The non-perturbative renormalization group approach to KPZ

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Kyoto, August 2014

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Why NPRG?

Many exact results for KPZ in $d = 1 \Rightarrow$ RG is not needed.

But not so many results for KPZ in $d > 1$!

We know that there exists a phase transition between a smooth and a rough phase for $d > 2$, but...

- Is there an upper critical dimension d_c (meaning of d_c?)
- Can we explain "generic" scaling in the rough phase?
- Can we compute the critical exponents and the correlation function (and the probability distribution)?

$\Rightarrow$ need a versatile and reliable method

$\Rightarrow$ RG is the method of choice...

... but perturbative RG is known to fail in the rough phase for $d > 1$...

$\Rightarrow$ non-perturbative RG

Field theory for KPZ

(NP)RG works on correlation (and response) functions

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Derive from KPZ equation a generating function(al) $\mathcal{Z}$ of correlation functions in terms of a functional integral

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Introduce a response field $\tilde{h}$ that allows us to enforce the equation of motion and that takes care of the fluctuations induced by the noise term

$$\partial_t h = \nu \nabla^2 h + \frac{\lambda}{2} (\nabla h)^2 + \eta \iff \mathcal{Z} = \int \mathcal{D}[h, i\tilde{h}]\, e^{-\mathcal{S}[h,\tilde{h}] + \int (jh + \tilde{j}\tilde{h})}$$

$$\mathcal{S}[h, \tilde{h}] = \int d^d x\, dt\, \left\{ \tilde{h} \left[\partial_t h - \nu\, \nabla^2 h - \frac{\lambda}{2} (\nabla h)^2 \right] - D\, \tilde{h}^2 \right\}$$

$$\langle h(x_1, t_1) \dots \tilde{h}(x_{n+p}, t_{n+p}) \rangle = \frac{1}{\mathcal{Z}[j, \tilde{j}]} \frac{\delta^{n+p} \mathcal{Z}[j, \tilde{j}]}{\delta j(x_1, t_1) \dots \delta \tilde{j}(x_{n+p}, t_{n+p})}$$

Perturbative RG before non-perturbative RG

Perturbation theory I: treat the non linear term

$$\mathcal{S}[h,\tilde{h}] = \int d^d x\, dt\, \left\{ \tilde{h} \left[\partial_t h - \nu \nabla^2 h - \frac{\lambda}{2} (\nabla h)^2 \right] - D\, \tilde{h}^2 \right\}$$

as a perturbation and expand (around Edwards-Wilkinson).

Perturbation theory II: use the Cole-Hopf formulation and expand the non gaussian term.

In the two cases, the rough phase is unreachable (for $d > 1$)

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The recourse to other methods is unavoidable

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Other methods

- Study of discrete models, (Tang *et al.* 1992, E. Marinari *et al.* 2012, Kelling and Ódor, PRE 2011);

- Direct integration, (Miranda and Reis 2008);

- Real space RG (Castellano *et al.* 1998-99);

- Perturbative FRG (Le Doussal and Wiese, PRE 72,2005);

- Mode-Coupling Theory, (Frey, Täuber and Hwa, PRE 1996, Colaiori and MoorePRL2001);

- Self-Consistent Expansion, (Schwartz and Edwards 1992, Schwartz and Katzav 2008).

And, of course, numerical simulations and experiments!

The non-perturbative RG for the Ising model

$$\mathcal{Z} = \int \mathcal{D}\phi(x)\; e^{-H[\phi]+\int_x J\phi}$$

with

$$H[\phi] = \int d^d x \left(\frac{1}{2}(\partial_x \phi)^2 + \frac{1}{2} r\phi^2 + g\phi^4 \right)$$

We want to compute:

→ Helmoltz free energy (up to a $-kT$ factor): $\mathcal{W}[J] = \ln \mathcal{Z}[J]$

→ Gibbs free energy (Legendre transform):

$$\Gamma[M] + \mathcal{W}[J] = \int_x J(x)M(x) \quad \text{with} \quad M(x) = \langle \phi(x) \rangle = \frac{\delta \mathcal{W}[J]}{\delta J(x)}$$

Perturbation expansion = expansion of $\exp(-g \int \phi^4)$

Wilson's idea:

Organize the summation over the fluctuations in a different way.

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Block-spins *à la* Kadanoff-Wilson

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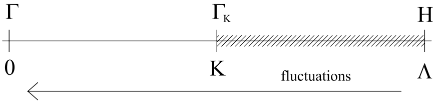

$\Downarrow$

Summation over rapid modes $\rightarrow$ effective hamiltonian for the slow modes

$$\text{rapid modes} = \phi_>(q) = \phi(q > k)$$
$$\text{slow modes} = \phi_<(q) = \phi(q < k)$$

$$\mathcal{Z} = \int \mathcal{D}\phi_<(x)\, \mathcal{D}\phi_>(x)\; e^{-H[\phi_<,\phi_>] + \int_x J(\phi_< + \phi_>)}$$

$$\mathcal{Z} = \int \mathcal{D}\phi_<(x)\; e^{-H_k[\phi_<] + \int_x J\phi_<}$$

$\Downarrow$

Flow equations of functions (or even functionals)

Integration over the "rapid" modes: The modern way

Idea: deform the model.

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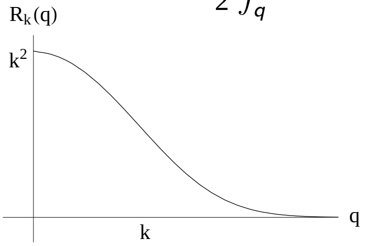
Build a one-parameter family of models, indexed by a scale k.

Integrate over the rapid modes only $\rightarrow$ freeze the slow modes $\rightarrow$ make them non-critical $\rightarrow$ give them a "large mass"

$$\mathcal{Z}[J] \rightarrow \mathcal{Z}_k[J] = \int D\phi \; \exp\left\{ - H[\phi] - \Delta H_k[\phi] + \int_x J(x)\phi(x) \right\}$$

$$\Delta H_k[\phi] = \frac{1}{2} \int_q R_k(q)\, \phi(q)\phi(-q)$$



The one-parameter family of models

Define:

- $\mathcal{Z}_k[J] = \int D\phi \;\exp\big\{ - H[\phi] - \Delta H_k[\phi] + \int_x J(x)\phi(x)\big\}$
- $\mathcal{W}_k[J] = \ln \mathcal{Z}_k[J]$
- $\Gamma_k[M] + \mathcal{W}_k[J] = \int_x J_x M_x - \frac{1}{2}\int_q R_k(q) M_q M_{-q}$

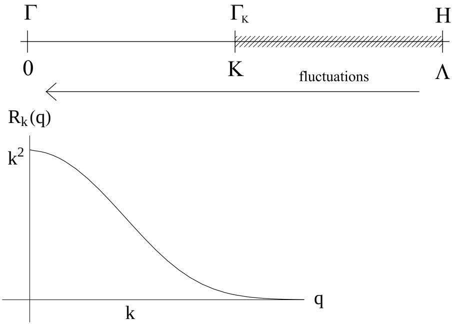
$\rightarrow$ when $k = \Lambda$ all fluctuations are frozen $\Rightarrow$ mean field is exact:
$\forall q, \quad R_{k=\Lambda}(q) \sim \Lambda^2, \quad \Rightarrow \quad \Gamma^{\mathrm{Leg}}_{k=\Lambda} = H + \Delta H_{k=\Lambda}$

$$\Rightarrow \quad \text{work with } \; \Gamma_k[M] = \Gamma^{\mathrm{Leg}}_k[M] - \Delta H_k[M]$$

$$\Rightarrow \quad \Gamma_{k=\Lambda}[M] = H[M]$$

$\rightarrow$ when $k = 0$ all fluctuations are integrated out and the original model is retrieved
$\forall q, \quad R_{k=0}(q) = 0, \quad \Rightarrow \quad \mathcal{Z}_{k=0}[J] = \mathcal{Z}[J]$ and $\Gamma_{k=0} = \Gamma$

To summarize:



$$\mathcal{Z}_k[J] = \int D\phi \;\exp\Big\{ - H[\phi] - \Delta H_k[\phi] + \int_x J(x)\phi(x) \Big\}$$

$$\left\{ \begin{array}{l} R_{k=\Lambda}(q) \sim \Lambda^2 \quad (\text{or } \infty) \\ \\ R_{k=0}(q) = 0 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} \Gamma_{k=\Lambda}[M] = H[\phi = M] \\ \\ \Gamma_{k=0}[M] = \Gamma[M] \end{array} \right. \tag{1}$$

then $\Gamma_{k=\Lambda}[M]$ interpolates between the microphysics at $k = \Lambda$ and the macrophysics at $k = 0$.

Exact flow equation: Wetterich's equation

The flow equation for $\Gamma_k[M]$ writes:

$$\partial_k \Gamma_k[M] = \frac{1}{2} \int_q \partial_k R_k(q) G_k[q; M] \qquad (2)$$

where $G_k[q; M]$ is the full 2-point function (propagator):
$G_k[q; M] = (\Gamma_k^{(2)} + R_k)^{-1}$ with $\Gamma_k^{(2)}[q; M] = \frac{\delta^2 \Gamma_k[M]}{\delta M(q) \delta M(-q)}$

Some properties of the Wetterich's equation:
– differential formulation of field theory
– involves only one integral
– the initial condition is the (microscopic) bare theory
– good properties of decoupling of the massive and rapid modes
– starting point of non-perturbative approximation schemes (not linked to an expansion in a coupling constant)

BUT
– leads to very few exact results;
– difficult to implement for gauge theories in high energy physics.

Two approximation schemes:

• The derivative expansion:

$$\Gamma_k[M] = \int d^dx \Big\{ U_k\big(M(x)\big) + \tfrac{1}{2} Z_k\big(M(x)\big)(\nabla M)^2 + \dots \Big\}$$

→ extremely accurate critical exponents, calculation of non-universal quantities, work for equilibrium and out of equilibrium systems, but... not appropriate for KPZ.

• The Blaizot-Mendez-Wschebor (BMW) approximation: flow of the two-point function $\Gamma_k^{(2)}(p)$ and approximation on $\Gamma_k^{(3)}$ and $\Gamma_k^{(4)}$

→ extremely accurate determination of the two-point function, but... impossible to implement for KPZ because of the symmetries.

Symmetries of the KPZ field theory

$$\mathcal{Z}[j,\tilde{j}] = \int \mathcal{D}[h, i\tilde{h}]\, e^{-\mathcal{S}[h,\tilde{h}]+\int(jh+\tilde{j}\tilde{h})}$$
$$\mathcal{S}[h,\tilde{h}] = \int d^d x\, dt\, \left\{\tilde{h}\left[\partial_t h - \nu\,\nabla^2 h - \tfrac{\lambda}{2}\,(\nabla h)^2\right] - D\,\tilde{h}^2\right\}$$

gauged shift symmetry:

$$h(t,\vec{x}) \to h(t,\vec{x}) + c(t) \;\Rightarrow\; \Gamma^{(1,1)}(\omega, \vec{p}=0) = i\omega$$

gauged Galilean symmetry (infinitesimal)

$$\begin{cases} h(t,\vec{x}) & \to\; \vec{x}\cdot\partial_t\vec{v}(t) + h(t,\vec{x}+\lambda\vec{v}(t)) \\ \tilde{h}(t,\vec{x}) & \to\; \tilde{h}(t,\vec{x}+\lambda\vec{v}(t)) \end{cases}$$

$$\Downarrow$$

$$i\omega\partial_{\vec{p}}\Gamma^{(2,1)}(\omega,\vec{p}=\vec{0};\omega_1,\vec{p}_1) = \lambda\vec{p}_1\Big(\Gamma^{(1,1)}(\omega+\omega_1,\vec{p}_1) - \Gamma^{(1,1)}(\omega_1,\vec{p}_1)\Big)$$

time reversal symmetry in $d=1$

$$\begin{cases} h(t,\vec{x}) & \to -h(-t,\vec{x}) \\ \tilde{h}(t,\vec{x}) & \to \tilde{h}(-t,\vec{x}) + \frac{\nu}{D}\nabla^2 h(-t,\vec{x}) \end{cases} \;\Rightarrow 2\,\mathrm{Re}\Gamma_\kappa^{(1,1)} = -\frac{\nu}{D}p^2\Gamma_\kappa^{(0,2)}$$

The quest for a symmetry-preserving scheme

→ Find a "geometric interpretation" of the Galilean symmetry:

Definition: $f(\vec{x})$ is a scalar if $\int d^d x\, f(\vec{x})$ is Galilean invariant

$$\Longrightarrow \begin{cases} \tilde{h}\ ,\ \nabla^2 h \ \rightarrow \ \text{scalars} \\ h\ ,\ \partial_t h \ \rightarrow \ \text{not scalars} \end{cases} \tag{3}$$

Analogy with fluid mechanics: introduce covariant time derivatives

$$\tilde{D}_t \equiv \partial_t - \lambda \nabla h \cdot \nabla \quad , \quad D_t h \equiv \partial_t h - \frac{\lambda}{2}(\nabla h)^2$$

Building blocks of a (gauged) Galilean invariant quantity:
three scalars: $\tilde{h}$, $\nabla_i \nabla_j h$, $D_t h$ with two operators $\tilde{D}_t$, ∇.

For instance: $\mathcal{S} = \int_{x,t} \left\{ \tilde{h}\Big(D_t h - \nu\, \nabla^2 h\Big) - D\, \tilde{h}^2 \right\}$

Computing the two-point functions $\Gamma^{(0,2)}(\omega,p)$ and $\Gamma^{(1,1)}(\omega,p)$

Define $\psi(t,\vec{x}) = \langle h(t,\vec{x})\rangle$ and $\tilde{\psi}(t,\vec{x}) = \langle \tilde{h}(t,\vec{x})\rangle$
$\Rightarrow \Gamma_k = \Gamma_k[\psi(t,\vec{x}),\tilde{\psi}(t,\vec{x})]$

Propose an ansatz for Γ_k consisting of an expansion at second order in the response field $\tilde{\psi}$:

$$\Gamma_k^{\mathrm{ans}}[\psi,\tilde{\psi}] = \int_{t,\vec{x}} \left\{ \tilde{\psi} f_k^{\lambda} D_t \psi - \frac{1}{2}\left[\nabla^2\psi f_k^{\nu}\tilde{\psi} + \tilde{\psi} f_k^{\nu}\nabla^2\psi\right] - \tilde{\psi} f_k^{D}\tilde{\psi} \right\}$$

with $f_k^X =$ three arbitrary functions : $f_k^X \equiv f_k^X(-\tilde{D}_t^2, -\nabla^2)$.

$$\begin{aligned}
\Gamma_k^{(2,0)}(\omega,\vec{p}) &= 0,\\
\Gamma_k^{(1,1)}(\omega,\vec{p}) &= i\omega\, f_k^{\lambda}\left(\omega^2,\vec{p}^{\,2}\right) + \vec{p}^{\,2}\, f_k^{\nu}(\omega^2,\vec{p}^{\,2}),\\
\Gamma_k^{(0,2)}(\omega,\vec{p}) &= -2\, f_k^{D}(\omega^2,\vec{p}^{\,2}).
\end{aligned}$$

This is the most general form of $\Gamma_k^{(1,1)}$ and $\Gamma_k^{(0,2)}$ compatible with the symmetries.
Infinitely many other $\Gamma^{(n,1)}$ and $\Gamma^{(n,2)}$ are in the ansatz to preserve all the symmetries.

Integration of the RG flow: I

We look for scale invariance $\Rightarrow$ fixed point of the RG flow $\Rightarrow$ we must work with dimensionless renormalized quantities

$$\hat{f}_k^D(\hat{\varpi}^2, \hat{p}^2) = f_k^D(\omega^2, p^2)/D_k \qquad \hat{p} = p/k$$
$$\hat{f}_k^\nu(\hat{\varpi}^2, \hat{p}^2) = f_k^\nu(\omega^2, p^2)/\nu_k \qquad \hat{\varpi} = \omega/(D_k k^2)$$
$$\hat{f}_k^\lambda(\hat{\varpi}^2, \hat{p}^2) = f_k^\lambda(\omega^2, p^2)$$

$\rightarrow$ two (running) "anomalous dimensions"

$$\eta^D(k) = -k\partial_k \ln D_k \ , \ \eta^\nu(k) = -k\partial_k \ln \nu_k$$

from which follows the two critical exponents:

$$z = 2 - \eta_\nu^* \ , \ \chi = (2 - d + \eta_D^* - \eta_\nu^*)/2$$

$\rightarrow$ one dimensionless coupling $\hat{g}_k = \lambda^2 D_k / {\nu_k}^3 k^{d-2}$ whose flow is:

$$k\partial_k \hat{g}_k = \hat{g}_k(d - 2 + 3\eta^\nu(k) - \eta^D(k))$$

Thus, for a fixed point with $g^* \neq 0 \Rightarrow$ $z + \chi = 2$.

Integration of the RG flow: II

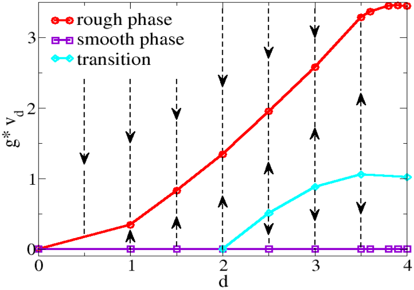
• In $d = 1$, "time-reversal" symmetry $\Rightarrow$ $f_k^D(\omega^2, p^2) = f_k^\nu(\omega^2, p^2)$ and $f_k^\lambda(\omega^2, p^2) = 1 \Rightarrow$ only one independent function (called $f = f_k^D = f_k^\nu$

• In $d > 1$, we make a further approximation on the three functions (NLO): $f_k^X(\omega^2, p^2) \to f_k^X(p^2)$ (on the r.h.s. of the flow equations)

• Initial condition of the RG flow, $k = \Lambda$:
$\Gamma_\Lambda[\psi, \tilde{\psi}] = \mathcal{S}[h = \psi, \tilde{h} = \tilde{\psi}] \Rightarrow f_\Lambda^D = D,\ f_\Lambda^\nu = \nu,\ f_\Lambda^\lambda = 1.$

But we could (must?) take generic initial conditions ! (Role of the irrelevant operators?)

Results



d	χ NLO	χ num.
1	1/2	1/2
2	0.373	0.384
3	0.180	0.304

See results by T. Halpin-Healy in this conference for many results of χ in $2+1$ dimensions ($0.380 \leq \chi \leq 0.389$).

Results in $d = 1$: fixed point and scaling

Only one function left $\begin{cases} \nu_k = D_k & \hat{f}_k^\nu = \hat{f}_k^D \equiv \hat{f}_k(\omega^2, p^2) \\ \hat{f}_k^\lambda = 1 & \eta_k^\nu = \eta_k^D \equiv \eta_k \end{cases}$

Dimensionless flow equations

$$\begin{aligned} k\partial_k \hat{f}_k(\hat{\varpi}, \hat{p}) &= \eta_k \hat{f}_k + \hat{p}\, \partial_{\hat{p}} \hat{f}_k + (2 - \eta_k)\hat{\varpi}\, \partial_{\hat{\varpi}} \hat{f}_k + I_k(\hat{\varpi}, \hat{p}) \\ k\partial_k \hat{g}_k &= \hat{g}_k(2\eta_k - 1) \end{aligned}$$

- There exists a fixed point: $(\hat{f}_*(\hat{\varpi}, \hat{p}), \hat{g}^*)$
- When $k \to 0$ at fixed p or ω $\hat{p} = p/k$ and/or $\hat{\varpi} = \omega/(k^2 D_k) \gg 1$

$$\hat{f}_*(\hat{\varpi}, \hat{p}) = \frac{1}{\hat{p}^{1/2}} \hat{\zeta} \left(\frac{\hat{\varpi}}{\hat{p}^{3/2}} \right)$$

Results in $d = 1$: Comparison with exact scaling functions

$$C(\varpi, p) = -\frac{\Gamma^{(0,2)}(\varpi, p)}{|\Gamma^{(1,1)}(\varpi, p)|^2} = \frac{2}{p^{7/2}} \frac{\hat{\zeta}\left(\frac{\hat{\varpi}}{\hat{p}^{3/2}}\right)}{\frac{\hat{\varpi}^2}{\hat{p}^3} + \hat{\zeta}^2\left(\frac{\hat{\varpi}}{\hat{p}^{3/2}}\right)}$$

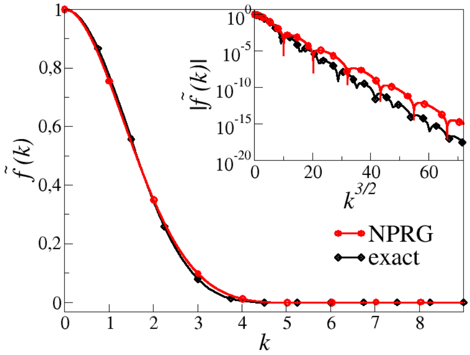
$$\equiv \frac{2}{D_0 p^{7/2}} \mathring{F}\left(\frac{\varpi}{D_0 p^{3/2}}\right) \qquad D_k = D_0 k^{-\eta_*^D}$$

Normalisations: Scaling function g defined by

$$C(t, x) = \alpha\, t^{2/3} g(\beta\, x / t^{3/2})$$

with arbitrary constants α and β fixed by comparison with Prähofer and Spohn, J. Stat. Phys., 115, (2004). They define three functions:

$$\begin{aligned} f(y) &= \; g''(y)/4 \\ \tilde{f}(k) &= \; 2\int_0^\infty dy\, \cos(ky) f(y) \\ \mathring{f}(\tau) &= \; 2\int_0^\infty dk\, \cos(k\tau) \tilde{f}(k^{2/3}) \end{aligned}$$



Asymtotic behavior:

$$\tilde{f} \sim \cos(a_0 k^{3/2}) e^{-b_0 k^{3/2}} \quad \text{for}\, k \to \infty$$

	a_0	b_0
NPRG	0.28(5)	0.49(1)
exact	1/2	1/2

Universal amplitude ratio:

$$g_0 = 2\Gamma(1/3)/\pi^2 \int_0^\infty d\tau \, \tau^{2/3} \, \mathring{f}(\tau)$$

	g_0
exact	1.15039
NPRG	1.19(1)

In d dimensions:

$$C(t,x) = x^{2\chi} F(t/x^z) \;\Rightarrow\; \text{with } F(y) = \begin{cases} F_0 & y \to 0 \\ F_\infty y^{2\chi/z} & y \to \infty \end{cases}$$

then

$$R = \frac{F_\infty}{F_0^{2/z} \lambda^{2\chi/z}}$$

KCW find in 2+1 dimensions $R = 0.940(2)$.

Conclusions

• KPZ with long-range correlated noise :

$$\langle \eta(t,x)\eta(t',x')\rangle = 2D(x-x')\delta(t-t')$$

with $D(p) = D(1 + wp^{2\rho})$ has been studied by us with NPRG. Very rich structure, highly non trivial!

• Anisotropic KPZ has also been studied by Kloss, anet and Wschebor (not yet published).

BUT... much remains to be done in 2+1 3+1 (and beyond):

• Improve the approximation (but time consuming): critical exponents, amplitude ratios, existence of an upper critical dimension

• Compute the height PDF in 2+1

• Study the Cole-Hopf version of KPZ

• Study the (stochastic) Navier-Stokes equation (paper in preparation)