(t)) = \widetilde{\mathrm{E}}[Z(t)^n]$$

$$\begin{aligned}
m_n(t,x) &= \widetilde{\mathrm{E}}[(x + i\widetilde{B}(t))^n] \\
&= \int_{-\infty}^{\infty} (x+iy)^n p(t, y|0) dy \\
&= \int_{-\infty}^{\infty} (x+iy)^n \frac{e^{-y^2/2t}}{\sqrt{2\pi t}} dy \\
(x + iy = iw &\Leftrightarrow y = w + ix) \\
&= \int_{-\infty-ix}^{\infty-ix} (iw)^n \frac{e^{-(ix+w)^2/2t}}{\sqrt{2\pi t}} dw \\
&= \int_{-\infty}^{\infty} (iw)^n \frac{e^{-(ix+w)^2/2t}}{\sqrt{2\pi t}} dw \\
&\equiv \mathcal{I}[W^n|(t,x)].
\end{aligned}$$

$$\mathcal{I}[f(W)|(t,x)] \equiv \int_{-\infty}^{\infty} f(iw) q(t, w|x) dw$$

$$\text{with} \quad q(t, w|x) = \frac{e^{-(ix+w)^2/2t}}{\sqrt{2\pi t}}.$$

$$\mathcal{I}[f(W)|(t,x)] \equiv \int_{-\infty}^{\infty} f(iw)q(t,w|x)dw$$

$$\text{with} \quad q(t,w|x) = \frac{e^{-(ix+w)^2/2t}}{\sqrt{2\pi t}}.$$

Lemma 1

(i) $\mathcal{I}[W^n|(t,x)] = m_n(t,x), n \in \mathbb{N}_0.$

(ii) If f is a polynomial function, then $\mathcal{I}[f(W)|(t,B(t))]$ is a martingale.

1.2 Krattenthaler's Determinantal Identity

[Krattenthaler99] C. Krattenthaler, Advanced determinant calculus, Séminaire Lotharingien Combin. **42** (The Andrews Festschrift) B42q (1999).

[Krattenthaler05] C. Krattenthaler, Advanced determinant calculus: a complement, Linear Algebra Appl. **411**, 68-166 (2005).

Lemma 2 (Krattenthaler)

Let $x_1, x_2, \ldots, x_N$, $a_2, a_3, \ldots, a_N$, c be indeterminates. If $\pi_0, \pi_1, \ldots, \pi_{N-1}$ are Laurent polynomials with $\deg \pi_k \le k$ and $\pi_k(c/x) = \pi_k(x)$ for $k = 0, 1, \ldots, N-1$, then

$$\det_{1\le j,k\le N} \Big[(x_j + a_N)(x_j + a_{N-1}) \cdots (x_j + a_{k+1}) \\ \times (c/x_j + a_N)(c/x_j + a_{N-1}) \cdots (c/x_j + a_{k+1}) \cdot \pi_{k-1}(x_j)\Big] \\ = \prod_{1\le j<k\le N} (x_j - x_k)(1 - c/(x_j x_k)) \prod_{j=1}^{N} a_j^{j-1} \prod_{j=1}^{N} \pi_{j-1}(-a_j).$$

Let

$$\begin{aligned}
h(\boldsymbol{x}) &= \det_{1\le j,k\le N}[x_j^{k-1}] = \prod_{1\le j<k\le N}(x_k - x_j) \quad \textbf{(Vandermonde determinant)},\\
\mathbb{W}_N(S) &= \{\boldsymbol{x} = (x_1,\dots,x_N)\in S^N : x_1 < \cdots < x_N\} \quad \textbf{(Weyl chamber)},\\
\mathbb{W}_N &= \mathbb{W}_N(\mathbb{R}),\\
\mathbb{A}_{[0,2\pi r)^N} &= \{\boldsymbol{x} = (x_1,\dots,x_N)\in \mathbb{R}^N : 0 \le x_1 < x_2 < \cdots < x_N < x_1 + 2\pi r\} \quad \textbf{(Weyl alcove)}.
\end{aligned}$$

For $\boldsymbol{x} \in \mathbb{R}^N, \boldsymbol{u} \in \mathbb{W}_N$,

$$\det_{1\le j,k\le N}\left[\prod_{1\le \ell\le N, \ell\ne k}\frac{x_j - u_\ell}{u_k - u_\ell}\right] = \frac{h(\boldsymbol{x})}{h(\boldsymbol{u})}.$$

Extensions of this identity.

- trigonometric/hyperbolic extensions

For $\boldsymbol{x} \in [0, 2\pi r)^N$, $\boldsymbol{u} \in \mathbb{W}_N([0, 2\pi r))$, $r > 0$,

$$\det_{1 \le j,k \le N} \left[\prod_{1 \le \ell \le N, \ell \ne k} \frac{\sin((x_j - u_\ell)/2r)}{\sin((u_k - u_\ell)/2r)} \right] = \prod_{1 \le j < k \le N} \frac{\sin((x_k - x_j)/2r)}{\sin((u_k - u_j)/2r)}.$$

For $\boldsymbol{x} \in \mathbb{R}^N$, $\boldsymbol{u} \in \mathbb{W}_N$, $r > 0$,

$$\det_{1 \le j,k \le N} \left[\prod_{1 \le \ell \le N, \ell \ne k} \frac{\sinh((x_j - u_\ell)/2r)}{\sinh((u_k - u_\ell)/2r)} \right] = \prod_{1 \le j < k \le N} \frac{\sinh((x_k - x_j)/2r)}{\sinh((u_k - u_j)/2r)}.$$

- elliptic extension

Assume that $\boldsymbol{u} = (u_1, \dots, u_N) \in \mathbb{A}_{[0,2\pi r)^N}$ and $\overline{u}_\delta = \delta + \sum_{j=1}^N u_j \in (0, 2\pi r)$. Let $\overline{x}_\delta = \delta + \sum_{j=1}^N x_j$. Then

$$\det_{1 \le j,k \le N} \left[\frac{\vartheta_1((\overline{u}_\delta + x_j - u_k)/2\pi r; \tau)}{\vartheta_1(\overline{u}_\delta/2\pi r; \tau)} \prod_{1 \le \ell \le N, \ell \ne k} \frac{\vartheta_1((x_j - u_\ell)/2\pi r; \tau)}{\vartheta_1((u_k - u_\ell)/2\pi r; \tau)} \right]$$
$$= \frac{\vartheta_1(\overline{x}_\delta/2\pi r; \tau)}{\vartheta_1(\overline{u}_\delta/2\pi r; \tau)} \prod_{1 \le j < k \le N} \frac{\vartheta_1((x_j - x_k)/2\pi r; \tau)}{\vartheta_1((u_j - u_k)/2\pi r; \tau)},$$

where

$$\vartheta_1(v;\tau) = i \sum_{n \in \mathbb{Z}} (-1)^n q^{(n-(1/2))^2} z^{2n-1} = \sum_{n=1}^{\infty} (-1)^{n-1} e^{\pi i \tau (n-(1/2))^2} \sin\{(2n-1)\pi v\}.$$

[RS06] H. Rosengren, M. Schlosser, Elliptic determinant evaluations and the Macdonald identities for affine root systems, Compositio Math. **142**, 937-961 (2006).

[K13] M. Katori, Elliptic determinantal process of type A, arXiv:1311.4146.

1.3 A Combination of the Above Two

- The previous integral transformation $\mathcal{I}[f(W)|(t,x)]$ is extended to the linear integral transformation of functions of $\boldsymbol{x} \in R^N$ so that, if $F^{(k)}(\boldsymbol{x}) = \prod_{j=1}^N f_j^{(k)}(x_j)$, $k=1,2$, then

$$\mathcal{I}\left[F^{(k)}(\boldsymbol{W})\,\middle|\{(t_\ell, x_\ell)\}_{\ell=1}^N\right] = \prod_{j=1}^N \mathcal{I}\left[f_j^{(k)}(W_j)\middle|(t_j, x_j)\right], \quad k=1,2,$$

  and

$$\begin{aligned}&\mathcal{I}\Big[c_1 F^{(1)}(\boldsymbol{W}) + c_2 F^{(2)}(\boldsymbol{W})\,\Big|\{(t_\ell, x_\ell)\}_{\ell=1}^N\Big] \\ &\quad = c_1\mathcal{I}\Big[F^{(1)}(\boldsymbol{W})\,\Big|\{(t_\ell, x_\ell)\}_{\ell=1}^N\Big] + c_2\mathcal{I}\Big[F^{(2)}(\boldsymbol{W})\,\Big|\{(t_\ell, x_\ell)\}_{\ell=1}^N\Big],\end{aligned}$$

  $c_1, c_2 \in \mathbb{C}$, for $0 < t_j < \infty, 1 \le j \le N$, where $\boldsymbol{W} = (W_1, \ldots, W_N) \in R^N$.

- In particular, if $t_\ell = t, 1 \le {}^\forall \ell \le N$, we write $\mathcal{I}[\cdot|\{(t_\ell, x_\ell)\}_{\ell=1}^N]$ simply as $\mathcal{I}[\cdot|(t, \boldsymbol{x})]$ with $\boldsymbol{x} = (x_1, \ldots, x_N)$.

For $\boldsymbol{x} \in \mathbb{R}^N, \boldsymbol{u} \in \mathbb{W}_N$,

$$\begin{aligned}
\frac{h(\boldsymbol{x})}{h(\boldsymbol{u})} &= \frac{1}{h(\boldsymbol{u})} \det_{1\leq j,k\leq N}[x_j^{k-1}] \\
&= \frac{1}{h(\boldsymbol{u})} \det_{1\leq j,k\leq N}[m_{k-1}(t,x_j)] \\
&\quad \text{Since } m_{k-1}(t,x_j) \text{ is a monic polynomial of } x_j \text{ of order } k-1. \\
&= \frac{1}{h(\boldsymbol{u})} \det_{1\leq j,k\leq N}\left[\mathcal{I}[(W_j)^{k-1}|(t,x_j)]\right] \\
&= \mathcal{I}\left[\frac{1}{h(\boldsymbol{u})} \det_{1\leq j,k\leq N}[(W_j)^{k-1}]\middle|(t,\boldsymbol{x})\right] \\
&= \mathcal{I}\left[\frac{h(\boldsymbol{W})}{h(\boldsymbol{u})}\middle|(t,\boldsymbol{x})\right] \\
&= \mathcal{I}\left[\det_{1\leq j,k\leq N}[\Phi_\xi^{u_k}(W_j)]\middle|(t,\boldsymbol{x})\right] \\
&\quad \text{By Krattenthaler's identity with } \Phi_\xi^{u_k}(W_j) \equiv \prod_{1\leq \ell\leq N, \ell\neq k} \frac{W_j - u_\ell}{u_k - u_\ell} \\
&= \det_{1\leq j,k\leq N}\left[\mathcal{I}[\Phi_\xi^{u_k}(W)|(t,x_j)]\right].
\end{aligned}$$

Let

$$\mathcal{M}_\xi^{u_k}(t,x) \equiv \mathcal{I}[\Phi_\xi^{u_k}(W)|(t,x)],$$

where

$$\Phi_\xi^{u_k}(W) \equiv \prod_{1\le \ell \le N, \ell \neq k} \frac{W-u_\ell}{u_k-u_\ell} \quad \text{(a polynomial of } W \text{ of order } N-1\text{)}.$$

$$\mathcal{M}_\xi^{u_k}(t,B(t)), \quad t\ge 0, \quad 1\le k\le N: \quad \text{martingales}$$

$$\frac{h(\boldsymbol{x})}{h(\boldsymbol{u})} = \det_{1\le j,k\le N} [\mathcal{M}_\xi^{u_k}(t,x_j)]$$

Consider N independent BMs, $B_1(t), B_2(t), \dots, B_N(t), t\ge 0$.

$$\frac{h(\mathbf{B}(t))}{h(\boldsymbol{u})} = \det_{1\le j,k\le N} [\mathcal{M}_\xi^{u_k}(t,B_j(t))] \equiv \mathcal{D}_\xi(t,\mathbf{B}(t)),$$

where $\mathbf{B}(t) = (B_1(t),\dots,B_N(t))$, $\boldsymbol{u} = (u_1,\dots,u_N) \in \mathbb{W}_N$.

We call it a Determinantal Martingale.

2. Dyson's BM Model with β=2 = Noncolliding BM

$N \in \{2, 3, \dots\}$.
$\boldsymbol{X}(t) = (X_1(t), X_2(t), \dots, X_N(t)), t \geq 0$.

- **System of Stochastic Differential Equations (SDEs)**

$$X_j(t) = u_j + W_j(t) + \sum_{\substack{1 \leq k \leq N, \\ k \neq j}} \int_0^t \frac{ds}{X_j(s) - X_k(s)}, \quad 1 \leq j \leq N, \quad t \geq 0,$$

$$\boldsymbol{X}(0) = \boldsymbol{u} = (u_1, \dots, u_N) \in \mathbb{W}_N.$$

$W_j(t)$, $1 \leq j \leq N$: N independent BMs started at 0.

- **(Backward) Kolmogorov equation for transition probability density**

$$\frac{\partial}{\partial t} p_N(t, \boldsymbol{y}|\boldsymbol{x}) = \frac{1}{2} \Delta_N p_N(t, \boldsymbol{y}|\boldsymbol{x}) + \sum_{\substack{1 \leq j,k \leq N, \\ j \neq k}} \frac{1}{x_j - x_k} \frac{\partial}{\partial x_j} p_N(t, \boldsymbol{y}|\boldsymbol{x}),$$

$$\Delta_N \equiv \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2}, \quad p_N(0, \boldsymbol{y}|\boldsymbol{x}) = \prod_{j=1}^N \delta(y_j - x_j), \quad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{W}_N.$$

- Let $\delta_x(\cdot)$ be a point-mass measure (Dirac measure) on $x \in \mathbb{R}$.

$$\begin{aligned}
\mathfrak{M} &= \text{the space of integer-valued Radom measures on } \mathbb{R} \\
&= \Big\{ \xi(\cdot) = \sum_{j \in \mathbb{I}} \delta_{x_j}(\cdot) \text{ with an index set } \mathbb{I} : \\
&\qquad \xi(K) = \sharp\{x_j : x_j \in K\} < \infty \text{ for all compact subset } K \subset \mathbb{R} \Big\}. \\
\mathfrak{M}_0 &= \{\xi \in \mathfrak{M} : \xi(\{x\}) \leq 1 \text{ for any } x \in \mathbb{R}\} \quad \text{(without any multiple points).}
\end{aligned}$$

- Consider the Dyson model as an $\mathfrak{M}_0$-valued process (having unlabeled configurations).

$$\Xi(t, \cdot) = \sum_{j=1}^{N} \delta_{X_j(t)}(\cdot) \in \mathfrak{M}_0, \quad t \geq 0,$$

with the initial condiguration $\xi(\cdot) = \sum_{j=1}^{N} \delta_{u_j}(\cdot) \in \mathfrak{M}_0$.

- The expectation for Ξ started at ξ is denoted by $\mathbb{E}_\xi[\cdots]$.

2.1 From *h*-Transform to Determinantal Martingale Representation (DMR)

- Consider N independent BMs, $\mathbf{B}(t) = (B_1(t), B_2(t), \ldots, B_N(t)), t \geq 0$. They start at $\boldsymbol{u} \in \mathbb{W}_N$;

$$\mathbf{B}(0) = \boldsymbol{u} = (u_1, u_2, \ldots, u_N) \in \mathbb{W}_N.$$

- Consider a first exit time from the Weyl chamber $\mathbb{W}_N$,

$$\begin{aligned} \tau &= \inf\{t > 0 : \mathbf{B}(t) \notin \mathbb{W}_N\} \\ &= \text{the first collision time of } B_j(t)\text{'s started at } \boldsymbol{u} \in \mathbb{W}_N. \end{aligned}$$

- Expectation of these **free BMs** is denoted by $\mathrm{E}_{\boldsymbol{u}}[\cdots]$.
- The indicator of a condition ω is denoted by

$$\mathbf{1}_{(\omega)} = \begin{cases} 1, & \text{if the condition } \omega \text{ is satisfied} \\ 0, & \text{otherwise} \end{cases}$$

Proposition 3 (Grabiner)

Assume that $\xi = \sum_{j=1}^{N} \delta_{u_j} \in \mathfrak{M}_0$ ($\Leftrightarrow \boldsymbol{u} \in \mathbb{W}_N$). For any $(\mathcal{F}_\Xi)_t$-measurable bonded function F, $0 \leq t \leq T < \infty$, the following equality is established,

$$\mathbb{E}_\xi [F(\Xi(\cdot))] = \mathrm{E}_{\boldsymbol{u}} \left[F\left(\sum_{j=1}^{N} \delta_{B_j}(\cdot) \right) \mathbf{1}_{(\tau > T)} \frac{h(\mathbf{B}(T))}{h(\boldsymbol{u})} \right].$$

[Grabiner99] D. J. Grabiner, Brownian motion in a Weyl chamber, non-colliding particles, and random matrices, Ann. Inst. Henri Poincaré, Probab. Stat. **35**, 177-204 (1999).

$(\mathcal{F}_\Xi)_t$-measurable bonded function, $0 \leq t \leq T < \infty$

It is sufficient to consider the cases that F is given by

$$F(\Xi(\cdot)) = \prod_{m=1}^{M} g_m(\boldsymbol{X}(t_m))$$

for an arbitrary $M \in \mathbb{N} \equiv \{1, 2, \dots\}$, an arbitrary increasing sequence of times $0 \leq t_1 < t_2 < \cdots < t_M \leq T < \infty$, with bounded **symmetric functions** g_m's.

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$$\mathbb{E}_\xi[F(\Xi(\cdot))] = \mathrm{E}_{\boldsymbol{u}}\left[F\left(\sum_{j=1}^{N} \delta_{B_j}(\cdot)\right) \mathbf{1}_{(\tau > T)} \frac{h(\mathbf{B}(T))}{h(\boldsymbol{u})}\right].$$

harmonic transform of the absorbing BM in $\mathbb{W}_N$

Equivalent

Dyson's BM model with $\beta = 2$

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harmonic transform of the absorbing BM in $\mathbb{W}_N$

Equivalent

Dyson's BM model with $\beta = 2$

- Our consideration is the following.

$$\mathrm{E}_{\boldsymbol{u}}\left[F\left(\sum_{j=1}^{N}\delta_{B_j}(\cdot)\right)\mathbf{1}_{(\tau>T)}\frac{h(\mathbf{B}(T))}{h(\boldsymbol{u})}\right]$$

$$=\mathrm{E}_{\boldsymbol{u}}\left[F\left(\sum_{j=1}^{N}\delta_{B_j}(\cdot)\right)\boxed{\frac{h(\mathbf{B}(T))}{h(\boldsymbol{u})}}\right]$$

Signed Measure

By the reflection principle of BM (by the same idea of the Karlin-McGregor formula, but we do not need it).

$$=\mathrm{E}_{\boldsymbol{u}}\left[F\left(\sum_{j=1}^{N}\delta_{B_j}(\cdot)\right)\mathcal{D}_{\xi}(T,\mathbf{B}(T))\right],$$

where $\mathcal{D}_{\xi}(T,\mathbf{B}(T))=\det_{1\le j,k\le N}[\mathcal{M}_{\xi}^{u_k}(T,B_j(T))]$.

Theorem 4

Assume that $\xi = \sum_{j=1}^{N} \delta_{u_j} \in \mathfrak{M}_0$. For any $(\mathcal{F}_{\Xi})_t$-measurable bounded function F, $0 \leq t \leq T < \infty$, the equality

$$\mathbb{E}_{\xi}[F(\Xi(\cdot))] = \mathrm{E}_{\boldsymbol{u}} \left[F \left(\sum_{j=1}^{N} \delta_{B_j}(\cdot) \right) \mathcal{D}_{\xi}(T, \mathbf{B}(T)) \right]$$

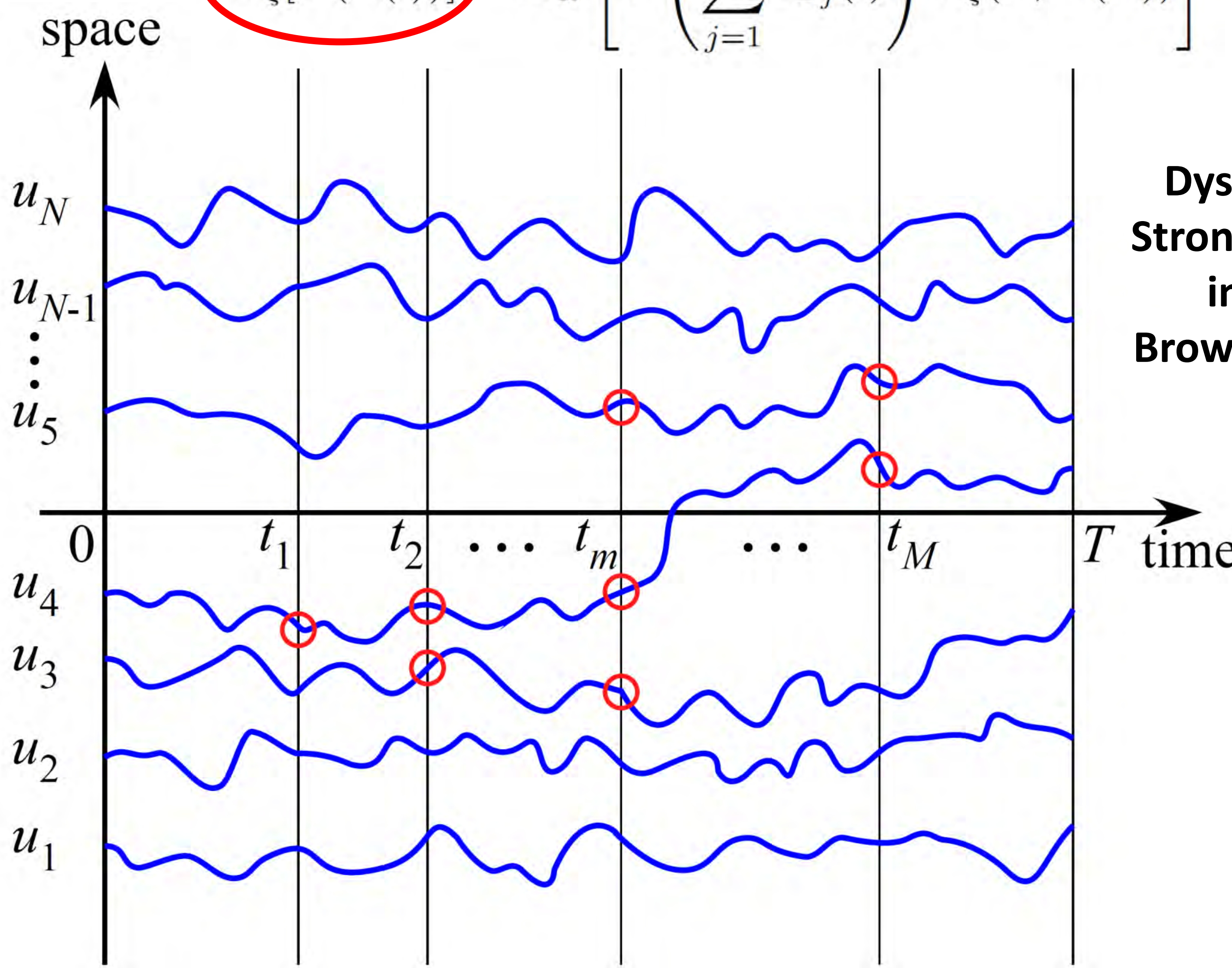
holds, where

$$\mathcal{D}_{\xi}(t, \boldsymbol{x}) = \det_{1 \leq j,k \leq N} [\mathcal{M}_{\xi}^{u_k}(t, x_j)], \quad x \in \mathbb{R}^N, \quad t \geq 0.$$

[K-Tanemura13] M. Katori, H. Tanemura, Complex Brownian motion representation of the Dyson model, Electron. Commun. Probab. **18** (4), 1-16 (2013).

[K14] M. Katori, Determinantal martingales and noncolliding diffusion processes, Stoch. Proc. Appl. **124**, 3724-3768 (2014).

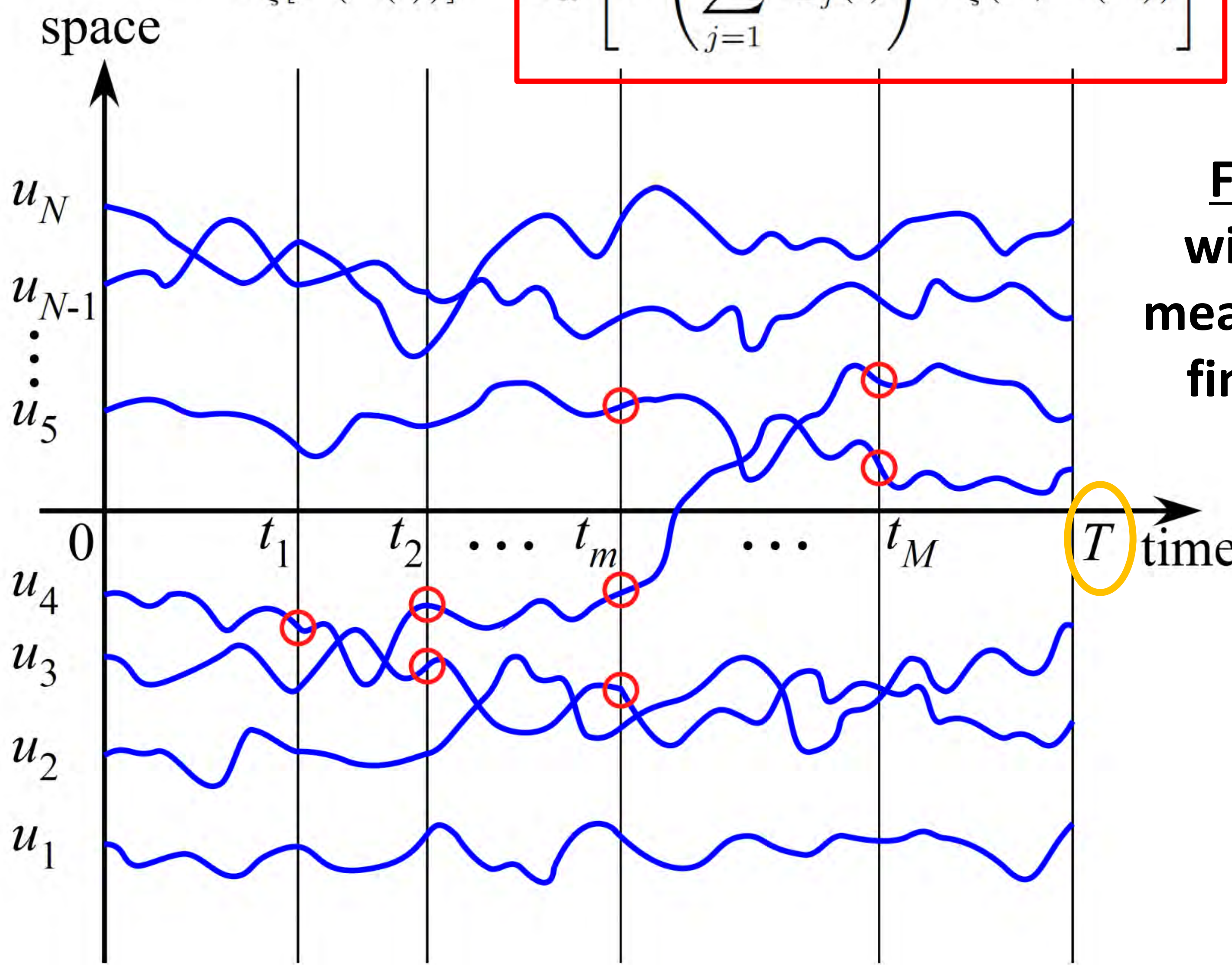
$$\mathbb{E}_{\xi}[F(\Xi(\cdot))] = \mathrm{E}_{\boldsymbol{u}}\left[F\left(\sum_{j=1}^{N}\delta_{B_j}(\cdot)\right)\mathcal{D}_{\xi}(T,\mathbf{B}(T))\right]$$



Dyson's model: Strongly repulsive-interacting Brownian motions

We observe particles at points ◯ on the spatio-temporal plane.

$$\mathbb{E}_\xi[F(\Xi(\cdot))] = \mathrm{E}_{\boldsymbol{u}}\left[F\left(\sum_{j=1}^{N}\delta_{B_j}(\cdot)\right)\mathcal{D}_\xi(T,\mathbf{B}(T))\right]$$



<u>Free BMs</u>
with signed measure at the final time *T*

We observe particles at points ○ on the spatio-temporal plane.

$$\frac{h(\mathbf{B}(t))}{h(\boldsymbol{u})} = \det_{1 \le j,k \le N} [\mathcal{M}_\xi^{u_k}(t, B_j(t))] \equiv \mathcal{D}_\xi(t, \mathbf{B}(t)),$$

where $\mathbf{B}(t) = (B_1(t), \dots, B_N(t))$, $\boldsymbol{u} = (u_1, \dots, u_N) \in \mathbb{W}_N$.

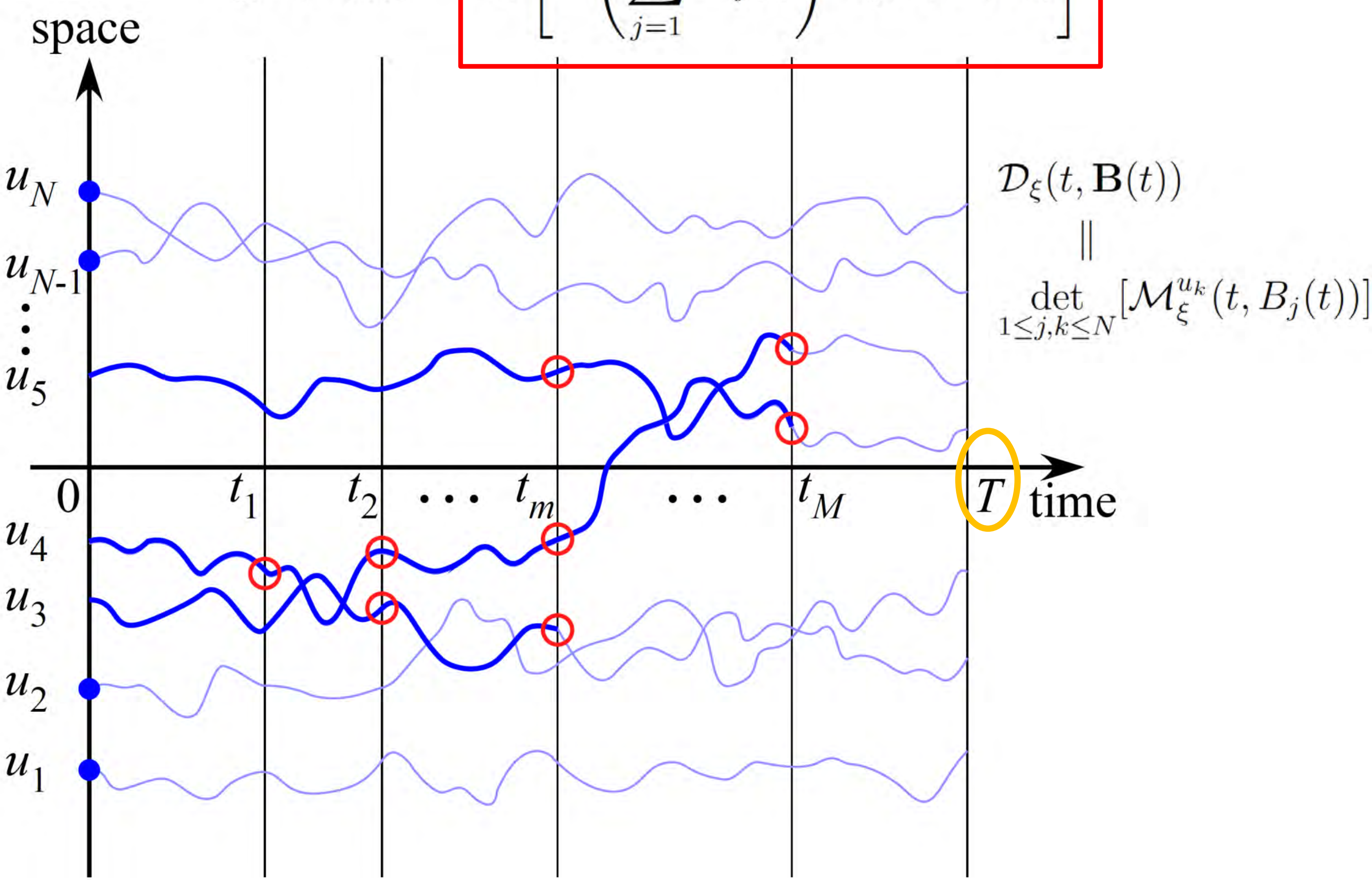
$\mathcal{M}_\xi^{u_k}(t, x)$ satisfies the following three properties.

(M1) $\mathcal{M}_\xi^{u_k}(t, B(t))$, $1 \le k \le N$, $t \in [0, \infty)$ are continuous-time martingales.

(M2) For any time $t \ge 0$, $\mathcal{M}_\xi^{u_k}(t, x), 1 \le k \le N$ are linearly independent functions of x.

(M3) For $1 \le j, k \le N$, $\lim_{t \downarrow 0} \mathrm{E}_{u_j}[\mathcal{M}_\xi^{u_k}(t, B(t))] = \delta_{jk}$.

$$\mathbb{E}_\xi[F(\Xi(\cdot))] = \mathrm{E}_{\boldsymbol{u}}\left[F\left(\sum_{j=1}^{N}\delta_{B_j}(\cdot)\right)\mathcal{D}_\xi(T,\mathbf{B}(T))\right]$$



We observe particles at points ○ on the spatio-temporal plane.

- For $n \in \mathbb{N}$, an index set $\{1, 2, \dots, n\}$ is denoted by $\mathbb{I}_n$.
- Fixed $N \in \mathbb{N}$ with $N' \in \mathbb{I}_N$, we write $\mathbb{J} \subset \mathbb{I}_N, \sharp\mathbb{J} = N'$, if $\mathbb{J} = \{j_1, \dots, j_{N'}\}, 1 \leq j_1 < \cdots < j_{N'} \leq N$.
- For $\boldsymbol{x} = (x_1, \dots, x_N) \in \mathbb{R}^N$, put $\boldsymbol{x}_{\mathbb{J}} = (x_{j_1}, \dots, x_{j_{N'}})$.
- In particular, we write $\boldsymbol{x}_{N'} = \boldsymbol{x}_{\mathbb{I}_{N'}}, 1 \leq N' \leq N$. (By definition $\boldsymbol{x}_N = \boldsymbol{x}$.)

Reducibility of DMR

Assume that $\xi(\cdot) = \sum_{j=1}^{N} \delta_{u_j}(\cdot) \in \mathfrak{M}_0$ ($\Leftrightarrow \boldsymbol{u} \in \mathbb{W}_N$). Let $1 \leq N' \leq N$. For $0 < t \leq T < \infty$ and an $\mathcal{F}_t$-measurable symmetric function $F_{N'}$ on $\mathbb{R}^{N'}$,

$$\sum_{\mathbb{J} \subset \mathbb{I}_N, \sharp\mathbb{J} = N'} \mathrm{E}_{\boldsymbol{u}} \left[F_{N'}(\mathbf{B}_{\mathbb{J}}(t)) \mathcal{D}_{\xi}(T, \mathbf{B}(T)) \right]$$
$$= \int_{\mathbb{W}_{N'}} \xi^{\otimes N'}(d\boldsymbol{v}) \mathrm{E}_{\boldsymbol{v}} \left[F_{N'}(\mathbf{B}_{N'}(t)) \mathcal{D}_{\xi}(T, \mathbf{B}_{N'}(T)) \right].$$

Determinantal Martingale with a Small Matrix

2.2 From DMR to Correlation Kernel

2.2.1 Density function (one-point correlation)

- Let $C_0(\mathbb{R})$ be the set of all continuous real-valued functions with compact supports on $\mathbb{R}$.
- **Density function** at a single time t is denoted by $\rho_\xi(t,x)$, which is also a function of the initial configuration $\xi \in \mathfrak{M}_0$.
- For given $\xi \in \mathfrak{M}_0$, it is defined as a continuous function of $x \in \mathbb{R}$ for $0 \le t \le T < \infty$ such that for any 'test function' $\chi \in C_0(\mathbb{R})$,

$$\mathbb{E}_\xi \left[\int_{\mathbb{R}} \chi(x) \Xi(t, dx) \right] = \int_{\mathbb{R}} dx\, \chi(x) \rho_\xi(t,x).$$

- The test function χ is symmetrized as $g(\boldsymbol{x}) = \sum_{j=1}^{N} \chi(x_j)$.
- It is applied as F to our DMR, and we obtain the equality

$$\mathbb{E}_\xi \left[\sum_{j=1}^{N} \chi(X_j(t)) \right] = \mathrm{E}_{\boldsymbol{u}} \left[\sum_{j=1}^{N} \chi(B_j(t)) \mathcal{D}_\xi(T, \mathbf{B}(T)) \right], \quad 0 \le t \le T < \infty.$$

- The LHS gives

$$\mathbb{E}_\xi \left[\sum_{j=1}^{N} \chi(X_j(t)) \right] = \mathbb{E}_\xi \left[\int_{\mathbb{R}} \chi(x) \Xi(t, dx) \right],$$

since $\Xi(t, \cdot) = \sum_{j=1}^{N} \delta_{X_j(t)}(\cdot)$. It is nothing but the LHS of the equation defining $\rho_\xi(t,x)$,

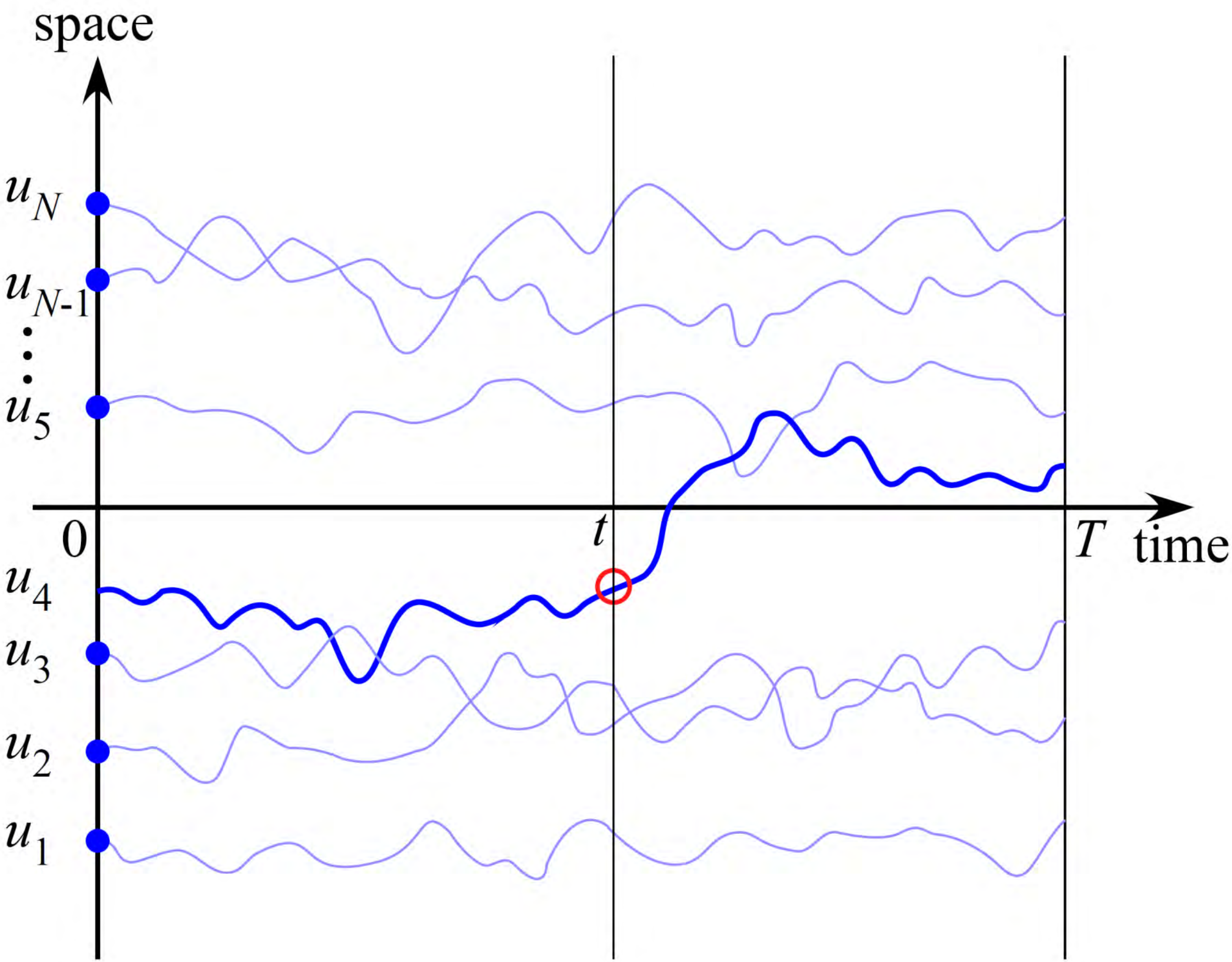
$$\mathbb{E}_\xi \left[\int_{\mathbb{R}} \chi(x) \Xi(t, dx) \right] = \int_{\mathbb{R}} dx\, \chi(x) \rho_\xi(t,x).$$

- So what we have to do is to calculate the RHS of our DMR.

- Since we observe the density, which is a one-particle function, **Reducibility** of DMR gives the follows,

$$\sum_{j=1}^{N} \mathrm{E}_{\boldsymbol{u}}[\chi(B_j(t))\mathcal{D}_\xi(T, \mathbf{B}(T))] = \int_{\mathbb{R}} \xi(dv)\mathrm{E}_v[\chi(B(t))\mathcal{M}_\xi^v(t, B(t))]$$

$$= \int_{\mathbb{R}} \xi(dv) \int_{\mathbb{R}} dx\, \chi(x)p(t, x|v)\mathcal{M}_\xi^v(t, x).$$


space
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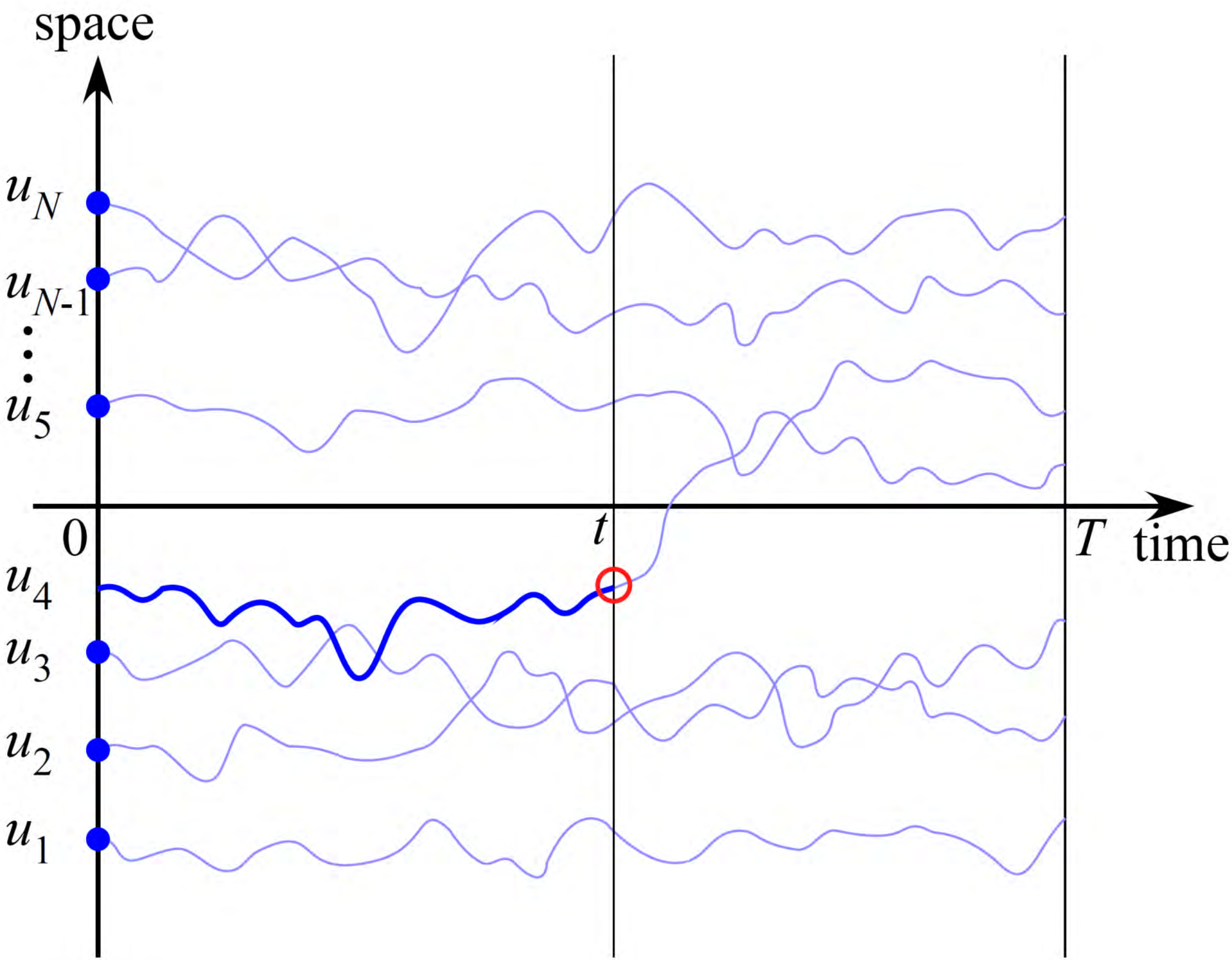

space
u_N
$u_{N\text{-}1}$
u_5
0
t
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time
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u_3
u_2
u_1

- Since we observe the density, which is a one-particle function, **Reducibility** of DMR gives the follows,

$$\sum_{j=1}^{N} \mathrm{E}_{\boldsymbol{u}}[\chi(B_j(t))\mathcal{D}_\xi(T,\mathbf{B}(T))] = \int_{\mathbb{R}} \xi(dv)\mathrm{E}_v[\chi(B(t))\mathcal{M}_\xi^v(t,B(t))]$$

$$= \int_{\mathbb{R}} \xi(dv) \int_{\mathbb{R}} dx\, \chi(x)p(t,x|v)\mathcal{M}_\xi^v(t,x).$$

N × N determinantal martingale is reduced to a single martingale

- Since we observe the density, which is a one-particle function, **Reducibility** of DMR gives the follows,

$$\sum_{j=1}^{N} \mathrm{E}_{\boldsymbol{u}}[\chi(B_j(t))\mathcal{D}_\xi(T, \mathbf{B}(T))] = \int_{\mathbb{R}} \xi(dv)\mathrm{E}_v[\chi(B(t))\mathcal{M}_\xi^v(t, B(t))]$$

$$= \int_{\mathbb{R}} \xi(dv) \int_{\mathbb{R}} dx\, \chi(x) p(t, x|v)\mathcal{M}_\xi^v(t, x).$$

The final time *T* is reduced to the observing time *t*

- Since we observe the density, which is a one-particle function, **Reducibility** of DMR gives the follows,

$$\sum_{j=1}^{N} \mathrm{E}_{\boldsymbol{u}}[\chi(B_j(t))\mathcal{D}_\xi(T, \mathbf{B}(T))] = \int_{\mathbb{R}} \xi(dv)\mathrm{E}_v[\chi(B(t))\mathcal{M}_\xi^v(t, B(t))]$$
$$= \int_{\mathbb{R}} \xi(dv) \int_{\mathbb{R}} dx\, \chi(x)p(t, x|v)\mathcal{M}_\xi^v(t, x).$$

- By Fubini's theorem, we can rewrite it as $\displaystyle\int_{\mathbb{R}} dx\, \chi(x) \int_{\mathbb{R}} \xi(dv)\, p(t, x|v)\mathcal{M}_\xi^v(t, x).$

- This should be equal to $\displaystyle\int_{\mathbb{R}} dx\, \chi(x)\rho_\xi(t, x).$

- Then we obtain the result $\rho_\xi(t, x) = \int_{\mathbb{R}} \xi(dv)p(t, x|v)\mathcal{M}_\xi^v(t, x).$

- For convenience, we introduce a function,

$$\mathcal{G}_\xi(s, x; t, y) = \int_{\mathbb{R}} \xi(dv) p(s, x|v) \mathcal{M}_\xi^v(t, y), \quad (s, x), (t, y) \in [0, \infty) \times \mathbb{R}.$$

- Then the above result is written as

$$\rho_\xi(t, x) = \mathcal{G}_\xi(t, x; t, x), \quad x \in \mathbb{R}, \quad t \in [0, \infty).$$

2.2.2 Two-time correlation function

- For $0 \le t_1 < t_2 \le T < \infty$, set $g_1(\boldsymbol{x}) = \sum_{j=1}^{N} \chi_1(x_j)$, $g_2(\boldsymbol{x}) = \sum_{j=1}^{N} \chi_2(x_j)$,

  where $\chi_m \in \mathrm{C}_0(\mathbb{R}), m = 1, 2$, and put $F(\Xi(\cdot)) = \prod_{m=1}^{2} g_m(\boldsymbol{X}(t_m))$.

- DMR gives

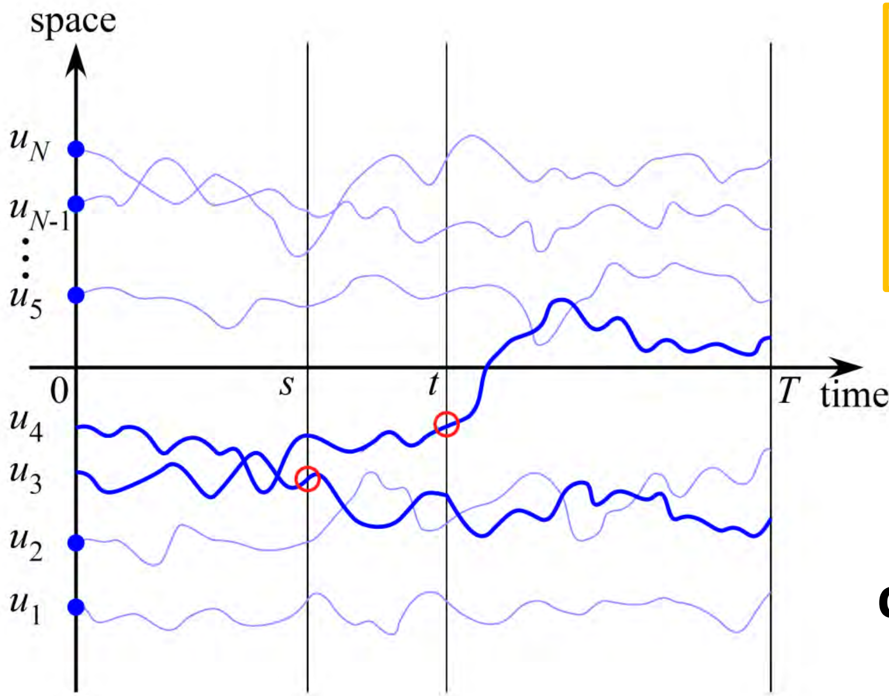
$$\mathbb{E}_\xi\left[\sum_{j=1}^{N}\sum_{k=1}^{N} \chi_1(X_j(t_1))\chi_2(X_k(t_2))\right] = \mathrm{E}_{\boldsymbol{u}}\left[\sum_{j=1}^{N}\sum_{k=1}^{N} \chi_1(B_j(t_1))\chi_2(B_k(t_2))\mathcal{D}_\xi(T, \mathbf{B}(T))\right], \ 0 \le t \le T < \infty.$$

- The LHS defines the **two-time correlation function** $\rho_\xi(s, x; t, y)$ as

$$\mathbb{E}_\xi\left[\sum_{j=1}^{N}\sum_{k=1}^{N} \chi_1(X_j(t_1))\chi_2(X_k(t_2))\right] = \int_{\mathbb{W}_2} dx_1 dx_2\, \chi_1(x_1)\chi_2(x_2)\rho_\xi(t_1, x_1; t_2, x_2).$$

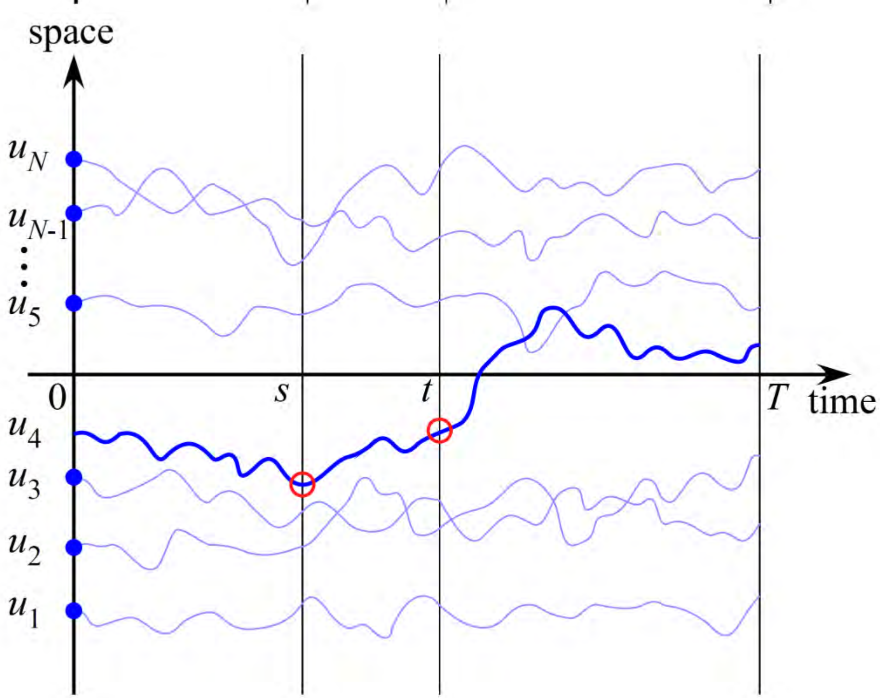
- On the other hand, the RHS gives

$$\begin{aligned}
&\sum_{j=1}^{N}\sum_{k=1}^{N} \mathrm{E}_{\boldsymbol{u}}[\chi_1(B_j(t_1))\chi_2(B_k(t_2))\mathcal{D}_\xi(T, \mathbf{B}(T))] \\
&\quad = \sum_{\substack{1\le j,k\le N, \\ j\ne k}} \mathrm{E}_{\boldsymbol{u}}[\chi_1(B_j(t_1))\chi_2(B_k(t_2))\mathcal{D}_\xi(T, \mathbf{B}(T))] \\
&\quad + \sum_{1\le j\le N} \mathrm{E}_{\boldsymbol{u}}[\chi_1(B_i(t_1))\chi_2(B_j(t_2))\mathcal{D}_\xi(T, \mathbf{B}(T))].
\end{aligned}$$

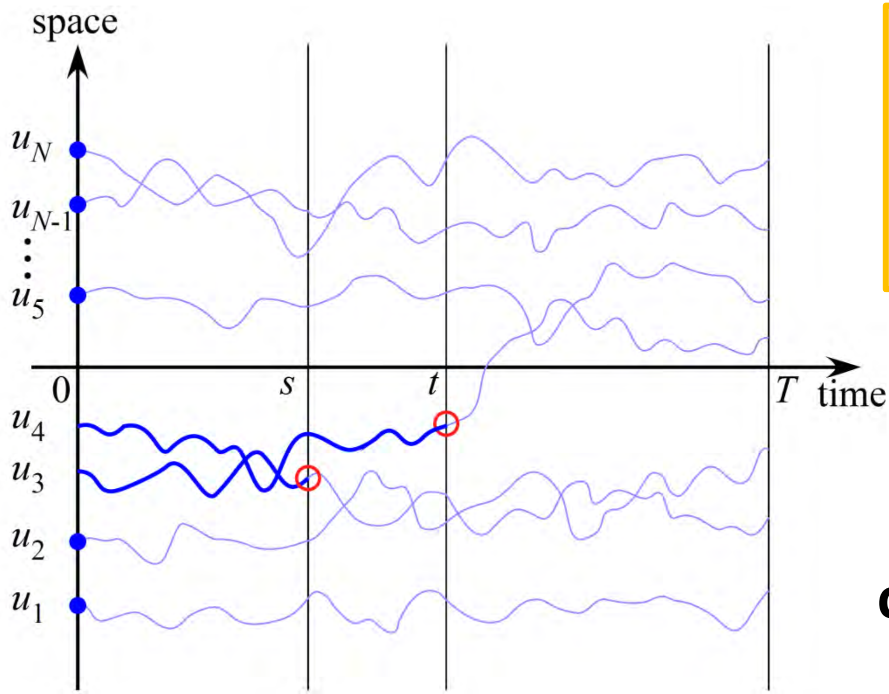


We observe at the TWO spatio-temporal points marked by ○.

The case that two paths contribute to the observation.

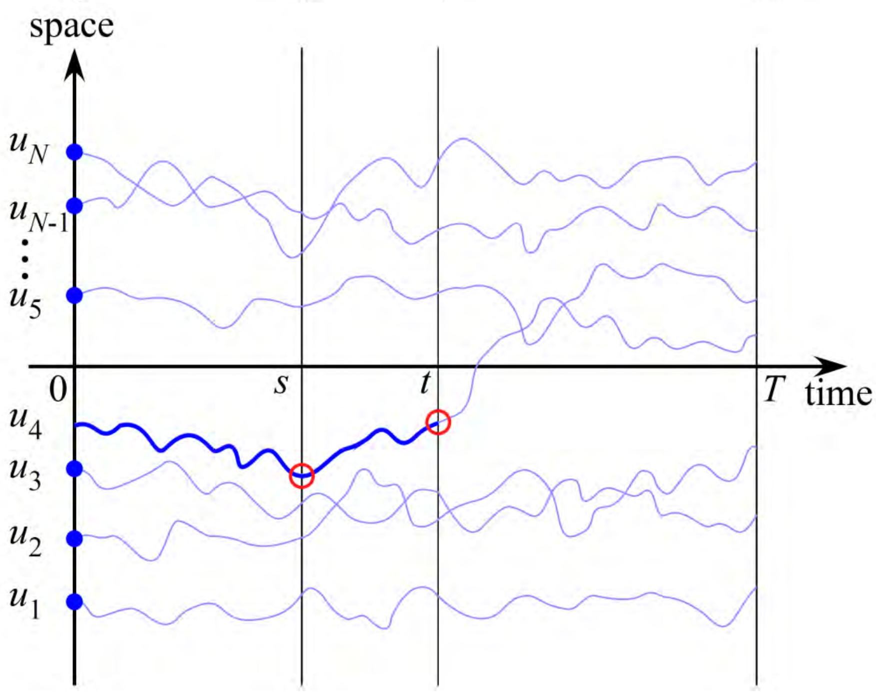


The case that only one path contributes to the observation.



We observe at the TWO spatio-temporal points marked by ○.

The case that two paths contribute to the observation.



The case that only one path contributes to the observation.

Reduced to 2 × 2 Det. Mar.

- By **Reducibility of DMR**, it becomes

$$\int_{\mathbb{W}_2} \xi^{\otimes 2}(d\boldsymbol{v}) \mathrm{E}_{(v_1,v_2)} \left[\chi_1(B_1(t_1))\chi_2(B_2(t_2)) \det \begin{pmatrix} \mathcal{M}_\xi^{v_1}(T, B_1(T)) & \mathcal{M}_\xi^{v_1}(T, B_2(T)) \\ \mathcal{M}_\xi^{v_2}(T, B_1(T)) & \mathcal{M}_\xi^{v_2}(T, B_2(T)) \end{pmatrix} \right]$$

$$+ \int_{\mathbb{R}} \xi(dv) \mathrm{E}_v[\chi_1(B(t_1))\chi_2(B(t_2))\mathcal{M}_\xi^v(T, B(T))]$$

← Reduced to a single martingale

$$= \int_{\mathbb{W}_2} \xi^{\otimes 2}(d\boldsymbol{v}) \mathrm{E}_{(v_1,v_2)} \left[\chi_1(B_1(t_1))\chi_2(B_2(t_2)) \det \begin{pmatrix} \mathcal{M}_\xi^{v_1}(t_1, B_1(t_1)) & \mathcal{M}_\xi^{v_1}(t_2, B_2(t_2)) \\ \mathcal{M}_\xi^{v_2}(t_1, B_1(t_1)) & \mathcal{M}_\xi^{v_2}(t_2, B_2(t_2)) \end{pmatrix} \right]$$

$$+ \int_{\mathbb{R}} \xi(dv) \mathrm{E}_v[\chi_1(B(t_1))\chi_2(B(t_2))\mathcal{M}_\xi^v(t_2, B(t_2))].$$

- By Fubini's theorem, it is equal to

$$\int_{\mathbb{W}_2} dx_1 dx_2\, \chi_1(x_1)\chi_2(x_2) \det \begin{pmatrix} \mathcal{G}_\xi(t_1, x_1; t_1, x_1) & \mathcal{G}_\xi(t_1, x_1; t_2, x_2) \\ \mathcal{G}_\xi(t_2, x_2; t_1, x_1) & \mathcal{G}_\xi(t_2, x_2; t_2, x_2) \end{pmatrix}$$

$$+ \int_{\mathbb{W}_2} dx_1 dx_2\, \chi_1(x_1)\chi_2(x_2) \mathcal{G}_\xi(t_1, x_1; t_2, x_2) p(t_2 - t_1, x_2|x_1)$$

$$= \int_{\mathbb{W}_2} dx_1 dx_2\, \chi_1(x_1)\chi_2(x_2) \det \begin{pmatrix} \mathcal{G}_\xi(t_1, x_1; t_1, x_1) & \mathcal{G}_\xi(t_1, x_1; t_2, x_2) \\ \mathcal{G}_\xi(t_2, x_2; t_1, x_1) - p(t_2 - t_1; x_2|x_1) & \mathcal{G}_\xi(t_2, x_2; t_2, x_2) \end{pmatrix}.$$

- By **Reducibility of DMR**, it becomes

$$\int_{\mathbb{W}_2} \xi^{\otimes 2}(d\boldsymbol{v})\mathrm{E}_{(v_1,v_2)}\left[\chi_1(B_1(t_1))\chi_2(B_2(t_2))\det\begin{pmatrix} \mathcal{M}_\xi^{v_1}(T,B_1(T)) & \mathcal{M}_\xi^{v_1}(T,B_2(T)) \\ \mathcal{M}_\xi^{v_2}(T,B_1(T)) & \mathcal{M}_\xi^{v_2}(T,B_2(T)) \end{pmatrix}\right]$$

$$+\int_{\mathbb{R}} \xi(dv)\mathrm{E}_v[\chi_1(B(t_1))\chi_2(B(t_2))\mathcal{M}_\xi^v(T,B(T))]$$

$$= \int_{\mathbb{W}_2} \xi^{\otimes 2}(d\boldsymbol{v})\mathrm{E}_{(v_1,v_2)}\left[\chi_1(B_1(t_1))\chi_2(B_2(t_2))\det\begin{pmatrix} \mathcal{M}_\xi^{v_1}(t_1,B_1(t_1)) & \mathcal{M}_\xi^{v_1}(t_2,B_2(t_2)) \\ \mathcal{M}_\xi^{v_2}(t_1,B_1(t_1)) & \mathcal{M}_\xi^{v_2}(t_2,B_2(t_2)) \end{pmatrix}\right]$$

$$+\int_{\mathbb{R}} \xi(dv)\mathrm{E}_v[\chi_1(B(t_1))\chi_2(B(t_2))\mathcal{M}_\xi^v(t_2,B(t_2))].$$

- By Fubini's theorem, it is equal to

$$\int_{\mathbb{W}_2} dx_1 dx_2\, \chi_1(x_1)\chi_2(x_2)\det\begin{pmatrix} \mathcal{G}_\xi(t_1,x_1;t_1,x_1) & \mathcal{G}_\xi(t_1,x_1;t_2,x_2) \\ \mathcal{G}_\xi(t_2,x_2;t_1,x_1) & \mathcal{G}_\xi(t_2,x_2;t_2,x_2) \end{pmatrix}$$

$$+\int_{\mathbb{W}_2} dx_1 dx_2\, \chi_1(x_1)\chi_2(x_2)\mathcal{G}_\xi(t_1,x_1;t_2,x_2)p(t_2-t_1,x_2|x_1)$$

$$= \int_{\mathbb{W}_2} dx_1 dx_2\, \chi_1(x_1)\chi_2(x_2)\det\begin{pmatrix} \mathcal{G}_\xi(t_1,x_1;t_1,x_1) & \mathcal{G}_\xi(t_1,x_1;t_2,x_2) \\ \mathcal{G}_\xi(t_2,x_2;t_1,x_1) - p(t_2-t_1;x_2|x_1) & \mathcal{G}_\xi(t_2,x_2;t_2,x_2) \end{pmatrix}.$$

- The result

$$\int_{\mathbb{W}_2} dx_1 dx_2\, \chi_1(x_1)\chi_2(x_2) \det \begin{pmatrix} \mathcal{G}_\xi(t_1, x_1; t_1, x_1) & \mathcal{G}_\xi(t_1, x_1; t_2, x_2) \\ \mathcal{G}_\xi(t_2, x_2; t_1, x_1) - p(t_2 - t_1; x_2|x_1) & \mathcal{G}_\xi(t_2, x_2; t_2, x_2) \end{pmatrix}$$

should be equal to $\int_{\mathbb{W}_2} dx_1 dx_2\, \chi_1(x_1)\chi_2(x_2)\rho_\xi(t_1, x_1; t_2, x_2)$.

- Then the two-time correlation function is obtained as

$$\rho_\xi(s, x; t, y) = \det \begin{pmatrix} \mathbb{K}_\xi(s, x; s, x) & \mathbb{K}_\xi(s, x; t, y) \\ \mathbb{K}_\xi(t, y; s, x) & \mathbb{K}_\xi(t, y; t, y) \end{pmatrix}$$

for $0 \leq s < t < \infty$, $x, y \in \mathbb{R}$, where

$$\mathbb{K}_\xi(s, x; t, y) = \mathcal{G}_\xi(s, x; t, y) - \mathbf{1}_{(s>t)} p(s - t, x|y)$$

with

$$\mathcal{G}_\xi(s, x; t, y) = \int_{\mathbb{R}} \xi(dv) p(s, x|v) \mathcal{M}_\xi^v(t, y).$$

- The result

$$\int_{\mathbb{W}_2} dx_1 dx_2\, \chi_1(x_1)\chi_2(x_2) \det \begin{pmatrix} \mathcal{G}_\xi(t_1, x_1; t_1, x_1) & \mathcal{G}_\xi(t_1, x_1; t_2, x_2) \\ \mathcal{G}_\xi(t_2, x_2; t_1, x_1) - p(t_2 - t_1; x_2|x_1) & \mathcal{G}_\xi(t_2, x_2; t_2, x_2) \end{pmatrix}$$

should be equal to $\int_{\mathbb{W}_2} dx_1 dx_2\, \chi_1(x_1)\chi_2(x_2)\rho_\xi(t_1, x_1; t_2, x_2)$.

- Then the two-time correlation function is obtained as

$$\rho_\xi(s, x; t, y) = \det \begin{pmatrix} \mathbb{K}_\xi(s, x; s, x) & \mathbb{K}_\xi(s, x; t, y) \\ \mathbb{K}_\xi(t, y; s, x) & \mathbb{K}_\xi(t, y; t, y) \end{pmatrix}$$

for $0 \leq s < t < \infty$, $x, y \in \mathbb{R}$, where

$$\mathbb{K}_\xi(s, x; t, y) = \mathcal{G}_\xi(s, x; t, y) - \mathbf{1}_{(s>t)} p(s - t, x|y)$$

with

$$\mathcal{G}_\xi(s, x; t, y) = \int_{\mathbb{R}} \xi(dv) p(s, x|v) \mathcal{M}_\xi^v(t, y).$$

2.2.3 General results

Theorem 5

(i) For any finite and fixed initial configuration without multiple points, that is, for $\xi \in \mathfrak{M}_0, \xi(\mathbb{R}) = N \in \mathbb{N}$, Dyson's Brownian motion model is **determinantal** in the sense that all spatio-temporal correlation functions are given by determinants as

$$\rho_\xi\left(t_1, \boldsymbol{x}_{N_1}^{(1)}; \dots ; t_M, \boldsymbol{x}_{N_M}^{(M)}\right) = \det_{\substack{1 \le i \le N_m, 1 \le j \le N_n, \\ 1 \le m,n \le M}} \left[\mathbb{K}_\xi(t_m, x_i^{(m)}; t_n, x_j^{(n)})\right],$$

$0 \le t_1 < \cdots < t_M < \infty$, $1 \le N_m \le N$, $\boldsymbol{x}_{N_m}^{(m)} \in \mathbb{R}^{N_m}, 1 \le m \le M \in \mathbb{N}$. Here the integral kernel, $\mathbb{K}_\xi : ([0, \infty) \times \mathbb{R})^2 \mapsto \mathbb{R}$, is a function of initial configuration ξ and is called the **correlation kernel**.

(ii) The **correlation kernel** is given by

$$\mathbb{K}_\xi(s, x; t, y) = \mathcal{G}_\xi(s, x; t, y) - \mathbf{1}_{(s>t)} p(s - t, x|y),$$

with

$$\mathcal{G}_\xi(s, x; t, y) = \int_{\mathbb{R}} \xi(dv) p(s, x|v) \mathcal{M}_\xi^v(t, y).$$

- For any integer $M \in \mathbb{N}$, a sequence of times $\boldsymbol{t} = (t_1, \dots, t_M) \in [0, \infty)^M$ with $0 \le t_1 < \cdots < t_M < \infty$, and a sequence of functions $\boldsymbol{f} = (f_{t_1}, \dots, f_{t_M}) \in \mathrm{C}_0(\mathbb{R})^M$, the **moment generating function** of multi-time distribution of the present Dyson model is defined by

$$\Psi_\xi^{\boldsymbol{t}}[\boldsymbol{f}] \equiv \mathbb{E}_\xi \left[\exp \left\{ \sum_{m=1}^{M} \int_{\mathbb{R}} f_{t_m}(x) \Xi(t_m, dx) \right\} \right].$$

- It is expanded with respect to 'test functions' $\chi_{t_m}(\cdot) \equiv e^{f_{t_m}(\cdot)} - 1, 1 \le m \le M$ as

$$\Psi_\xi^{\boldsymbol{t}}[\boldsymbol{f}] = \sum_{\substack{0 \le N_m \le N, \\ 1 \le m \le M}} \int_{\prod_{m=1}^{M} \mathbb{W}_{N_m}} \prod_{m=1}^{M} d\boldsymbol{x}_{N_m}^{(m)} \prod_{j=1}^{N_m} \chi_{t_m}\left(x_j^{(m)}\right) \rho_\xi\left(t_1, \boldsymbol{x}_{N_1}^{(1)}; \dots; t_M, \boldsymbol{x}_{N_M}^{(M)}\right),$$

where $\boldsymbol{x}_{N_m}^{(m)}$ denotes $(x_1^{(m)}, \dots, x_{N_m}^{(m)})$.

- Given an integral kernel, $\mathbf{K}(s,x;t,y)$, $(s,x),(t,y) \in [0,\infty) \times \mathbb{R}$, a **Fredholm determinant** is defined as

$$\operatorname*{Det}_{\substack{(s,t)\in\{t_1,\dots,t_M\}^2,\\ (x,y)\in\mathbb{R}^2}} \left[\delta_{st}\delta_x(\{y\}) + \mathbf{K}(s,x;t,y)\chi_t(y)\right]$$

$$= \sum_{\substack{0\le N_m\le N,\\ 1\le m\le M}} \int_{\prod_{m=1}^M \mathbb{W}_{N_m}} \prod_{m=1}^{M} d\boldsymbol{x}_{N_m}^{(m)} \prod_{j=1}^{N_m} \chi_{t_m}\left(x_j^{(m)}\right) \det_{\substack{1\le j\le N_m, 1\le k\le N_n,\\ 1\le m,n\le M}} \left[\mathbf{K}(t_m, x_j^{(m)}; t_n, x_k^{(n)})\right]$$

for a sequence of functions $\boldsymbol{\chi} = (\chi_{t_1},\dots,\chi_{t_M}) \in \mathrm{C}_0(\mathbb{R})^M$, where $d\boldsymbol{x}_{N_m}^{(m)} = \prod_{j=1}^{N_m} dx_j^{(m)}$, $1 \le m \le M$.

$$\Psi_\xi^{\boldsymbol{t}}[\boldsymbol{f}] = \operatorname*{Det}_{\substack{(s,t)\in\{t_1,\dots,t_M\}^2,\\ (x,y)\in\mathbb{R}^2}} \left[\delta_{st}\delta_x(\{y\}) + \mathbb{K}_\xi(s,x;t,y)\chi_t(y)\right],$$

$$\mathbb{K}_\xi(s,x;t,y) = \int_{\mathbb{R}} \xi(dv) p(s,x|v) \mathcal{M}_\xi^v(t,y) - 1_{(s>t)} p(s-t,x|y).$$

2.3 Extension to Include Multiple Points in Initial Configuration

- For general $\xi = \sum_{i=1}^{N} \delta_{u_i} \in \mathfrak{M}$ with $\xi(\mathbb{R}) = N < \infty$, define $\operatorname{supp} \xi = \{x \in \mathbb{R} : \xi(x) > 0\}$ and let $\xi_*(\cdot) = \sum_{v \in \operatorname{supp} \xi} \delta_v(\cdot)$.

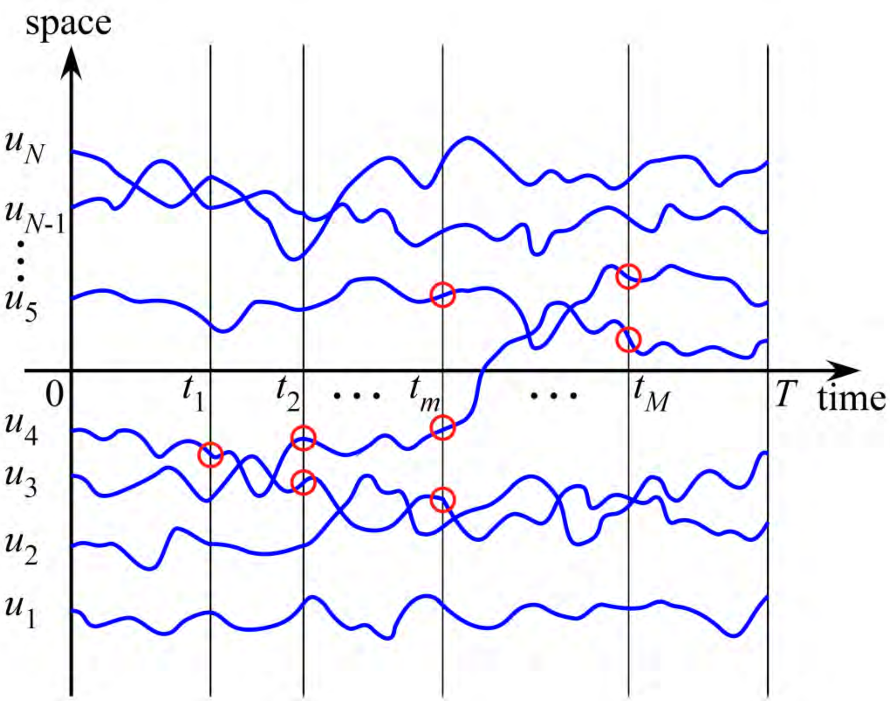
- For $s \in [0, \infty)$, $v, x \in \mathbb{R}$, $z, \zeta \in \mathbb{C}$, let $\phi_\xi^v((s,x); z, \zeta) = \dfrac{p(s,x|\zeta)}{p(s,x|v)} \dfrac{1}{z-\zeta} \displaystyle\prod_{i=1}^{N} \dfrac{z - x_i}{\zeta - x_i}$,

  and
$$\Phi_\xi^v((s,x); z) = \frac{1}{2\pi\sqrt{-1}} \oint_{C(\delta_v)} d\zeta\, \phi_\xi^v((s,x); z, \zeta),$$
  where $C(\delta_v)$ is a closed contour on the complex plane $\mathbb{C}$ encircling a point v on $\mathbb{R}$ once in the positive direction.

- Define
$$\mathcal{M}_\xi^u((s,x)|(t,y)) = \mathcal{I}\left[\Phi_\xi^u((s,x); W) \middle| (t,y)\right], \quad (s,x),(t,y) \in [0,\infty) \times \mathbb{R}.$$

- Then it is easy to see that the previous $\mathbb{K}_\xi$ is rewritten as
$$\mathbb{K}_\xi(s,x;t,y) = \int_{\mathbb{R}} \xi_*(dv) p(s,x|v) \mathcal{M}_\xi^v((s,x)|(t,y)) - \mathbf{1}(s>t) p(s-t,x|y),$$
  $(s,x),(t,y) \in [0,\infty) \times \mathbb{R}$. This expression is valid for general $\xi = \sum_{i=1}^{N} \delta_{u_i} \in \mathfrak{M}$ with $\xi(\mathbb{R}) = N < \infty$

2.4 Extension to Infinite Particle Systems



Observables depend on N' paths with $N' < N = \xi(\mathbb{R})$

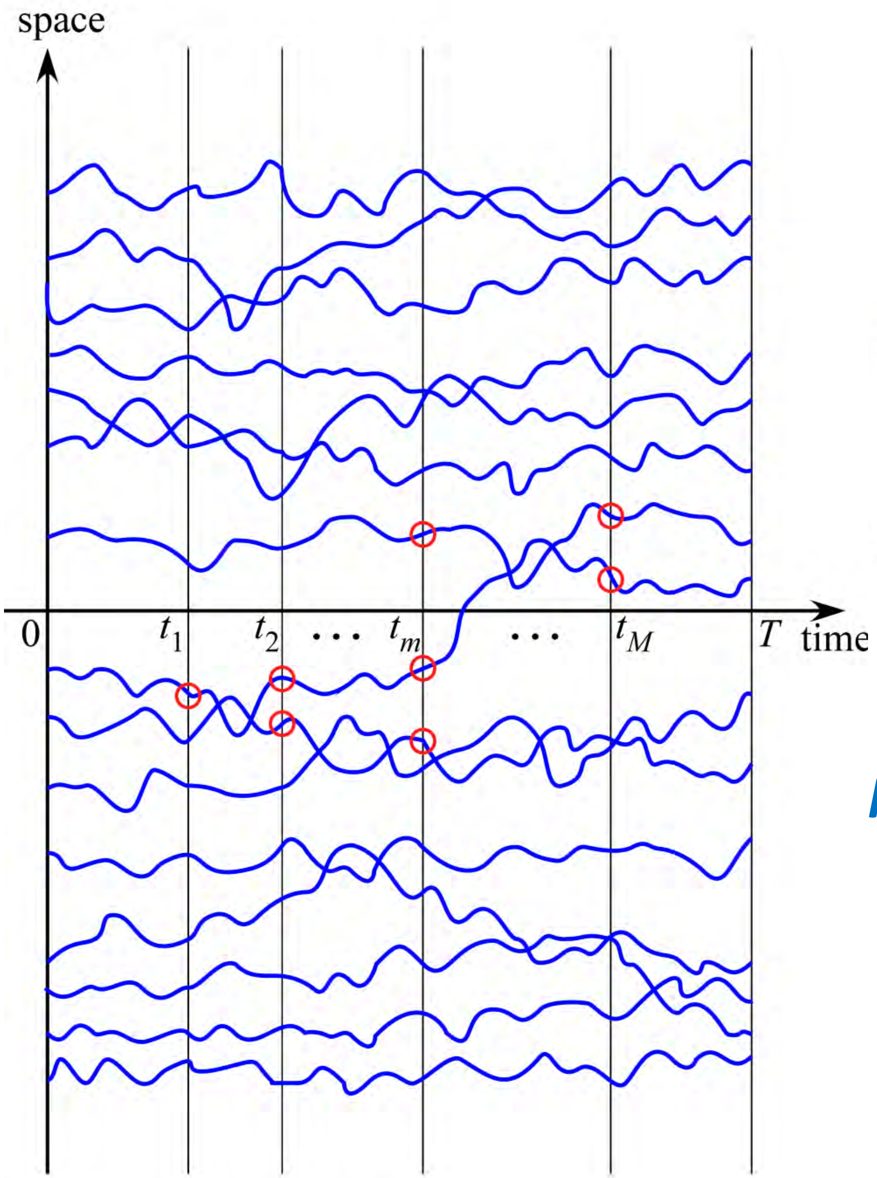
$$\mathcal{D}_\xi(T, \mathbf{B}(T)) \to \mathcal{D}_\xi(T, \mathbf{B}_{N'}(T))$$

with $N' < N = \xi(\mathbb{R})$

***N'* is fixed**

***N* increases**

2.4 Extension to Infinite Particle Systems



Observables depend on N' paths with $N' < N = \xi(\mathbb{R})$

$$\mathcal{D}_\xi(T, \mathbf{B}(T)) \to \mathcal{D}_\xi(T, \mathbf{B}_{N'}(T))$$
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***N'* is fixed**

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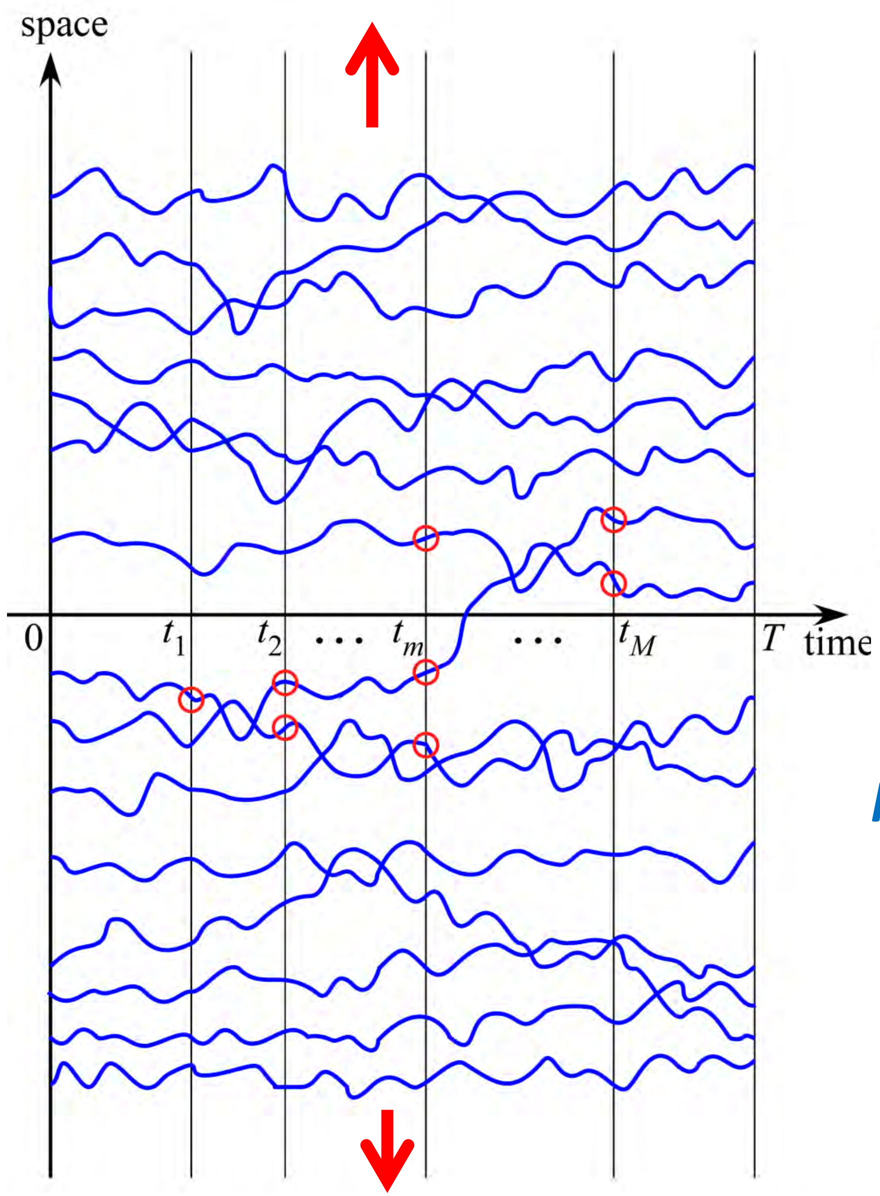
2.4 Extension to Infinite Particle Systems



Observables depend on N' paths with $N' < N = \xi(\mathbb{R})$

$$\mathcal{D}_\xi(T, \mathbf{B}(T)) \to \mathcal{D}_\xi(T, \mathbf{B}_{N'}(T))$$

with $N' < N = \xi(\mathbb{R})$

***N'* is fixed**

***N* increases to infinity**

- If the observables depend on at most N' paths, the determinantal martingale

$$\mathcal{D}_\xi(T, \mathbf{B}_{N'}(T)) = \det_{1 \leq j,k \leq N'} [\mathcal{M}_\xi^{u_k}(T, B_j(T))]$$

is a determinant of an $N' \times N'$ matrix.

- The element of the matrix is given by the 'martingale functions'

$$\mathcal{M}_\xi^{u_k}(t, x) = \mathcal{I}[\Phi_\xi^{u_k}(W)|(t, x)], \quad k \in \{1, 2, \dots, N\}.$$

- They are the integral transforms of the functions

$$\Phi_\xi^{u_k}(z) = \prod_{\substack{1 \leq \ell \leq N, \\ \ell \neq k}} \frac{z - u_\ell}{u_k - u_\ell}, \quad k \in \{1, 2, \dots, N\}.$$

- The functions are polynomials of z of order $N - 1$.
  The zeros are given by $\{u_\ell\}_{\ell=1}^N \setminus \{u_k\}, k \in \{1, 2, \dots, N\}$.

- If the observables depend on at most N' paths, the determinantal martingale

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  The zeros are given by $\{u_\ell\}_{\ell=1}^N \setminus \{u_k\}, k \in \{1, 2, \dots, N\}$.

$N \to \infty$

$$\Phi_\xi^{u_k}(z) = \prod_{\substack{\ell \in \mathbb{Z}, \\ \ell \ne k}} \frac{z - u_\ell}{u_k - u_\ell}, \quad k \in \mathbb{Z}.$$

the Weierstrass canonical products of entire functions (with genus $p = 0$)

3. Summary and Future Problems

3.1 From `Fluctuation-Correlation Theorem' to `Fluctuation-Response (Dissipation) Theorem'

$$\mathbb{K}_\xi(s,x;t,y) = \int_{\mathbb{R}} \xi(dv) p(s,x|v) \mathcal{M}_\xi^v(t,y) - \mathbf{1}_{(s>t)} p(s-t,x|y)$$

Correlation
(Correlation Kernel)

Fluctuation
(Martingale Function)

Future Problems

- Consider some **Response Functions** to some **Perturbations**.
- The response functions may be expressed by correlation functions.
- Then the above will give **'Fluctuation-Response (Dissipation)' Theorem** for the log-gases.

3.2 Log-Gases and Theory of Entire Functions

- **Weierstrass primary factors**

$$G(x,p) = \begin{cases} 1-x, & \text{if } p=0, \\ (1-x)\exp\left[x + \dfrac{x^2}{2} + \cdots + \dfrac{x^p}{p}\right], & \text{if } p \in \mathbb{N}. \end{cases}$$

- **Hadamard theorem**

Let $f(z)$ be an entire function. Its zeros are given by $\{u_\ell, \ell \in \mathbb{I}\}$ ($\mathbb{I}$ is an index set.)

$$\text{order:} \quad \rho = \limsup_{r\to\infty} \frac{\log\log \max_{|z|=r} |f(z)|}{\log r}, \quad (|f(z)| \sim e^{r^\rho} \quad \text{as } r \to \infty),$$

$$\text{convergence exponent:} \quad \text{for any } \alpha > \rho, \quad \sum_{\ell\in\mathbb{I}} \frac{1}{|u_\ell|^\alpha} < \infty.$$

Then

$$f(z) = z^m e^{P_q(z)} \prod_{\ell\in\mathbb{I}, u_\ell \neq 0} G\left(\frac{z}{u_\ell}, p\right),$$

where $p \leq \rho$, $P_q(z)$ is a polynomial in z of order $q \leq \rho$, m is the mutiplicity of the zero at the origin.

- genus $p=0 \Longrightarrow$ the Dyson model (noncolliding BM)

$$f(z)=f_\xi(z)=\prod_{\ell\in\mathbb{Z}}\left(1-\frac{z}{u_\ell}\right), \quad \xi=\sum_{\ell\in\mathbb{Z}}\delta_{u_\ell},$$

$$\Phi_\xi^{u_k}(z)=\frac{1}{z-u_k}\frac{f_\xi(z)}{f_\xi'(u_k)}=\prod_{\ell\in\mathbb{Z},\ell\neq k}\frac{z-u_\ell}{u_k-u_\ell}.$$

- genus $p=1 \Longrightarrow$ the Airy process ($\mathrm{Ai}(z)$ has the order $\rho=3/2$)

$$f(z)=f_\xi(z)=e^{c_0+c_1 z}\prod_{\ell\in\mathbb{N}}\left[\left(1-\frac{z}{u_\ell}\right)e^{z/u_\ell}\right],$$

$$\Phi_\xi^{u_k}(z)=\frac{1}{z-u_k}\frac{f_\xi(z)}{f_\xi'(u_k)}=e^{c_1(z-u_k)}\exp\left[\sum_{\ell\in\mathbb{N}}\frac{z-u_k}{u_\ell}\right]\prod_{\ell\in\mathbb{N},\ell\neq k}\frac{z-u_\ell}{u_k-u_\ell}.$$

$$X_j(t)=u_j+W_j(t)+\frac{t^2}{4}+\sum_{k\in\mathbb{N},k\neq j}\int_0^t\frac{ds}{X_j(s)-X_k(s)}+\left(c_1+\sum_{\ell\in\mathbb{I}}\frac{1}{u_\ell}\right)t, \quad j\in\mathbb{N}, \quad t\geq 0,$$

where $W_j(t), j\in\mathbb{N}$ are independent BMs.

[K-Tanemura09] M. Katori, H. Tanemura, Zeros of Airy function and relaxation process, J. Stat. Phys. **136**, 1177-1204 (2009).

- The Fredholm determinantal formula of Borodin and Corwin for the O'Connell process (the Whittaker measure).

[BC14] A. Borodin, I. Corwin, Macdonald processes, Probab. Theory Relat. Fields **158**, 225-400 (2014).

This result is related to the following entire function (a 'geometric lifting' of the previous $\Phi_\xi^{u_k}(z)$ with $\xi(\mathbb{R}) = N < \infty$)

$$\Phi_\xi^{u_k}(z) = \Gamma(1 - a(u_k - z)) \prod_{\substack{1 \leq \ell \leq N, \\ \ell \neq k}} \frac{\Gamma(a(u_\ell - u_k))}{\Gamma(a(u_\ell - z))}, \quad a > 0.$$

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{\ell \in \mathbb{N}} \left[\left(1 + \frac{z}{\ell}\right) e^{-z/\ell} \right].$$

[K12] M. Katori, System of complex Brownian motions associated with the O'Connell process, J. Stat. Phys. **149**, 411-431 (2012).

- Elliptic extension of the Dyson model.

$$\Phi_\xi^{u_k}(z) = \frac{\vartheta_1((\overline{u}_\delta + z - u_k)/2\pi r; iNt_*/2\pi r^2)}{\vartheta_1(\overline{u}_\delta/2\pi r; iNt_*/2\pi r^2)} \prod_{\substack{1 \leq \ell \leq N, \\ \ell \neq k}} \frac{\vartheta_1((z - u_\ell)/2\pi r; iNt_*/2\pi r^2)}{\vartheta_1((u_k - u_\ell)/2\pi r; iNt_*/2\pi r^2)}.$$

$$\vartheta_1(v; \tau) = -iq^{1/4} q_0 z \prod_{\ell \in \mathbb{N}} (1 - q^{2\ell} z^2)(1 - q^{2\ell-2}/z^2), \quad z = e^{\pi i v}, \quad q = e^{\pi i \tau}.$$

[K13] M. Katori, Elliptic determinantal process of type A, arXiv:math.PR/1311.4146.

3.3 Toward an Abstract Theory of Determinantal Processes

$$\mathcal{M}_{\xi}^{u_k}(t, B(t)), k \in \mathbb{I} : \text{martingales}$$

$$\begin{aligned} \mathcal{M}(t) &= \text{continuous martingale} \\ &= B(\langle \mathcal{M}, \mathcal{M} \rangle_t) \quad \text{(a time change of BM)} \end{aligned}$$

DDS (Dambis, Dubins-Schwarz) Brownian motion

$$\left. \begin{array}{ll} \text{BMs } B_j(t), & j \in \mathbb{I} \\ \text{time changes } \tau_k(t), & k \in \mathbb{I} \end{array} \right\} \Rightarrow \mathcal{D}(t) = \det_{j,k \in \mathbb{I}} [B_j(\tau_k(t))].$$