RG derivation of relativistic fluid dynamic equations for a viscous fluid

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Introduction

- Relativistic hydrodynamics for a perfect fluid is widely and successfully used in the RHIC phenomenology. T. Hirano, D.Teaney, …

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- A growing interest in dissipative hydrodynamics.

  hadron corona (rarefied states); Hirano et al …

  Generically, an analysis using dissipative hydrodynamics is needed even to show the dissipative effects are small.

  A.Muronga and D. Rischke; A. K. Chaudhuri and U. Heinz,; R. Baier, P. Romatschke and U. A. Wiedemann; R. Baier and P. Romatschke (2007) and the references cited in the last paper.

However,

is the theory of relativistic hydrodynamics for a viscous fluid fully established?

The answer is

No!

unfortunately.

Fundamental problems with relativistic hydro-dynamical equations for viscous fluid

a. Ambiguities in the form of the equation, even in the same frame and equally derived from Boltzmann equation: Landau frame; unique, Eckart frame; Eckart eq. v.s. Grad-Marle-Stewart eq.; Muronga v.s. R. Baier et al

b. Instability of the equilibrium state in the eq.'s in the Eckart frame, which affects even the solutions of the causal equations, say, by Israel-Stewart. W. A. Hiscock and L. Lindblom ('85, '87); R. Baier et al ('06, '07)

c. Usual 1st-order equations are acausal as the diffusion eq. is, except for Israel-Stewart and those based on the extended thermodynamics with relaxation times, but the form of causal equations is still controversial.

---- The purpose of the present talk ---

For analyzing the problems a and b first,
we derive hydrodynaical equations for a viscous fluid from Boltzmann equation on the basis of a mechanical reduction theory (so called the RG method) and a natural ansatz on the origin of dissipation.
We also show that the new equation in the Eckart frame is stable.
We emphasize that the definition of the flow and the physical nature of the respective local rest frame is not trivial as is taken for granted in the literature, which is also true even in the second-order equations.

Contents

The basics of rel. fluid dynamics

References

- D. H. Rischke, nucl-th/9809044
- P. Romatschke, arXiv:0902.3636v3[hep-ph]
- J. M. Stewart, ``*Non-Equilibrium Relativistic Kinetic Theory*", Lecture Notes in Physics 10 (Springer-Verlag), 1971
- S. R. de Groot, W.A. van Leeuwen and Ch. G. van Weert, ``*Relativistic Kinetic Theory*", North-Holland (1980)
- C. Cercignani and G. M. Kremer, ``The Relativistic Boltzmann Equation: Theory and Applications", PMP22, (Birkhaeuser, 2002)

Special relativity

$$x^{\mu} = (ct = x^{0}, \mathbf{x})$$

$$ds^{2} = g_{\mu\nu} dx^{\mu} dx^{\nu} \qquad g^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$$

$$\gamma = \frac{1}{\sqrt{1-\frac{v^{2}}{c^{2}}}}$$

Linear transformations of space-time which keep any inner products with this metric tensor are Lorentz transformation.

$$\mathbf{p} = \gamma m \mathbf{v}$$

$$E_{p} = \sqrt{(pc)^{2} + (mc^{2})^{2}} = cp^{0} = \gamma mc^{2}$$

$$\gamma_{v'} d^{3}x' = \gamma_{v} d^{3}x = d^{3}x_{0}$$

$$\frac{d^{3}p'}{E_{p'}} = \frac{d^{3}p}{E_{p}} \longleftrightarrow \frac{d^{3}p'}{\gamma'} = \frac{d^{3}p}{\gamma} \Longrightarrow d^{3}x'd^{3}p' = d^{3}xd^{3}p$$

$$1 = \int f(\mathbf{x}, \mathbf{p}, t) d^{3}xd^{3}p$$

The distribution function is Lorentz-invariant!

Basics

1. The fluid dynamic equations as conservation (balance) equations

$$\partial_\mu N_i^\mu \equiv 0\ ,\quad i = 1, \ldots, n\ ,$$ local conservation of charges

$$\partial_\mu T^{\mu\nu} \equiv 0\ ,\quad \nu = 0, \ldots, 3\ .$$ local conservation of energy-mom.

2.Tensor decomposition and choice of frame

u^μ ; **arbitrary** normalized **time-like** vector $u \cdot u = 1$

Def. space-like projection $\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu\ ,\ \Delta^{\mu\nu} u_\nu = 0\ ,\ \Delta^{\mu\alpha}\Delta_\alpha^\nu = \Delta^{\mu\nu}$

space-like vector

space-like traceless tensor

$$N_i^\mu = n_i\, u^\mu + \nu_i^\mu\ ,$$

$$T^{\mu\nu} = \epsilon\, u^\mu u^\nu - p\, \Delta^{\mu\nu} + q^\mu u^\nu + q^\nu u^\mu + \pi^{\mu\nu}$$

$n_i \equiv N_i \cdot u$; net density of charge *i* in the **Local Rest Frame**

$\nu_i^\mu \equiv \Delta_\nu^\mu N_i^\nu$; net flow in LRF

$\epsilon \equiv u_\mu T^{\mu\nu} u_\nu$; energy density in LRF $p \equiv -\frac{1}{3} T^{\mu\nu}\Delta_{\mu\nu}$; isotropic pressure in LRF

$q^\mu \equiv \Delta^{\mu\alpha} T_{\alpha\beta} u^\beta$; heat flow in LRF

$\pi^{\mu\nu} \equiv \left[\frac{1}{2}\left(\Delta_\alpha^\mu \Delta_\beta^\nu + \Delta_\beta^\mu \Delta_\alpha^\nu\right) - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}\right] T^{\alpha\beta}$; stress tensor in LRF

Define u^{μ} so that it has a physical meaning.

A. Particle frame (Eckart frame)

space-like

$$u_{\mathrm{E}}^{\mu} \equiv \frac{N_i^{\mu}}{\sqrt{N_i \cdot N_i}}$$; parallel to particle current of i $\longrightarrow$ $0 = N_i^{\mu} \Delta_{\mu\nu} = \nu_i^{\mu}$

B. Energy frame (Landau-Lifshitz frame)

$$u_{\mathrm{L}}^{\mu} \equiv \frac{T_{\nu}^{\mu} u_{\mathrm{L}}^{\nu}}{\sqrt{u_{\mathrm{L}}^{\alpha} T_{\alpha}^{\beta} T_{\beta\gamma} u_{\mathrm{L}}^{\beta}}}$$; flow of the energy-momentum density

$\longrightarrow$ $q^{\mu} = 0$

$$T_{\nu}^{\mu} u^{\nu} = \varepsilon u^{\mu} + q^{\mu}$$

$$N_i^{\mu} = n_i\, u^{\mu} + \nu_i^{\mu} \quad ,$$
$$T^{\mu\nu} = \epsilon\, u^{\mu} u^{\nu} - p\, \Delta^{\mu\nu} + q^{\mu} u^{\nu} + q^{\nu} u^{\mu} + \pi^{\mu\nu}$$

Ideal fluid dynamics

P. Romatschke, arXiv:0902.3636v3[hep-ph]

$$T^{\mu\nu}_{(0)} = \epsilon \left(c_0 g^{\mu\nu} + c_1 u^\mu u^\nu \right) + p \left(c_2 g^{\mu\nu} + c_3 u^\mu u^\nu \right)$$

Constraints: $T^{00}_{(0)} = \varepsilon$ energy density in LRF

$$T^{ij}_{(0)} = p\delta^{ij}$$

$$\downarrow$$

$$T^{\mu\nu}_{(0)} = \epsilon \, u^\mu u^\nu - p \, \Delta^{\mu\nu}$$

Typical hydrodynamic equations for a viscous fluid
--- Choice of the frame and ambiguities in the form ---

Fluid dynamics = a system of balance equations

$$\partial_\mu T^{\mu\nu} = 0, \qquad \partial_\mu N^\mu = 0.$$

energy-momentum : $T^{\mu\nu}$ number : N^μ

$$T^{\mu\nu} \equiv \epsilon u^\mu u^\nu - p\Delta^{\mu\nu} + \delta T^{\mu\nu} \qquad N^\mu \equiv n u^\mu + \delta N^\mu$$

Dissipative part

Eckart eq. no dissipation in the number flow; ⇒ Describing the flow of matter

$$\delta T^{\mu\nu} = u^\mu T\lambda\left(\frac{1}{T}\nabla^\nu T - Du^\nu\right) + u^\nu T\lambda\left(\frac{1}{T}\nabla^\mu T - Du^\mu\right) + 2\eta\frac{1}{2}\left(\nabla^\mu u^\nu + \nabla^\nu u^\mu - \frac{2}{3}\Delta^{\mu\nu}\nabla\cdot u\right) + \zeta\Delta^{\mu\nu}\nabla\cdot u$$

$$\delta N^\mu = 0.$$

with $D \equiv u^\mu \partial_\mu$, $\nabla^\mu \equiv \Delta^{\mu\nu}\partial_\nu$, $\Delta_p^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu \equiv \Delta^{\mu\nu}$,

--- Involving time-like derivative ---.

Landau-Lifshits no dissipation in energy flow ⇒ describing the energy flow.

$$\delta T^{\mu\nu} = 2\eta\frac{1}{2}\left(\nabla^\mu u^\nu + \nabla^\nu u^\mu - \frac{2}{3}\Delta^{\mu\nu}\nabla\cdot u\right) + \zeta\Delta^{\mu\nu}\nabla\cdot u$$

$$\delta N^\mu = -\lambda\frac{nT}{\epsilon + p}\left(\frac{1}{T}\nabla^\mu T - \frac{1}{\epsilon + p}\nabla^\mu p\right)$$

⇔

$\delta T^{\mu\nu}u_\nu = 0$, No dissipative energy-density nor energy-flow

$u_\mu \delta N^\mu = 0$ No dissipative particle density

--- Involving only space-like derivatives ---

with transport coefficients:

ζ ; Bulk viscocity, η; Shear viscocity

λ ;Heat conductivity

The explicit form of Eckart equation

The dissipative part of the energy-momentum tensor may be determined from the local form of the second law of thermodynamics.

put $$S^\mu = s\, u^\mu + \beta\, q^\mu$$

$$T\partial \cdot S = (T\beta - 1)\partial \cdot q + q \cdot (\dot{u} + T\partial\beta) + \pi^{\mu\nu}\,\partial_\mu u_\nu + \Pi\,\theta \geq 0$$

$u_\nu \partial_\mu T^{\mu\nu} = 0 \quad \partial \cdot N = 0$

$$\beta \equiv 1/T \ ,$$
$$\Pi \equiv \zeta\,\theta \ ,$$
$$q^\mu \equiv \kappa\, T\, \Delta^{\mu\nu}\,(\partial_\nu \ln T - \dot{u}_\nu) \ ,$$
$$\pi^{\mu\nu} \equiv 2\,\eta \left[\frac{1}{2}\left(\Delta^\mu_\alpha \Delta^\nu_\beta + \Delta^\mu_\beta \Delta^\nu_\alpha\right) - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}\right] \partial^\alpha u^\beta$$

Then

$$\partial \cdot S = \frac{\Pi^2}{\zeta T} - \frac{q \cdot q}{\kappa\, T^2} + \frac{\pi^{\mu\nu}\pi_{\mu\nu}}{2\,\eta\, T} \geq 0$$

Non-relativistic limit

Y.Minami, T.K., K.Tsumura(2010);
frame-independence

See also, Landau-Lifshitz.

$$\epsilon \to \rho c^2$$

ρ ;the mass density

$$T^{00} \sim c^2\rho \qquad T^{i0} \sim c\rho v^i \qquad T^{ij} \sim \rho v^i v^j - P g^{ij} + \eta\left(\partial^i v^j + \partial^j v^i - \frac{2}{3} g^{ij}\nabla\cdot \boldsymbol{v}\right) + \zeta g^{ij}\nabla\cdot\boldsymbol{v}.$$

$$\partial_\mu T^{\mu j} = 0 \longrightarrow \frac{\partial(\rho\boldsymbol{v})}{\partial t} + \nabla(\rho\boldsymbol{v}\boldsymbol{v}) = -\nabla P + \eta\nabla^2\boldsymbol{v} + \left(\zeta + \frac{1}{3}\eta\right)\nabla(\nabla\cdot\boldsymbol{v}).$$

(Navier-Stokes eq.)

$$u_\mu\partial_\nu T^{\mu\nu} = 0 \longrightarrow \frac{\partial s}{\partial t} + \nabla\cdot(s\boldsymbol{v} + \boldsymbol{J}_s^D) = 2\frac{\eta}{T}([\nabla\boldsymbol{v}]^s)^2 + \frac{\zeta}{T}(\nabla\cdot\boldsymbol{v})^2 + \frac{\kappa}{T^2}(\nabla T)^2$$

$$\mu = (w - TS)/n$$

$$d\left(\frac{w}{n}\right) = Td\left(\frac{S}{n}\right) + \frac{dP}{n}$$

$$w = \varepsilon + P$$

Enthalpy density

$$\partial_\mu N^\mu = 0 \longrightarrow \frac{\partial n}{\partial t} + \nabla\cdot(n\vec{v}) = 0$$

Acausality problem

P. Romatschke, arXiv:0902.3636v3[hep-ph]

Fluctuations around the equilibrium:

$$\epsilon = \epsilon_0 + \delta\epsilon(t,x) \qquad u^\mu = (1,\vec{0}) + \delta u^\mu(t,x)$$

Linearized equation;

$$(\epsilon + p)Du^y - \nabla^y p + \Delta^y_\nu \partial_\mu \Pi^{\mu\nu} = (\epsilon_0 + p_0)\partial_t \delta u^y + \partial_x \Pi^{xy}$$

$$\Pi^{xy} = \eta\left(\nabla^x u^y + \nabla^y u^x\right) + \left(\zeta - \frac{2}{3}\eta\right)\Delta^{xy}\nabla_\alpha u^\alpha = -\eta_0\, \partial_x \delta u^y$$

$$\Downarrow$$

$$\partial_t \delta u^y - \frac{\eta_0}{\epsilon_0 + p_0}\partial_x^2 \delta u^y = 0$$

Diffusion equation!

The signal runs with an infinite speed.

Non-local thermodynamics (Maxwell-Cattaneo) → Mueller-Israel-Stewart

P. Romatschke, arXiv:0902.3636v3[hep-ph]

$$\tau_\pi \partial_t \Pi^{xy} + \Pi^{xy} = -\eta_0 \partial_x \delta u^y$$

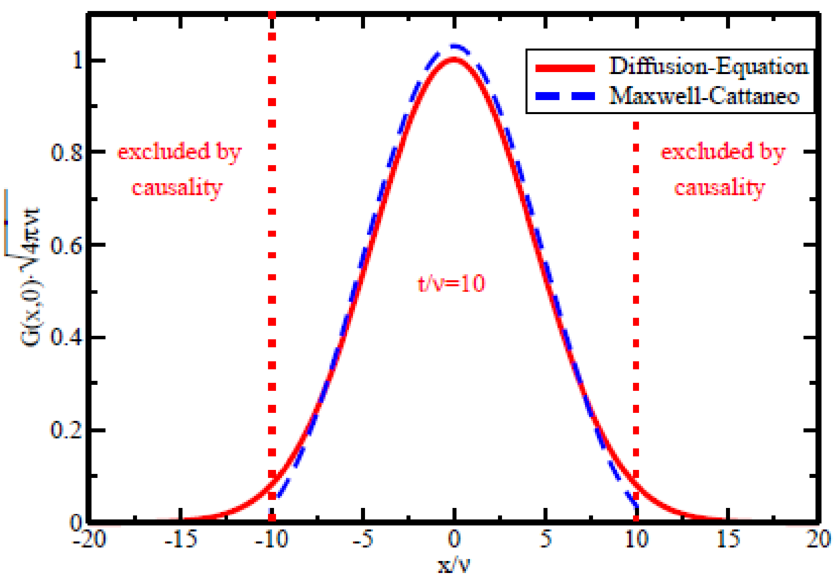
↓ Telegrapher's equation

$$\partial_t \delta u^y + \frac{1}{\epsilon_0 + p_0} \partial_x \pi^{xy} = 0, \quad \tau_\pi \partial_t \pi^{xy} + \pi^{xy} = -\eta_0 \partial_x \delta u^y$$

$$\left[\partial_t^2 + \frac{\partial_t}{\tau_\pi} - \frac{\nu}{\tau_\pi}\partial_x^2\right] G(\mathbf{x}, \mathbf{x}') = \frac{1}{\tau_\pi}\delta^2(\mathbf{x} - \mathbf{x}')$$

$$G(\mathbf{x}, \mathbf{x}') = \theta(t - t')\,\theta\left(\frac{(t-t')^2\nu}{\tau_\pi} - (x - x')^2\right)\frac{e^{-\frac{t-t'}{2\tau_\pi}}}{\sqrt{4\nu\tau_\pi}} I_0\left(\sqrt{\frac{(t-t')^2}{4\tau_\pi^2} - \frac{(x-x')^2}{4\nu\tau_\pi}}\right)$$

Diffusion Eq. vs. Maxwell-Cattaneo



Compatibility of the definition of the flow and the LRF

In the kinetic approach, one needs a matching condition.

Seemingly plausible ansatz are;

$$\epsilon \equiv u_\mu T^{\mu\nu} u_\nu = \epsilon_0 \equiv u_\mu T_0^{\mu\nu} u_\nu$$
$$n \equiv u \cdot N = n_0 \equiv u \cdot N_0$$

Is this always correct, irrespective of the frames?

Particle frame is the same local equilibrium state as the energy frame?

Note that the entropy density S(x) and the pressure P(x) etc can be quite Different from those in the equilibrium.

Eg. $\exists$ the bulk viscosity

Local equilibrium $\longrightarrow$ No dissipation!

D. H. Rischke, nucl-th/9809044

Distribution function in LRF:

$$f_0(k,x) = \frac{g}{(2\pi)^3}\left[\exp\{y_0(k,x)\} \pm 1\right]^{-1} \qquad y_0(k,x) \equiv [k \cdot u(x) - \mu(x)]/T(x)$$

Non-local distribution function;

$$f(k,x) \equiv \frac{g}{(2\pi)^3}\left[\exp\{y(k,x)\} \pm 1\right]^{-1}$$

$$y(k,x) \simeq y_0(k,x) + \varepsilon_1(x) + k \cdot \varepsilon_2(x) + k_\mu k_\nu\, \varepsilon_3^{\mu\nu}(x)$$

The problem of causality:

$$C_v \partial T / \partial t = -\partial q / \partial x$$

Fourier's law; $$q = -\lambda \partial T / \partial x$$

Then $$C_v \partial T / \partial t = \lambda \nabla^2 T$$

Causality is broken; the signal propagate with an infinite speed.

Modification; $$\tau_q \frac{\partial}{\partial t} q(t, x) + q(t, x) = -\lambda \frac{\partial}{\partial x} T(t, x)$$

Extended thermodynamics

Nonlocal thermodynamics $$q(t, x) = \int \mathrm{d}s \left[\theta(t-s) \frac{1}{\tau_q} \mathrm{e}^{-\frac{1}{\tau_q}(t-s)} \lambda\right] \left[-\frac{\partial}{\partial x} T(s, x)\right]$$

Memory effects; i.e., non-Markovian

Derivation(Israel-Stewart): Grad's 14-moments method

+ ansats so that Landau/Eckart eq.'s are derived.

Problematic

The problems:

- Foundation of Grad's 14 moments method
- ad-hoc constraints on $\delta T^{\mu\nu}$ and δN^{μ} consistent with the underlying dynamics?

The purpose of the present work:

(1) The renormalization group method is applied to derive rel. hydrodynamic equations as a construction of an invariant manifold of the Boltzmann equation as a dynamical system.

(2) Our generic equations include the Landau equation in the energy frame, but is different from the Eckart in the particle frame and stable, even in the first order.

(3) Apply dissipative rel. hydro. to obetain the spetral function of density fluctuations and discuss critical phenomena around QCD critical point.

The problem with the constraint in particle frame:

K. Tsumura, K.Ohnishi, T.K. Phys. Lett. B646 (2007) 134-140

$$T^{\mu\nu} = (\epsilon + 3\zeta\tilde{X})\,u^\mu u^\nu - (p + \zeta\tilde{X})\,\Delta^{\mu\nu} + \lambda T\, u^\mu \tilde{X}^\nu + \lambda T\, u^\nu \tilde{X}^\mu + 2\eta X^{\mu\nu}$$

$N^\mu = m\,n\,u^\mu$ with $\tilde{X} \equiv -\{1/3\,(4/3-\gamma)^{-1}\}^2\,\nabla\cdot u$

i.e., $\delta N^\mu = 0.$ $\tilde{X}^\mu \equiv \nabla^\mu \ln T$

$$\begin{cases} 5.\ \delta T^\mu{}_\mu = 0, \\ 2.\ u_\mu\,\delta N^\mu = 0, \\ 3.\ \Delta_{\mu\nu}\,\delta N^\nu = 0. \end{cases}$$

trivial (2., 3.)

Grad-Marle-Stewart constraints

c.f. $\delta T^{\mu\nu}u_\nu = 0,$

Landau

Eckart's constraints :

$$\begin{cases} 1.\ u_\mu u_\nu\,\delta T^{\mu\nu} = 0, \\ 2.\ u_\mu\,\delta N^\mu = 0, \\ 3.\ \Delta_{\mu\nu}\,\delta N^\nu = 0, \end{cases}$$

trivial (2., 3.)

still employed by I-S and Betz et al.

Phenomenological Derivation

$$\partial_\mu T^{\mu\nu} = 0, \qquad \partial_\mu J^\mu = 0,$$

$$\partial_\mu S^\mu \geq 0,$$

$$S^\mu = S^\mu(T^{\mu\nu},\, J^\mu).$$

$$u^\mu = J^\mu/(J^\nu\, J_\nu)^{\frac{1}{2}},$$ particle frame

$$u^\mu = T^{\mu\nu}\, u_\nu/(u_\rho T^{\rho\sigma} T_{\sigma\tau}\, u^\tau)^{\frac{1}{2}}$$ energy frame

Generic form of energy-momentum tensor and flow velocity:

$$\begin{aligned} T^{\mu\nu} &= (e+\delta e)\, u^\mu\, u^\nu - (p+\delta p)\, \Delta^{\mu\nu} + q^\mu\, u^\nu + q^\nu\, u^\mu + \pi^{\mu\nu}, \\ N^\mu &= (n+\delta n)\, u^\mu + \nu^\mu, \end{aligned}$$

with

$$\begin{aligned} e+\delta e &\equiv T_{ab}\, u^a\, u^b, \\ p+\delta p &\equiv -1/3\, T_{ab}\, \Delta^{ab}, \\ q^\mu &\equiv T_{ab}\, u^a\, \Delta^{b\mu}, \\ \pi^{\mu\nu} &\equiv T_{ab}\, \Delta^{ab\mu\nu}, \\ n+\delta n &\equiv N_a\, u^a, \\ \nu^\mu &\equiv N_a\, \Delta^{a\mu}, \end{aligned}$$

natural choice and parametrization

$$\begin{aligned} \delta e &= f_e\, \Pi, \\ \delta p &= \Pi, \\ \delta n &= f_n\, \Pi, \end{aligned}$$

Notice;

$$q^\mu\, u_\mu = 0,\ \nu^\mu\, u_\mu = 0,\ \text{and}\ u_\mu\, \pi^{\mu\nu} = \pi^{\mu\nu}\, u_\nu = \pi^\mu{}_\mu = 0.$$

From $$T\,S^\mu = p\,u^\mu + u_\nu\,T^{\mu\nu} - \mu\,N^\mu$$

$$\partial_\mu S^\mu = \Pi\left[f_e\,D\left(\frac{1}{T}\right) - \frac{1}{T}\nabla^\mu u_\mu - f_n\,D\left(\frac{\mu}{T}\right)\right]$$
$$+ q^\mu\left[\frac{1}{T}Du_\mu + \nabla_\mu\left(\frac{1}{T}\right)\right] - \nu^\mu\,\nabla_\mu\left(\frac{\mu}{T}\right) + \pi^{\mu\nu}\,\frac{1}{T}\nabla_\mu u_\nu$$

In particle frame;

$$\partial_\mu S^\mu = \Pi\left[f_e\,D\left(\frac{1}{T}\right) - \frac{1}{T}\nabla^\mu u_\mu - f_n\,D\left(\frac{\mu}{T}\right)\right] + q^\mu\left[\frac{1}{T}Du_\mu + \nabla_\mu\left(\frac{1}{T}\right)\right] + \pi^{\mu\nu}\,\frac{1}{T}\nabla_\mu u_\nu .$$

With the choice,

$$\Pi = \zeta T\left[f_e\,D\left(\frac{1}{T}\right) - \frac{1}{T}\nabla^\mu u_\mu - f_n\,D\left(\frac{\mu}{T}\right)\right],$$
$$q^\mu = -\lambda T^2\left[\frac{1}{T}Du^\mu + \nabla^\mu\left(\frac{1}{T}\right)\right],$$
$$\pi^{\mu\nu} = 2\,\eta\,\Delta^{\mu\nu\rho\sigma}\,\nabla_\rho u_\sigma,$$

we have

$$\partial_\mu S^\mu = \frac{\Pi^2}{\zeta T} - \frac{q^\mu q_\mu}{\lambda T^2} + \frac{\pi^{\mu\nu}\pi_{\mu\nu}}{2\eta T} \geq 0 \Rightarrow$$

f_e, f_n can be finite, not in contradiction with the fundamental laws!

$$\delta e = f_e \Pi, \quad \delta p = \Pi, \quad \delta n = f_n \Pi,$$

$$\Longleftrightarrow$$

$$u_\mu \delta T^{\mu\nu} u_\nu = \delta e = f_e \Pi$$

$$\delta T^\mu_\mu = \delta e - 3\delta p = (f_e - 3)\Pi.$$

Energy frame: $q^\mu = 0,$

$$\Pi = \zeta T \left[f_e D\left(\frac{1}{T}\right) - \frac{1}{T}\nabla^\mu u_\mu - f_n D\left(\frac{\mu}{T}\right) \right],$$

$$\pi^{\mu\nu} = 2\eta \Delta^{\mu\nu\rho\sigma} \nabla_\rho u_\sigma,$$

$$\nu^\mu = \lambda \left(\frac{nT}{e+p}\right)^2 \nabla^\mu \left(\frac{\mu}{T}\right),$$

coincide with the Landau equation with f_e=f_n=0.

Microscopic derivation gives the explicit form of f_e and f_n in each frame:

particle frame; $f_e = 3$ together with $f_n = 0$ $\longrightarrow$ $\begin{cases} \delta T^\mu_\mu = 0 \\ u_\mu \delta T^{\mu\nu} u_\nu = 3\Pi \neq 0 \end{cases}$

energy frame; $f_e = f_n = 0$

Relativistic Boltzmann equation

S.R. de Groot et al, ``Relativistic Kinetic Theory; Principles and Applications" (North-Holland, 1980),
C. Cercignani and G.M. Kremer, ``The Relativistic Boltzmann Equation: Theory and Applications" (Birkhaeuser, 2002)

For single classical gas,

$$p^\mu \, \partial_\mu \, f_p(x) = C[f]_p(x),$$

$$C[f]_p(x) \equiv \frac{1}{2!} \sum_{p_1} \frac{1}{p_1^0} \sum_{p_2} \frac{1}{p_2^0} \sum_{p_3} \frac{1}{p_3^0} \, \omega(p\,,\,p_1|p_2\,,\,p_3) \left(f_{p_2}(x) \, f_{p_3}(x) - f_p(x) \, f_{p_1}(x) \right),$$

where $\omega(p\,,\,p_1|p_2\,,\,p_3) \propto \delta^4(p + p_1 - p_2 - p_3)$

$$\omega(p\,,\,p_1|p_2\,,\,p_3) = \omega(p_2\,,\,p_3|p\,,\,p_1) = \omega(p_1\,,\,p|p_3\,,\,p_2) = \omega(p_3\,,\,p_2|p_1\,,\,p)$$

collision invariants: $\sum_p \frac{1}{p^0} \, C[f]_p(x) = \sum_p \frac{1}{p^0} \, p^\mu \, C[f]_p(x) = 0$

Conservation law of the particle number and the energy-momentum

$$S^\mu \equiv -\sum_p \frac{1}{p^0} \, p^\mu \, f_p(x) \, (\ln f_p(x) - 1)$$

H-theorem. → if $\ln f_p(x)$ is a linear combination of

The collision invariants, the system is local equilibrium

Maxwell distribution (N.R.)

Juettner distribution (Rel.)

The separation of scales in the relativistic heavy-ion collisions

Liouville ⇨ Boltzmann ⇨ Fluid dyn.

Hamiltonian

Navier-Stokes eq.

Slower dynamics （Reduction of dynamical Systems)

on the basis of the RG method; Chen-Goldenfeld-Oono('95),T.K.('95)

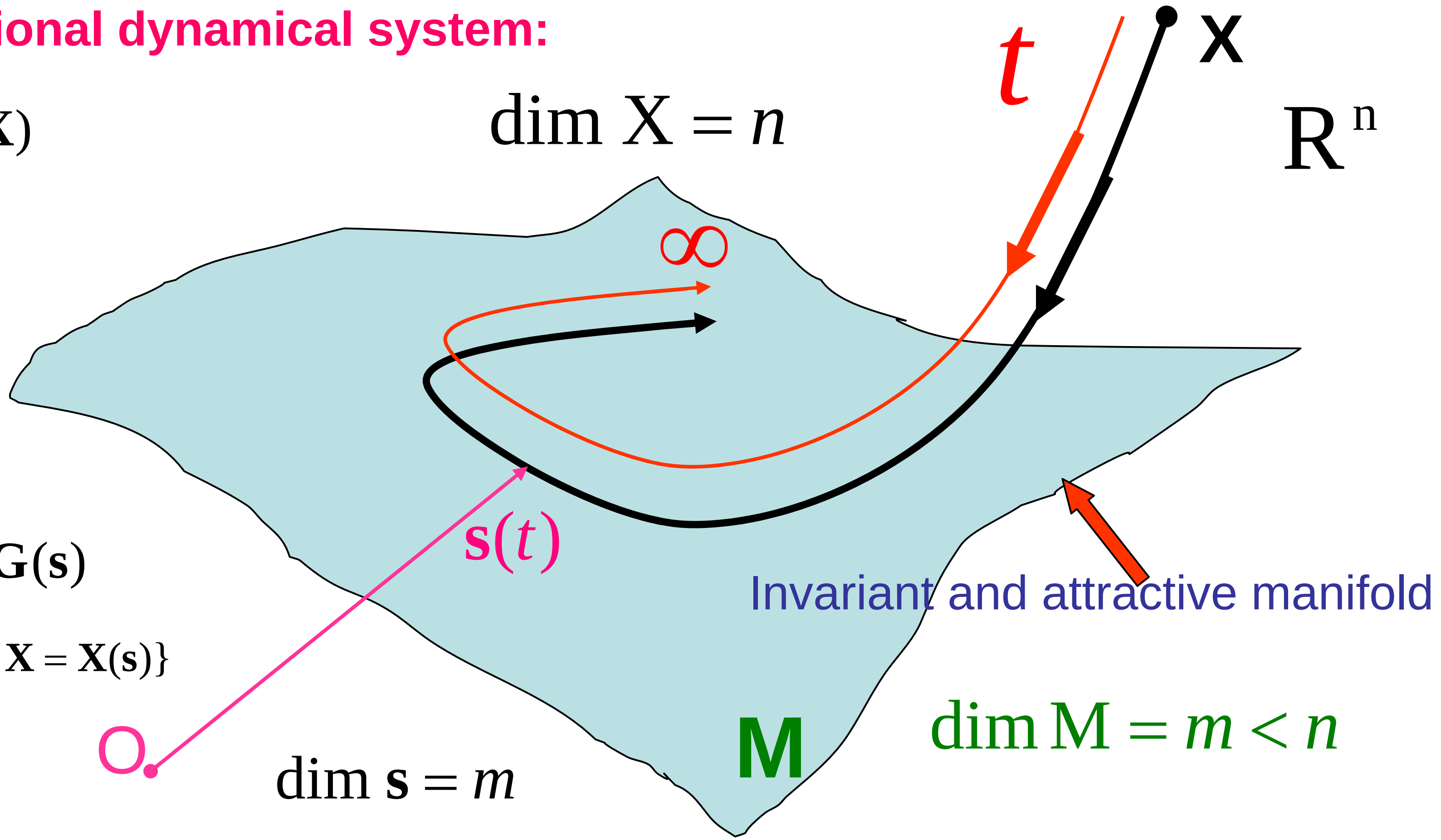
C.f. Y. Hatta and T.K. ('02) , K.Tsumura and TK ('05);
Tsumura, Ohnishi, T.K. ('07)

Geometrical image of reduction of dynamics

n-dimensional dynamical system:

$$\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X})$$

$$\begin{cases} \dfrac{d\mathbf{s}}{dt} = \mathbf{G}(\mathbf{s}) \\ \mathrm{M} = \{\mathbf{X} \mid \mathbf{X} = \mathbf{X}(\mathbf{s})\} \end{cases}$$



eg.

In Field theory, $\mathbf{X} = (g_1, g_2, \ldots, g_n) \equiv \boldsymbol{g} \rightarrow \boldsymbol{s} = (s_1, s_2, \ldots, s_m)$

renormalizable ⟺ $\exists$ **Invariant manifold M** dim M $= m < n \leq \infty$

For dynamical systems:

$$\frac{dX}{dt} = F(X,t),$$

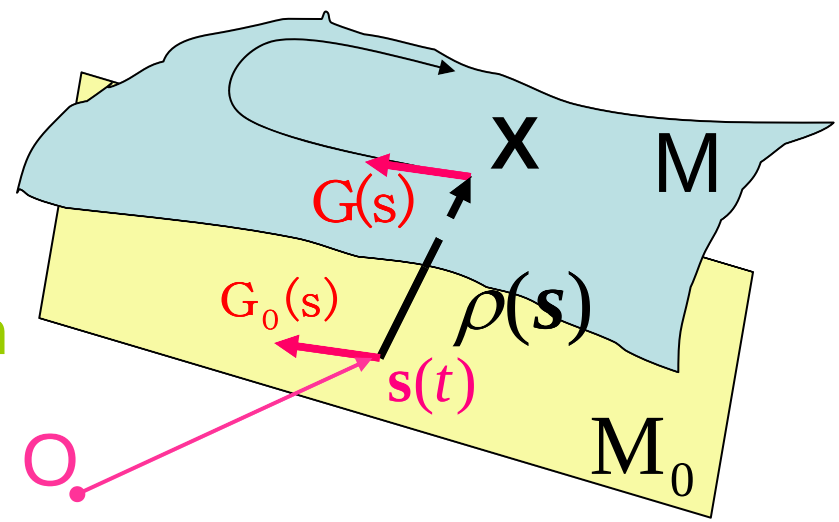
$$\downarrow$$

$$\frac{ds}{dt} = G(s),$$, reduced dynamics on M

$$X = R(s);$$, representation of M

Y.Kuramoto('89)

Y. Kuramoto ('89)



Perturbative reduction of dynamics

$$\frac{dX}{dt} = F(X,t),$$

$$\downarrow$$

$$\frac{ds}{dt} = G_0(s) + \gamma(s),$$, reduced dynamics on M

$$X = R_0(s) + \rho(s);$$, the invariant manifold M

Krylov-Bogoliubov-Mitropolosky

Geometrical image of perturbative reduction of dynamics

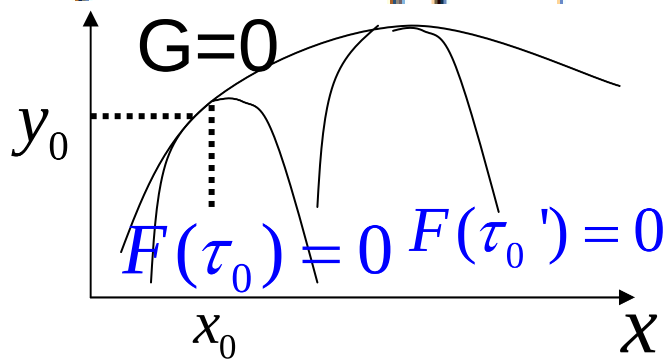
A geometrical interpretation:

T.K. ('95)

construction of the envelope of the perturbative solutions

Let $\{C_\tau\}_\tau$ be a family of curves parametrized by τ in the x-y plane;

$$C_\tau : F(x, y, \tau, \mathbf{C}(\tau)) = 0 \ ?$$



E: The envelope of C_τ $G(x, y) = 0.$

$$F_{\tau_0}(x_0, y_0, \tau_0, C(\tau_0)) \equiv \frac{\partial F(x_0, y_0, \tau_0)}{\partial \tau_0} + \frac{\partial C}{\partial \tau_0}\frac{\partial F(x_0, y_0, \tau_0, C(\tau_0))}{\partial C} = 0.$$

The envelop equation: $dF / d\tau_0 = 0$ ⟺ RG eq.

the solution is inserted to F with the condition

$\tau_0 = x_0$ ← the tangent point

⇒ $G(x, y) = F(x, y, \mathbf{C}(x))$

A simple example: resummation and extracting slowdynamics

T.K. ('95)

$$\frac{d^2x}{dt^2} + \epsilon\frac{dx}{dt} + x = 0,$$

the damped oscillator!

$$x(t) = \bar{A}\exp(-\frac{\epsilon}{2}t)\sin(\sqrt{1-\frac{\epsilon^2}{4}}t+\bar{\theta}),$$

$$x(t,t_0) = x_0(t,t_0) + \epsilon x_1(t,t_0) + \epsilon^2 x_2(t,t_0) + \ldots,$$

$$\ddot{x}_0 + x_0 = 0, \qquad \ddot{x}_{n+1} + x_{n+1} = -\dot{x}_n.$$

$$x(t_0,t_0) = W(t_0).$$

$$W(t_0) = W_0(t_0) + \epsilon W_1(t_0) + \epsilon^2 W_2(t_0) + \ldots,$$

$$x_0(t,t_0) = A(t_0)\sin(t+\theta(t_0)), \quad W_0(t_0) = x_0(t_0,t_0) = A(t_0)\sin(t_0+\theta(t_0)).$$

$$x_1(t,t_0) = -\frac{A}{2}\cdot(t-t_0)\sin(t+\theta), \qquad W_1(t_0) = 0$$

a secular term appears, invalidating P.T.

$$x_2(t) = \frac{A}{8}\{(t-t_0)^2 \sin(t+\theta) - (t-t_0)\cos(t+\theta)\}, \qquad W_2(t_0) = 0$$

Secular terms appear again!

Collecting the terms, we have

$$\boxed{\begin{aligned} x(t,t_0) = {} & A\sin(t+\theta) - \epsilon\frac{A}{2}(t-t_0)\sin(t+\theta) \\ & + \epsilon^2\frac{A}{8}\{(t-t_0)^2\sin(t+\theta) - (t-t_0)\cos(t+\theta)\} \end{aligned}}$$

With I.C.: $W(t_0) = W_0(t_0) = A(t_0)\sin(t_0 + \theta(t_0))$

; parameterized by the functions,

$A(t_0), \phi(t_0) \equiv t_0 + \theta(t_0)$

The secular terms invalidate the pert. theory, like the log-divergence in QFT!

$\{C_{t_0}\}_{t_0}$: $\quad \{x(t,t_0)\}_{t_0} \quad\quad x_E(t) = x(t,t) = W(t).$

$\frac{dx(t,t_0)}{dt_0} = 0, \quad t_0 = t. \quad \Longrightarrow \quad A(t_0)$ and $\theta(t_0)$

$$x_2(t) = \frac{A}{8}\{(t-t_0)^2 \sin(t+\theta) - (t-t_0)\cos(t+\theta)\}, \qquad W_2(t_0) = 0$$

Secular terms appear again!

Collecting the terms, we have

$$\boxed{\begin{aligned} x(t,t_0) &= A\sin(t+\theta) - \epsilon\frac{A}{2}(t-t_0)\sin(t+\theta) \\ &\quad + \epsilon^2\frac{A}{8}\{(t-t_0)^2\sin(t+\theta) - (t-t_0)\cos(t+\theta)\} \end{aligned}}$$

With I.C.: $W(t_0) = W_0(t_0) = A(t_0)\sin(t_0+\theta(t_0))$

; parameterized by the functions,

$A(t_0), \phi(t_0) \equiv t_0 + \theta(t_0)$

Let us try to construct the envelope function of the set of locally divergent functions, Parameterized by t_0 !

$\{C_{t_0}\}_{t_0}$: $\{x(t,t_0)\}_{t_0}$ $\quad x_E(t) = x(t,t) = W(t).$

$\frac{dx(t,t_0)}{dt_0} = 0$, $t_0 = t.$ $\Longrightarrow$ $A(t_0)$ and $\theta(t_0)$

$$\frac{dA}{dt_0} + \epsilon A = 0, \quad \frac{d\theta}{dt_0} + \frac{\epsilon^2}{8} = 0,$$

$$A(t_0) = \bar{A}\mathrm{e}^{-\epsilon t_0/2}, \quad \theta(t_0) = -\frac{\epsilon^2}{8}t_0 + \bar{\theta},$$

$$x_{_E}(t) = x(t,t) = W_0(t) = \bar{A}\exp(-\frac{\epsilon}{2}t)\sin((1-\frac{\epsilon^2}{8})t + \bar{\theta}),$$

$$\sqrt{1-\epsilon^2/4} = 1 - \epsilon^2/8 + O(\epsilon^4)$$

The envelop function $x_{_E}(t) = W_0(t)$ an approximate but

global solution in contrast to the pertubative solutions

which have secular terms and valid only in local domains.

c.f. Chen et al ('95)

Notice also the resummed nature!

More generic example

S.Ei, K. Fujii & T.K.('00)

$$\partial_t \mathbf{u} = A\mathbf{u} + \epsilon \mathbf{F}(\mathbf{u}), \quad |\epsilon| < 1$$

$$\mathbf{u}(t;t_0) = \mathbf{u}_0(t;t_0) + \epsilon \mathbf{u}_1(t;t_0) + \epsilon^2 \mathbf{u}_2(t;t_0) + \cdots$$

$$\begin{aligned} \mathbf{W}(t_0) &= \mathbf{W}_0(t_0) + \epsilon \mathbf{W}_1(t_0) + \epsilon^2 \mathbf{W}_2(t_0) + \cdots, \\ &= \mathbf{W}_0(t_0) + \boldsymbol{\rho}(t_0), \end{aligned}$$

$$\begin{aligned} (\partial_t - A)\mathbf{u}_0 &= 0, \\ (\partial_t - A)\mathbf{u}_1 &= \mathbf{F}(\mathbf{u}_0), \\ (\partial_t - A)\mathbf{u}_2 &= \mathbf{F}'(\mathbf{u}_0)\mathbf{u}_1, \end{aligned}$$

$$(\mathbf{F}'(\mathbf{u}_0)\mathbf{u}_1)_i = \sum_{j=1}^{n} \{\partial(F'(\mathbf{u}_0))_i / \partial(u_0)_j\} (u_1)_j$$

When A has semi-simple zero eigenvalues.

$$A\mathrm{U}_i = 0, \quad (i = 1, 2, \ldots, m).$$

We suppose that other eigenvalues have negative real parts;

$$A\mathrm{U}_\alpha = \lambda_\alpha \mathrm{U}_\alpha, \quad (\alpha = m+1, m+2, \cdots, n),$$

where $\mathrm{Re}\lambda_\alpha < 0$. One may assume without loss of generality that U_i's and U_α's are linearly independent.

The adjoint operator $A^\dagger$ has the same eigenvalues as A has;

$$A^\dagger \tilde{\mathrm{U}}_i = 0, \quad (i = 1, 2, \ldots, m),$$
$$A^\dagger \tilde{\mathrm{U}}_\alpha = \lambda^*_\alpha \tilde{\mathrm{U}}_\alpha, \quad (\alpha = m+1, m+2, \cdots, n).$$

Def. P **the projection onto the kernel** $\ker A$

$$P + Q = 1$$

Since we are interested in the asymptotic state as $t \to \infty$, we may assume that the lowest-order initial value belongs to $\ker A$:

$$\mathbf{W}_0(t_0) = \sum_{i=1}^{m} C_i(t_0)\mathbf{U}_i = \mathbf{W}_0[\boldsymbol{C}]. \quad \Longleftrightarrow \quad \mathrm{M}_0$$

$$\mathbf{u}_0(t;t_0) \;=\; \mathrm{e}^{(t-t_0)A}\mathbf{W}_0(t_0) \;=\; \textstyle\sum_{i=1}^{m} C_i(t_0)\mathbf{U}_i.$$

Parameterized with m variables,
Instead of n!

$$\boldsymbol{C} = {}^t(C_1, C_2, \cdots, C_m)$$

$$\mathbf{u}_1(t;t_0) = \mathrm{e}^{(t-t_0)A}[\mathbf{W}_1(t_0) + A^{-1}Q\mathbf{F}(\mathbf{W}_0(t_0))] \\ +(t-t_0)P\mathbf{F}(\mathbf{W}_0(t_0)) - A^{-1}Q\mathbf{F}(\mathbf{W}_0(t_0)).$$

The would-be rapidly changing terms can be eliminated by the choice;

$$\mathbf{W}_1(t_0) = -A^{-1}Q\mathbf{F}(\mathbf{W}_0(t_0)), \qquad P\mathbf{W}_1(t_0) = 0$$

Then, the secular term appears only the P space;

$$\mathbf{u}_1(t;t_0) = (t-t_0)P\mathbf{F} - A^{-1}Q\mathbf{F}$$

a deformation of the manifold ρ

Deformed (invariant) slow manifold: $M_1 = \{\mathbf{u} | \mathbf{u} = \mathbf{W}_0 - \epsilon A^{-1} Q \mathbf{F}(\mathbf{W}_0)\}$

$$\mathbf{u}(t; t_0) = \mathbf{W}_0 + \epsilon\{(t - t_0) P \mathbf{F} - A^{-1} Q \mathbf{F}\}$$

A set of locally divergent functions parameterized by t_0 !

The RG/E equation $\partial \mathbf{u} / \partial t_0 \big|_{t_0 = t} = \mathbf{0}$ gives the envelope, which is globally valid:

$$\dot{\mathbf{W}}_0(t) = \epsilon P \mathbf{F}(\mathbf{W}_0(t)),$$

which is reduced to an m-dimensional coupled equation,

$$\dot{C}_i(t) = \epsilon \langle \tilde{\mathbf{U}}_i, \mathbf{F}(\mathbf{W}_0[C]) \rangle, \quad (i = 1, 2, \cdots, m).$$

The global solution (the invariant manifod):

$$\mathbf{u}(t) = \mathbf{u}(t; t_0 = t) = \sum_{i=1}^{m} C_i(t) \mathbf{U}_i - \epsilon A^{-1} Q \mathbf{F}(\mathbf{W}_0[C]).$$

We have derived the invariant manifold and the **slow dynamics** on the manifold by the RG method.

Extension; (a) A Is not semi-simple. (2) Higher orders. (Ei, Fujii and T.K. Ann.Phys.('00))

Layered pulse dynamics for TDGL and NLS.

The RG/E equation $\left.\partial \mathbf{u} / \partial t_0\right|_{t_0=t} = \mathbf{0}$

gives the envelope, which is globally valid:

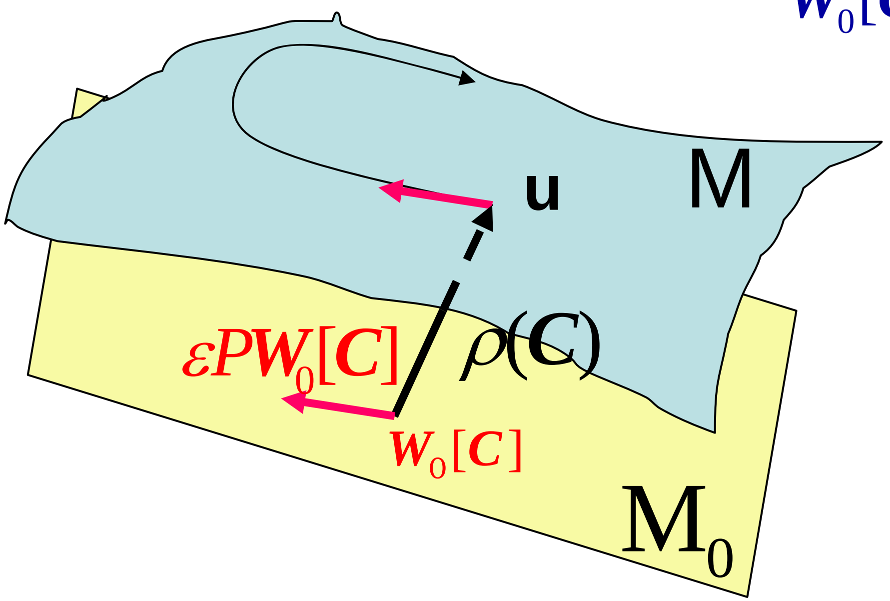
$$\dot{\mathbf{W}}_0(t) = \epsilon P \mathbf{F}(\mathbf{W}_0(t)),$$

which is reduced to an m-dimensional coupled equation,

$$\dot{C}_i(t) = \epsilon \langle \tilde{\mathrm{U}}_i, \mathbf{F}(\mathbf{W}_0[C]) \rangle, \quad (i = 1, 2, \cdots, m).$$

The global solution (the invariant manifod):

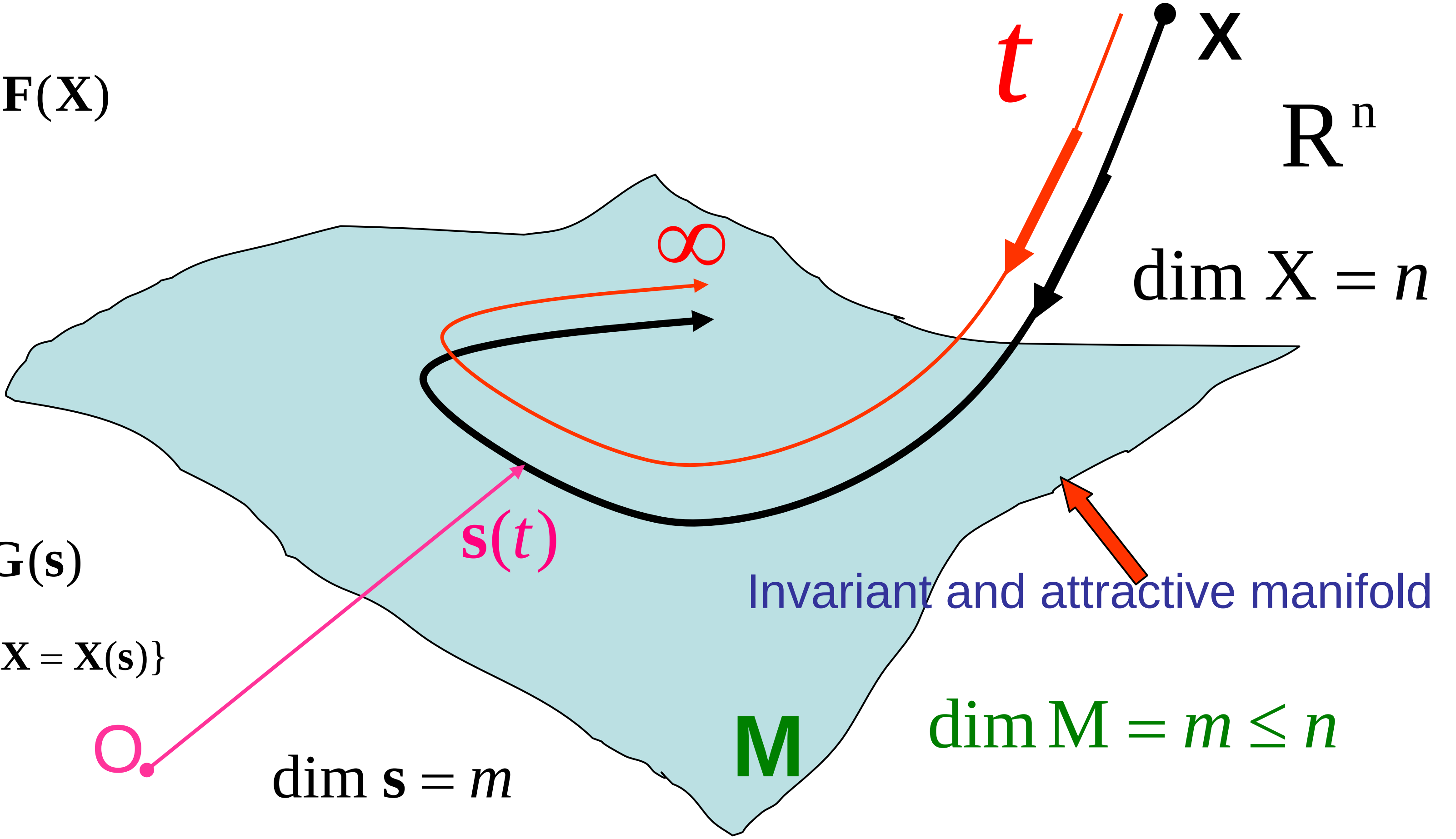
$$\mathbf{u}(t) = \mathbf{u}(t; t_0 = t) = \underbrace{\sum_{i=1}^{m} C_i(t) \mathrm{U}_i}_{W_0[C]} \underbrace{- \epsilon A^{-1} Q \mathbf{F}(\mathbf{W}_0[C])}_{\rho(C)}.$$



Geometrical image of reduction of dynamics

$$\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X})$$

$$\begin{cases} \dfrac{d\mathbf{s}}{dt} = \mathbf{G}(\mathbf{s}) \\ \mathrm{M}=\{\mathbf{X} \,|\, \mathbf{X} = \mathbf{X}(\mathbf{s})\} \end{cases}$$



eg.

$\mathbf{X} = f(\mathbf{r}, \mathbf{p})$; **distribution function in the phase space (infinite dimensions)**

$\mathbf{s} = \{u^{\mu}, T, n\}$; **the hydrodinamic quantities (5 dimensions), conserved quantities.**

Previous attempts to derive the dissipative hydrodynamics as a reduction of the dynamics

N.G. van Kampen, J. Stat. Phys. 46(1987), 709

unique but non-covariant form and hence not Landau either Eckart!

Cf. Chapman-Enskog method to derive Landau and Eckart eq.'s; see, eg, de Groot et al ('80)

Here,

In the covariant formalism,
in a unified way and systematically
derive dissipative rel. hydrodynamics at once!

Derivation of the relativistic hydrodynamic equation from the rel. Boltzmann eq. --- an RG-reduction of the dynamics

K. Tsumura, T.K. K. Ohnishi; Phys. Lett. B646 (2007) 134-140

c.f. Non-rel. Y.Hatta and T.K., Ann. Phys. 298 ('02), 24; T.K. and K. Tsumura, J.Phys. A:39 (2006), 8089

Ansatz of the origin of the dissipation= the spatial inhomogeneity,
leading to Navier-Stokes in the non-rel. case .

$\boldsymbol{a}_p^\mu$ would become a macro flow-velocity ⟵ **Coarse graining of space-time**

$$\partial^\mu = \frac{1}{a_p^2} a_p^\mu a_p^\nu \partial_\nu + (g^{\mu\nu} - \frac{a_p^\mu a_p^\nu}{a_p^2})\partial_\nu \equiv a_p^\mu \partial_\tau + \nabla^\mu \qquad \Delta^{\mu\nu} \equiv g^{\mu\nu} - \frac{a_p^\mu a_p^\nu}{a_p^2}$$

$$\frac{\partial}{\partial\tau} = \frac{1}{\boldsymbol{a}_p^2} \boldsymbol{a}_p^\mu \partial_\mu \equiv \boldsymbol{D},$$ time-like derivative $$\boldsymbol{\Delta}_p^{\mu\nu}\partial_\nu \equiv \boldsymbol{\nabla}^\mu \equiv \boldsymbol{\Delta}_p^{\mu\nu}\frac{\partial}{\partial\sigma^\nu}$$ space-like derivative

$$x^\mu \Longrightarrow \tau \quad \sigma^\mu$$

Rewrite the Boltzmann equation as,

$$\Longrightarrow \frac{\partial}{\partial\tau} f_p(\tau\,,\,\sigma) = \frac{1}{p\cdot\boldsymbol{a}_p} C[f]_p(\tau\,,\,\sigma) - \frac{1}{p\cdot\boldsymbol{a}_p} p\cdot\boldsymbol{\nabla} f_p(\tau\,,\,\sigma)$$

perturbation

Only spatial inhomogeneity leads to dissipation.

RG gives a resummed distribution function, from which $T^{\mu\nu}$ and N^μ are obtained.

Chen-Goldenfeld-Oono('95), T.K.('95), **S.-I. Ei, K. Fujii and T.K. (2000)**

Solution by the perturbation theory

0th

$$\frac{\partial}{\partial \tau} \tilde{f}_p^{(0)} = \frac{1}{p \cdot \boldsymbol{a}_p} C[f]_p \Big|_{f=\tilde{f}^{(0)}}$$

"slow"

$$\frac{\partial}{\partial \tau} \tilde{f}_p^{(0)} = 0 \Longrightarrow \frac{1}{p \cdot \boldsymbol{a}_p} C[f]_p \Big|_{f=\tilde{f}^{(0)}} = 0$$

$$\Longrightarrow \tilde{f}_p^{(0)}(\tau\,,\,\sigma\,;\,\tau_0) = (2\pi)^{-3} \exp\left[\frac{\mu(\sigma\,;\,\tau_0) - p^\mu\, u_\mu(\sigma\,;\,\tau_0)}{T(\sigma\,;\,\tau_0)}\right] \equiv f_p^{\mathrm{eq}}(\sigma\,;\,\tau_0)$$

$$\Longrightarrow \tilde{f}^{(0)}(\tau) = f^{\mathrm{eq}}$$

written in terms of the hydrodynamic variables.
Asymptotically, the solution can be written solely in terms of the hydrodynamic variables.

- Five conserved quantities $T(\sigma\,;\,\tau_0),\ \mu(\sigma\,;\,\tau_0),\ u_\mu(\sigma\,;\,\tau_0)$

reduced degrees of freedom

m = 5

$$\boxed{u^\mu(\sigma\,;\,\tau_0)\, u_\mu(\sigma\,;\,\tau_0) = 1}$$

- 0th invariant manifold $f_p^{(0)}(\tau_0\,,\,\sigma) = f_p^{\mathrm{eq}}(\sigma\,;\,\tau_0)$

$$\Longrightarrow f^{(0)}(\tau_0) = f^{\mathrm{eq}}$$

Local equilibrium

1st

$$\frac{\partial}{\partial \tau}\tilde{f}_p^{(1)} = \sum_q A_{pq}\,\tilde{f}_q^{(1)} + F_p$$

Evolution op.： $A_{pq} \equiv \frac{1}{p\cdot \boldsymbol{a}_p}\frac{\partial}{\partial f_q}C[f]_p\Big|_{f=f^{\mathrm{eq}}}$

inhomogeneous：

$$F_p \equiv -\frac{1}{p\cdot \boldsymbol{a}_p}\,p\cdot \boldsymbol{\nabla} f_p^{\mathrm{eq}}$$

Collision operator $L_{pq} \equiv f_p^{\mathrm{eq}\text{-}1}\,A_{pq}\,f_q^{\mathrm{eq}}$

$$L_{pq} = -\frac{1}{p\cdot \boldsymbol{a}_p}\frac{1}{2!}\sum_{p_1}\frac{1}{p_1^0}\sum_{p_2}\frac{1}{p_2^0}\sum_{p_3}\frac{1}{p_3^0}\,\omega(p\,,\,p_1|p_2\,,\,p_3)\,f_{p_1}^{\mathrm{eq}}\left(\delta_{pq}+\delta_{p_1q}-\delta_{p_2q}-\delta_{p_3q}\right)$$

The lin. op. L has good properties:

Def. **inner product:** $\langle\,\varphi\,,\,\psi\,\rangle \equiv \sum_p \frac{1}{p^0}\,(p\cdot \boldsymbol{a}_p)\,f_p^{\mathrm{eq}}\,\varphi_p\,\psi_p$

1. $\langle\,\varphi\,,\,L\,\psi\,\rangle = \langle\,L\,\varphi\,,\,\psi\,\rangle$ Self-adjoint

2. $\langle\,\varphi\,,\,L\,\varphi\,\rangle \leq 0$ for all φ Semi-negative definite

3. $L\,\varphi_0^{\alpha} = 0 \Longrightarrow \varphi_{0p}^{\alpha} = \begin{cases} p^{\mu} & \alpha=\mu, \\ m & \alpha = 4 \end{cases}$

L has 5 zero modes、other eigenvalues are negative.

1. Proof of self-adjointness

$$
\begin{aligned}
\langle \varphi , L\psi \rangle &= \sum_{pq} \frac{1}{p^0} (p\cdot a_p)\, f_p^{\mathrm{eq}} \varphi_p L_{pq} \psi_q \\
&= -\frac{1}{4}\frac{1}{2!} \sum_p \frac{1}{p^0} \sum_{p_1} \frac{1}{p_1^0} \sum_{p_2} \frac{1}{p_2^0} \sum_{p_3} \frac{1}{p_3^0} \omega(p\,,p_1|p_2\,,p_3) \\
&\qquad f_p^{\mathrm{eq}} f_{p_1}^{\mathrm{eq}} \left(\varphi_p + \varphi_{p_1} - \varphi_{p_2} - \varphi_{p_3}\right)\left(\psi_p + \psi_{p_1} - \psi_{p_2} - \psi_{p_3}\right) \\
&= \langle L\varphi\,,\psi\rangle .
\end{aligned}
$$

2. Semi-negativeness of the L

$$
\begin{aligned}
\langle \varphi , L\varphi \rangle &= -\frac{1}{4}\frac{1}{2!} \sum_p \frac{1}{p^0} \sum_{p_1} \frac{1}{p_1^0} \sum_{p_2} \frac{1}{p_2^0} \sum_{p_3} \frac{1}{p_3^0} \omega(p\,,p_1|p_2\,,p_3)\, f_p^{\mathrm{eq}} f_{p_1}^{\mathrm{eq}} \left(\varphi_p + \varphi_{p_1} - \varphi_{p_2} - \varphi_{p_3}\right)^2 \\
&\le 0 \text{ for all } \varphi
\end{aligned}
$$

3.Zero modes

$$
\varphi_{0p}^{\alpha} = \begin{cases} p^\mu & \alpha = \mu \\ m & \alpha = 4 \end{cases}
$$

en-mom.

Particle #

$$
\varphi_p + \varphi_{p_1} = \varphi_{p_2} + \varphi_{p_3}
$$

Collision invariants!
or conserved quantities.

Def. Projection operators:

$$\begin{cases} \left[P\,\psi\right]_p \equiv \sum_{\alpha\beta} \varphi_{0p}^{\alpha}\,\eta_{\alpha\beta}^{-1}\,\langle\,\varphi_0^{\beta}\,,\,\psi\,\rangle, \\ Q \equiv 1 - P. \end{cases}$$

metric

$$\eta^{\alpha\beta} \equiv \langle\,\varphi_0^{\alpha}\,,\,\varphi_0^{\beta}\,\rangle$$

$$\eta_{\alpha\beta}^{-1}\ ;\ \sum_{\gamma} \eta^{\alpha\gamma}\,\eta_{\gamma\beta}^{-1} = \delta_{\beta}^{\alpha}$$

$$\frac{\partial}{\partial\tau}\tilde{f}^{(1)} = A\,\tilde{f}^{(1)} + F$$

$$\Rightarrow\ \tilde{f}^{(1)}(\tau) = e^{(\tau-\tau_0)A}\left\{\underline{f^{(1)}(\tau_0)} + \underline{A^{-1}\,\bar{Q}\,F}\right\} + (\tau-\tau_0)\,\bar{P}\,F - A^{-1}\,\bar{Q}\,F.$$

The initial value yet not determined

fast motion
to be avoided

$$\bar{P} \equiv f^{\mathrm{eq}}\,P\,f^{\mathrm{eq}\,-1}, \qquad \bar{Q} \equiv f^{\mathrm{eq}}\,Q\,f^{\mathrm{eq}\,-1}.$$

$$f_{pq}^{\mathrm{eq}} \equiv f_p^{\mathrm{eq}}\,\delta_{pq}$$

$$\Rightarrow\ \tilde{f}^{(1)}(\tau) = (\tau-\tau_0)\,\bar{P}\,F - A^{-1}\,\bar{Q}\,F$$

eliminated by the choice

- Modification of the manifold：$f^{(1)}(\tau_0) = -A^{-1}\,\bar{Q}\,F$

Second order solutions

$$\frac{\partial}{\partial \tau}\tilde{f}^{(2)} = A\,\tilde{f}^{(2)} + I \qquad \text{with} \quad I_p \equiv \frac{1}{p\cdot \boldsymbol{a}_p}\, p\cdot \boldsymbol{\nabla}\left[A^{-1}\,\bar{Q}\,F\right]_p$$

$$\Rightarrow \quad \tilde{f}^{(2)}(\tau) = e^{(\tau-\tau_0)A}\left\{\underline{f^{(2)}(\tau_0)} + A^{-1}\,\bar{Q}\,I\right\} + (\tau-\tau_0)\,\bar{P}\,I - A^{-1}\,\bar{Q}\,I$$

The initial value not yet determined

fast motion

$$\Rightarrow \quad \tilde{f}^{(2)}(\tau) = (\tau-\tau_0)\,\bar{P}\,I - A^{-1}\,\bar{Q}\,I.$$

eliminated by the choice

- Modification of the invariant manifold in the 2nd order; $f^{(2)}(\tau_0) = -A^{-1}\,\bar{Q}\,I,$

Application of RG/E equation to derive slow dynamics

Collecting all the terms, we have;

- Invariant manifold (hydro dynamical coordinates) as the initial value:

$$f(\tau_0) = f^{\mathrm{eq}} + \varepsilon\left(-A^{-1}\bar{Q}F\right) + \varepsilon^2\left(-A^{-1}\bar{Q}I\right) + O(\varepsilon^3),$$

- The perturbative solution with secular terms:

$$\tilde{f}(\tau) = f^{\mathrm{eq}} + \varepsilon\left(\underline{(\tau-\tau_0)}\bar{P}F - A^{-1}\bar{Q}F\right) + \varepsilon^2\left(\underline{(\tau-\tau_0)}\bar{P}I - A^{-1}\bar{Q}I\right) + O(\varepsilon^3).$$

RG/E equation

$$\left.\frac{\mathrm{d}}{\mathrm{d}\tau_0}\tilde{f}_p(\tau, \sigma; \tau_0)\right|_{\tau_0=\tau} = 0;$$

The meaning of $\tau_0 = \tau$ ⇨ found to be the coarse graining condition

The novel feature in the relativistic case;

Choice of the flow a_p^μ **; eg.** $a_p^\mu = u^\mu$

$$\partial_\mu J^{\mu\alpha}_{\mathrm{hydro}} = 0,$$

$$J^{\mu\alpha}_{\mathrm{hydro}} \equiv \sum_p \frac{1}{p^0} p^\mu \varphi^\alpha_{0p} \left\{ f^{\mathrm{eq}}_p - \left[A^{-1} \bar{Q} F\right]_p \right\} = J^{\mu\alpha}_0 + \Delta J^{\mu\alpha},$$

$$J^{\mu\alpha}_0 \equiv \sum_p \frac{1}{p^0} p^\mu \varphi^\alpha_{0p} f^{\mathrm{eq}}_p$$

$$\Delta J^{\mu\alpha} \equiv -\sum_p \frac{1}{p^0} p^\mu \varphi^\alpha_{0p} \left[A^{-1} \bar{Q} F\right]_p$$ $\Rightarrow$ produce the dissipative terms!

The distribution function;

$$f(\tau_0) = f^{\mathrm{eq}} - A^{-1} \bar{Q} F - A^{-2} \bar{Q} H - A^{-1} \bar{Q} I$$

Notice that the distribution function as the solution is represented solely by the hydrodynamic quantities!

A generic form of the flow vector

$$\boldsymbol{a}_p^\mu = \frac{1}{p\cdot u}\Big((p\cdot u)\,\cos\theta + m\,\sin\theta\Big)\,u^\mu \equiv \theta_p^\mu$$

θ : a parameter

- $\boldsymbol{\Delta}_p^{\mu\nu} = g^{\mu\nu} - u^\mu\,u^\nu \equiv \Delta^{\mu\nu},\ \ \Delta^\mu{}_\rho\Delta^{\rho\nu} = \Delta^{\mu\nu}$
- $\boldsymbol{D} = u^\mu\,\partial_\mu \equiv D,\quad \boldsymbol{\nabla}^\mu = \Delta^{\mu\nu}\,\partial_\nu \equiv \nabla^\mu$
- $\langle\,\varphi\,,\,\psi\,\rangle = \sum_p \frac{1}{p^0}\Big((p\cdot u)\,\cos\theta + m\,\sin\theta\Big)\,f_p^{\rm eq}\,\varphi_p\,\psi_p \equiv \langle\,\varphi\,,\,\psi\,\rangle_\theta$

- **Projection op. onto space-like traceless second-rank tensor;**

$$P^{\mu\nu\rho\sigma} \equiv \frac{1}{2}\left(\Delta^{\mu\rho}\Delta^{\nu\sigma} + \Delta^{\mu\sigma}\Delta^{\nu\sigma} - \frac{2}{3}\Delta^{\mu\nu}\Delta^{\rho\sigma}\right)$$

$$P^{\mu\nu\alpha\beta}P_{\alpha\beta}{}^{\rho\sigma} = P^{\mu\nu\rho\sigma}$$

Examples

$$\boxed{\theta = 0}$$

$$\Longleftrightarrow \quad \boldsymbol{a}_p^\mu = u^\mu$$

$$\partial_\mu J^{\mu\alpha}_{\text{hydro.}} = 0 \qquad \boxed{p \equiv nT}$$

$$\Delta J^{\mu\alpha} = \begin{cases} -\zeta\,\Delta^{\mu\nu} X + 2\eta X^{\mu\nu} & \alpha = \nu \\ -T\lambda z\,\hat{h}^{-1} X^\mu & \alpha = 4. \end{cases}$$

satisfies the Landau constraints

$$u_\mu u_\nu \delta T^{\mu\nu} = 0, u_\mu \Delta_{\sigma\nu} \delta T^{\mu\nu} = 0$$

$$u_\mu \delta N^\mu = 0$$

$$X \equiv -\nabla_\mu u^\mu,$$

$$X_\mu \equiv \nabla_\mu \ln T - \hat{h}^{-1} \nabla_\mu \ln(nT),$$

$$X_{\mu\nu} \equiv \frac{1}{2}\left(\Delta_{\mu\rho}\Delta_{\nu\sigma} + \Delta_{\mu\sigma}\Delta_{\nu\rho} - \frac{2}{3}\Delta_{\mu\nu}\Delta_{\rho\sigma}\right)\nabla^\rho u^\sigma.$$

$$T^{\mu\nu} = \epsilon u^\mu u^\nu - (p + \zeta X)\Delta^{\mu\nu} + 2\eta X^{\mu\nu}$$

$$N^\mu = n u^\mu - \lambda \frac{nT}{\epsilon + p} X^\mu.$$

Landau frame and Landau eq.!

with the microscopic expressions for the transport coefficients;

Bulk viscosity
$$\zeta \equiv -\frac{1}{T}\sum_{pq}\frac{1}{p^0}\,f^{\rm eq}_p\,\Pi_p\,\mathcal{L}^{-1}_{pq}\,\Pi_q$$

Heat conductivity
$$\lambda \equiv -\frac{1}{3}\frac{1}{T^2}\sum_{pq}\frac{1}{p^0}\,f^{\rm eq}_p\,Q^\mu_p\,\mathcal{L}^{-1}_{pq}\,Q_{\mu q}$$

Shear viscosity
$$\eta \equiv -\frac{1}{10}\frac{1}{T}\sum_{pq}\frac{1}{p^0}\,f^{\rm eq}_p\,\Pi^{\mu\nu}_p\,\mathcal{L}^{-1}_{pq}\,\Pi_{\mu\nu q}$$

$$\mathcal{L}_{pq} \equiv (p\cdot\theta_p)\,L_{pq} \Longleftarrow \theta_p \text{ -independent}$$

c.f. $L_{pq} = -\frac{1}{p\cdot a_p}\frac{1}{2!}\sum_{p_1}\frac{1}{p_1^0}\sum_{p_2}\frac{1}{p_2^0}\sum_{p_3}\frac{1}{p_3^0}\,\omega(p\,,\,p_1|p_2\,,\,p_3)\,f^{\rm eq}_{p_1}\left(\delta_{pq}+\delta_{p_1q}-\delta_{p_2q}-\delta_{p_3q}\right)$

$(a_p{}^\mu = \theta_p{}^\mu)$

In a Kubo-type form;

$$\zeta \equiv \frac{1}{T}\int_0^\infty {\rm d}s\,\langle\,\Pi(0)\,,\,\Pi(s)\,\rangle_{\rm eq},$$
$$\lambda \equiv -\frac{1}{3}\frac{1}{T^2}\int_0^\infty {\rm d}s\,\langle\,Q^\mu(0)\,,\,Q_\mu(s)\,\rangle_{\rm eq},$$
$$\eta \equiv \frac{1}{10}\frac{1}{T}\int_0^\infty {\rm d}s\,\langle\,\Pi^{\mu\nu}(0)\,,\,\Pi_{\mu\nu}(s)\,\rangle_{\rm eq}.$$

$$\boxed{[\Pi(s)]_p \equiv \sum_q\left[{\rm e}^{s\,\mathcal{L}}\right]_{pq}\Pi_q}$$
$$\boxed{\langle\,\varphi\,,\,\psi\,\rangle_{\rm eq} \equiv \sum_p\frac{1}{p^0}\,f^{\rm eq}_p\,\varphi_p\,\psi_p}$$

C.f. Bulk viscosity may play a role in determining the acceleration of the expansion of the universe, and hence the dark energy!

Landau equation: $a_p^\mu = u^\mu$

Eckart (particle-flow) frame:

Setting $a_p^\mu = \frac{m}{p\cdot u} u^\mu$

$$T^{\mu\nu} = (\epsilon + 3\zeta\tilde{X})\, u^\mu u^\nu - (p + \zeta \tilde{X})\,\Delta^{\mu\nu} + \lambda T\, u^\mu \tilde{X}^\nu + \lambda T\, u^\nu \tilde{X}^\mu + 2\eta X^{\mu\nu}$$

$$N^\mu = m\, n\, u^\mu$$

i.e., $\delta N^\mu = 0.$

with

$$\tilde{X} \equiv -\{1/3\,(4/3-\gamma)^{-1}\}^2\, \nabla\cdot u$$
$$\tilde{X}^\mu \equiv \nabla^\mu \ln T$$

(i) This satisfies the GMS constraints but not the Eckart's.

(ii) Notice that only the space-like derivative is incorporated.

(iii) This form is different from Eckart's and Grad-Marle-Stewart's, both of which involve the time-like derivative.

Eckart's constraints :
$$\begin{cases} 1.\ u_\mu u_\nu\, \delta T^{\mu\nu} = 0, \\ 2.\ u_\mu\, \delta N^\mu = 0, \\ 3.\ \Delta_{\mu\nu}\, \delta N^\nu = 0, \end{cases} \Longleftrightarrow \begin{cases} 5.\ T^\mu{}_\mu = 0, \\ 2.\ u_\mu\, \delta N^\mu = 0, \\ 3.\ \Delta_{\mu\nu}\, \delta N^\nu = 0. \end{cases}$$

Grad-Marle-Stewart constraints

c.f. Grad-Marle-Stewart equation;

$$\begin{aligned} \delta T^{\mu\nu} &= -3\,(3T^{-1}C_T+1)^{-1}\,\zeta\, u^\mu u^\nu\, \nabla\cdot u + u^\mu T\lambda\left(\frac{1}{T}\nabla^\nu T - Du^\nu\right) + u^\nu T\lambda\left(\frac{1}{T}\nabla^\mu T - Du^\mu\right) \\ &\quad + 2\eta\,\frac{1}{2}\left(\nabla^\mu u^\nu + \nabla^\nu u^\mu - \frac{2}{3}\Delta^{\mu\nu}\,\nabla\cdot u\right) + (3T^{-1}C_T+1)^{-1}\,\zeta\,\Delta^{\mu\nu}\,\nabla\cdot u, \\ \delta N^\mu &= 0. \end{aligned}$$

Which equation is better, Stewart et al’s or ours?

The linear stability analysis around the thermal equilibrium state.

c.f. Ladau equation is stable. (Hiscock and Lindblom (’85))

The stability of the equations in the “Eckart(particle)” frame

K.Tsumura and T.K. ;
Phys. Lett. B 668, 425 (2008).

Y. Minami and T.K.,
Prog. Theor. Phys.122, 881 (2010)

Linear Stability Analysis

K.Tsumura and T.K. ;PLB 668, 425 (2008).

Def. $T(x) = T_0 + \delta T(x),\ \mu(x) = \mu_0 + \delta\mu(x)$ and $u^\mu(x) = u_0^\mu + \delta u^\mu(x)$

with $u_0 \cdot \delta u = 0 \longleftarrow u \cdot u = 1$

Actually, we will put $u_0 = 0.$

Equation of Motion:

$$0 = \partial_\mu T^{\mu\nu} = \partial_\mu \delta T^{\mu\nu} \quad \text{and} \quad 0 = \partial_\mu N^\mu = \partial_\mu \delta N^\mu$$

Ansatz for the solution; plane-wave solution

$$(\delta u^\mu,\ \delta T,\ \delta\mu) = (\delta\tilde{u}^\mu,\ \delta\tilde{T},\ \delta\tilde{\mu})\, \mathrm{e}^{-ik\cdot x}$$

$$\sum_{\beta=1}^{5} M_{\alpha\beta}\Phi_\beta = 0,$$ where $M_{\alpha\beta} = M_{\alpha\beta}(k^0, \vec{k})$

5x5 determinant $\det M_{\alpha\beta} = 0$

Dispersion relation; $\varpi \equiv k^0 = k^0(k)$ (generically complex.)

The stability condition: $\mathrm{Im}(k^0(k)) \le 0 \qquad \forall k$

$$M_{\alpha\beta} \equiv \begin{pmatrix} \mathcal{L}_1 & 0 & 0 & 0 & 0 \\ 0 & \mathcal{L}_1 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{L}_1 - \mathcal{L}_2\,(k^3)^2 & -i\,\mathcal{L}_3\,k^3 & -i\,\mathcal{L}_4\,k^3 \\ 0 & 0 & i\,\mathcal{L}_5\,k^3 & \mathcal{L}_6 & \mathcal{L}_7 \\ 0 & 0 & i\,\mathcal{L}_8\,k^3 & \mathcal{L}_9 & \mathcal{L}_{10} \end{pmatrix}$$

$$\mathcal{L}_1 \equiv (\epsilon + p)\,(-i\,k^0) + \eta\,|\boldsymbol{k}|^2 \qquad \mathcal{L}_2 \equiv -\eta\,\frac{1}{3} - \zeta\,(3\,\gamma - 4)^{-2} \qquad \mathcal{L}_3 \equiv -\frac{\partial p}{\partial T} + \lambda\,(-i\,k^0) \qquad \mathcal{L}_4 \equiv -\frac{\partial p}{\partial \mu}$$

$$\mathcal{L}_5 \equiv (\epsilon + p) - \zeta\,3\,(3\,\gamma - 4)^{-2}\,(-i\,k^0) \qquad \mathcal{L}_6 \equiv \frac{\partial \epsilon}{\partial T}\,(-i\,k^0) + \lambda\,|\boldsymbol{k}|^2 \qquad \mathcal{L}_7 \equiv \frac{\partial \epsilon}{\partial \mu}\,(-i\,k^0) \qquad \mathcal{L}_8 \equiv n$$

$$\mathcal{L}_9 \equiv \frac{\partial n}{\partial T}\,(-i\,k^0) \qquad \mathcal{L}_{10} \equiv \frac{\partial n}{\partial \mu}\,(-i\,k^0)$$

$$\det M_{\alpha\beta} = 0$$

$$\mathcal{L}_1^2 \Big[(\mathcal{L}_1 - |\boldsymbol{k}|^2\,\mathcal{L}_2)\,(\mathcal{L}_6\,\mathcal{L}_{10} - \mathcal{L}_7\,\mathcal{L}_9) - |\boldsymbol{k}|^2\,\mathcal{L}_5\,(\mathcal{L}_3\,\mathcal{L}_{10} - \mathcal{L}_4\,\mathcal{L}_9) - |\boldsymbol{k}|^2\,\mathcal{L}_8\,(\mathcal{L}_4\,\mathcal{L}_6 - \mathcal{L}_3\,\mathcal{L}_7) \Big] = 0.$$

Dispersion relations

Transverse mode: $k^0 = -i\,\eta\,|\boldsymbol{k}|^2/(\epsilon+p)$

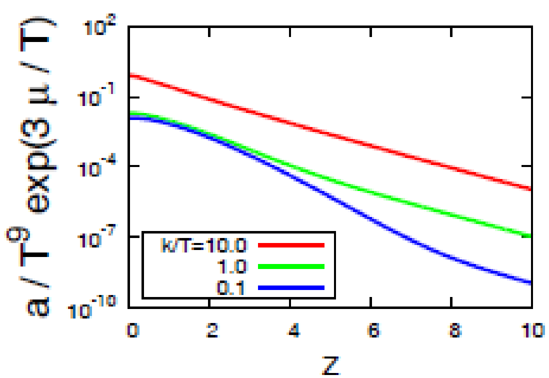
Longitudinal modes: $a\,\omega^3 + b\,\omega^2 + c\omega + d = 0, \text{ with } \omega = -ik^0$

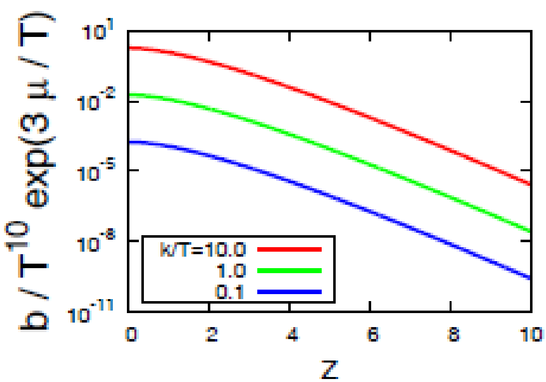
The condition for having all the roots in the left half plane of ϖ
(Routh-Hurwitz theorem)

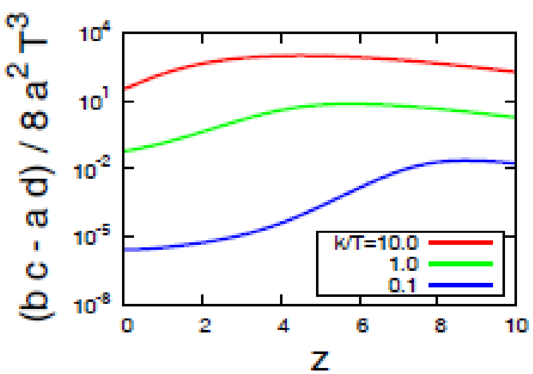
$$a > 0,\ b > 0,\ d > 0\ \text{and}\ bc - ad \geq 0.$$

However,

$$d \equiv |\boldsymbol{k}|^4\, n\, \lambda(\partial p/\partial \mu)_T > 0$$







The stability of the solutions in the “Eckart (particle)” frame:

K.Tsumura and T.K. ;PLB 668, 425 (2008).

(i) The Eckart and Grad-Marle-Stewart equations gives an instability, which has been known, and is now found to be attributed to the fluctuation-induced dissipation, proportional to Du^{μ}.

(ii) Our equation (TKO equation) seems to be stable, being dependent on the values of the transport coefficients and the EOS.

The numerical analysis shows that, the solution to our equation is stable at least for rarefied gasses.

The spectral function of the sound modes:

Y. Minami and T.K., Prog. Theor. Phys.122, 881 (2010)

$$u^\mu(\vec{r},t) = u_0^\mu + \delta u^\mu(\vec{r},t) \qquad n(\vec{r},t) = n_0 + \delta n(\vec{r},t)$$ etc

In the rest frame of the fluid,

$$u_0^\mu = (1, 0) \qquad \delta u^\mu(\vec{r},t) = (0, \vec{v}(\vec{r},t))$$

Inserting them into $T^{\mu\nu},\ N^\mu$, and taking the linear approx.

- Linearized Landau equation (Lin. Hydro in the energy frame);

$$\frac{\partial \delta n}{\partial t} + n_0 \nabla \cdot \vec{v} - \kappa \frac{n_0}{w_0}[\frac{T_0}{w_0}\nabla^2(\delta P) - \nabla^2(\delta T)] = 0$$

$$w_0 \frac{\partial \vec{v}}{\partial t} - \eta \nabla^2 \vec{v} - (\frac{1}{3}\eta + \zeta)\nabla(\nabla \cdot \vec{v}) + \nabla(\delta P) = 0$$

$$n_0 \frac{\partial \delta s}{\partial t} - \frac{\kappa}{T_0}\nabla^2(\delta T) + \frac{\kappa}{w_0}\nabla^2(\delta P) = 0$$

Rel. effects

with

$$\delta P(x) = \frac{w_0 c_s^2}{n_0 \gamma}\delta n(x) + \frac{w_0 c_s^2 \alpha_P}{\gamma}\delta T(x) \qquad \delta s(x) = -\frac{w_0 c_s^2 \alpha_P}{n_0^2 \gamma}\delta n(x) + \frac{\tilde{c}_n}{T_0}\delta T(x)$$

Solving δn as an initial value problem using Laplace transformation, we obtain $S_{nn}(\vec{k},\omega) = <\delta n(\vec{k},\omega)\delta n(\vec{k},t=0)>$, in terms of the initial correlation.

Spectral function of density fluctuations in the Landau frame

In the long-wave length limit, k→0

thermal mode

sound modes

$$\frac{S_{nn}(\vec{k},\omega)}{<(\delta n(\vec{k},t=0))^2>} = (1-\frac{1}{\gamma})\frac{2\Gamma_R k^2}{\omega^2+\Gamma_R^2 k^4} + \frac{1}{\gamma}(\frac{\Gamma_B k^2}{(\omega+c_s k)^2+\Gamma_B^2 k^4} + \frac{\Gamma_B k^2}{(\omega-c_s k)^2+\Gamma_B^2 k^4})$$

Rel. effects appear only in the width of the peaks.

Rel. effects appear only in the sound mode.

$$\Gamma_R = \chi \qquad \Gamma_B = \Gamma + \frac{1}{2}c_s^2 T_0(\kappa / w_0 - 2\chi\alpha_p)$$

$$\Gamma = \frac{1}{2}[\chi(\gamma-1)+\nu_l]$$

thermal diffusivity: $\chi = \frac{\kappa}{n_0 C_P}$

α_P :Isobaric thermal expansivity

c_s : sound velocity

γ : specific heat ratio

Long. kinetic viscosity : $\nu_l = (\zeta + \frac{4}{3}\eta)/w_0$

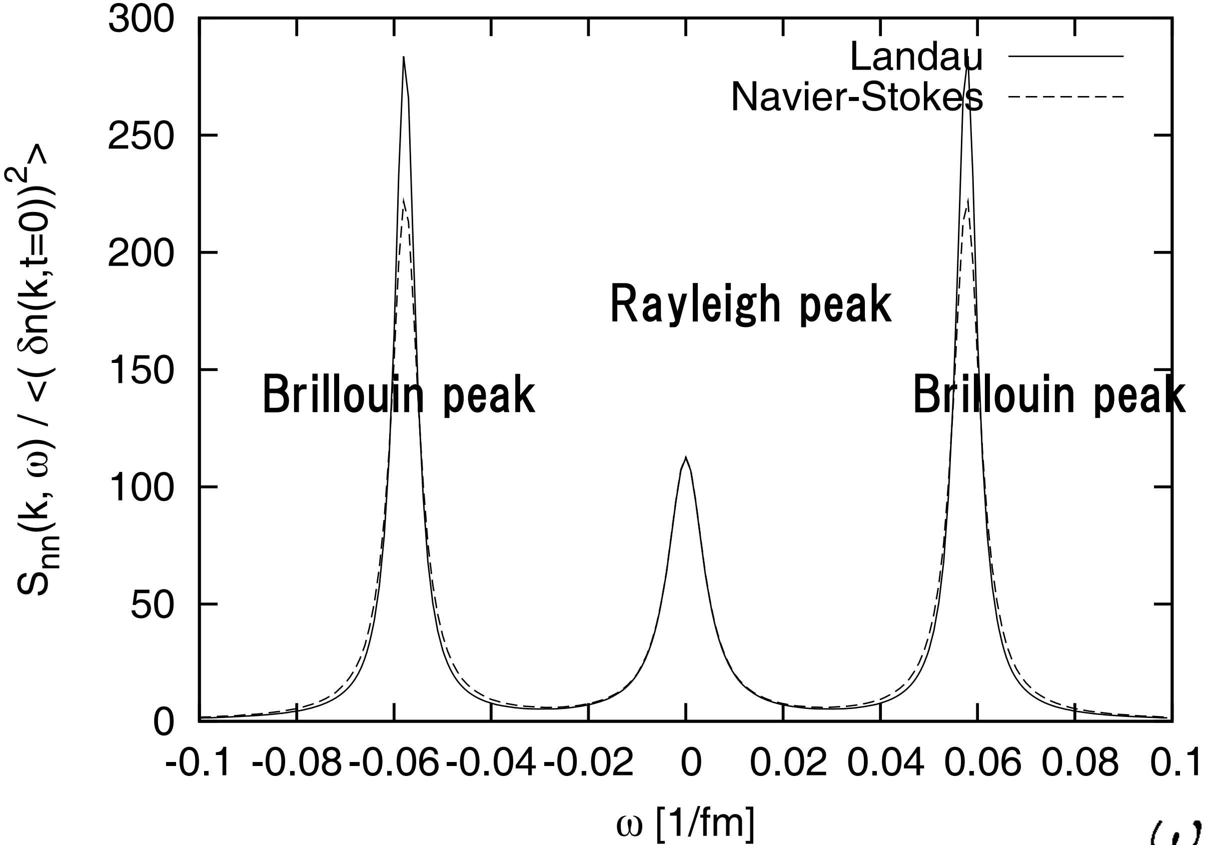
enthalpy

Notice: $\gamma = c_p / c_n = t^{-\tilde{\gamma}+\tilde{\alpha}} \to \infty$

As approaching the critical point, the ratio of specific heats diverges!

The strength of the sound modes vanishes out at the critical point.

Spectral function of density fluctuations in the Landau frame



ω [1/fm]

Eq. of State of ideal Massless particles

$$\epsilon_0 = 3P_0 = 3n_0T_0$$

$k = 0.1$ [1/fm]

$\mu_0 = 0$ [MeV]

$T_0 = 200$ [MeV]

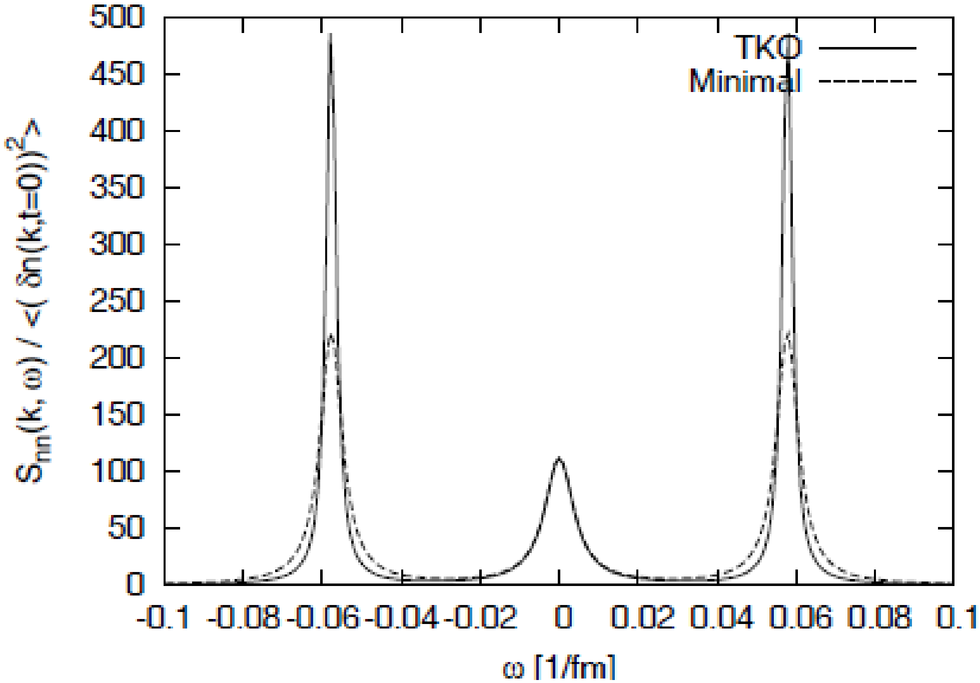
$\eta/s = 0.2$

$\zeta/s = 0.2$

$\kappa T_0/s = 0.4$

Relativistic effects appear only in the peak height and width of the Brillouin peaks.

Particle frame; Tsumura-Kunihiro-Ohnishi equation



$$\frac{S_{nn}(\boldsymbol{k},\omega)}{\langle(\delta n(\boldsymbol{k},t=0))^2\rangle} = \left(1-\frac{1}{\gamma}\right)\frac{2\chi k^2}{\omega^2+\chi^2 k^4} + \frac{1}{\gamma}\left[\frac{\Gamma_{\mathrm{B}}k^2}{(\omega-c_s k)^2+\Gamma_{\mathrm{B}}^2 k^4} + \frac{\Gamma_{\mathrm{B}}k^2}{(\omega+c_s k)^2+\Gamma_{\mathrm{B}}^2 k^4}\right]$$

$$\Gamma_{\mathrm{B}} = \frac{1}{2}\left[\chi(\gamma-1)+\nu_l^{\mathrm{TKO}} - \frac{\alpha p c_s^2}{n_0\tilde{c}_P}(\kappa T_0+3\zeta')\right]$$

$$\nu_l^{\mathrm{TKO}} \equiv \left(\zeta'+\frac{4}{3}\eta\right)/w_0.$$

Spectral function from I-S eq.

For $\tau_\kappa > \dfrac{\kappa T_0}{w_0}$ $\longleftarrow$ $\delta n(\boldsymbol{k},t) \sim \exp\left[-\dfrac{w_0 t}{(\beta_1 w_0 - 1)\kappa T_0}\right]$

$$\frac{S_{nn}(\boldsymbol{k},\omega)}{\langle(\delta n(\boldsymbol{k},t=0))^2\rangle} = \left(1-\frac{1}{\gamma}\right)\frac{2\chi k^2}{\omega^2+\chi^2 k^4} + \frac{1}{\gamma}\left[\frac{\Gamma_{\mathrm{B}} k^2}{(\omega - c_s k)^2 + \Gamma_{\mathrm{B}}^2 k^4} + \frac{\Gamma_{\mathrm{B}} k^2}{(\omega + c_s k)^2 + \Gamma_{\mathrm{B}}^2 k^4}\right] + O(k^2)\times\left[\frac{2/\beta_0\zeta}{\omega^2 + 1/(\beta_0\zeta)^2} + \frac{1/\beta_2\eta}{\omega^2+1/(2\beta_2\eta)^2} + \frac{2w_0/[(\beta_1 w_0 - 1)\kappa T_0]}{\omega^2 + w_0^2/[(\beta_1 w_0 - 1)\kappa T_0]^2}\right].$$

cf. Eckart equation;

$$\delta n(\boldsymbol{k},t) \sim \exp\left[\frac{w_0}{\kappa T_0} t\right]$$

Not damping!

No contribution in the long-wave length limit k→0.

Conversely speaking, the first-order hydro. equations have no problem to describe the hydrodynamic modes with long wave length, as it should.

Compatibility with the underlying kinetic equations?

Eckart constraints are not compatible with the Boltzmann equation, as proved

in K.Tsumura, T.K. and K.Ohnishi;PLB646 ('06), 134.

Proof that the Eckart equation constraints can not be compatible with the Boltzmann eq.

Preliminaries:

Collision operator

$$L_{pq} \equiv f_p^{\mathrm{eq}-1} A_{pq} f_q^{\mathrm{eq}} \qquad A_{pq} \equiv \frac{1}{p \cdot \boldsymbol{a}_p} \frac{\partial}{\partial f_q} C[f]_p \bigg|_{f=f^{\mathrm{eq}}}$$

L has 5 zero modes:

$$L \varphi_0^{\alpha} = 0 \qquad \varphi_{0p}^{\alpha} = \begin{cases} p^{\mu} & \alpha = \mu, \\ m & \alpha = 4 \end{cases}$$

The dissipative part;

$$-[A^{-1} \hat{Q} F]_p = f_p^{\mathrm{eq}} \phi_p$$

with

$$\phi_p \equiv -[L^{-1} Q f^{\mathrm{eq}-1} F]_p$$

due to the Q operator.

$$\langle \varphi_0^{\alpha}, \phi \rangle = 0 \ \text{ for } \alpha = 0, 1, 2, 3, 4$$

where

$$\langle \varphi, \psi \rangle \equiv \sum_p \frac{1}{p^0} (p \cdot \boldsymbol{a}_p) f_p^{\mathrm{eq}} \varphi_p \psi_p$$

$\Delta J^{\mu\alpha} \equiv -\sum_p \frac{1}{p^0} p^\mu \varphi^\alpha_{0p} \left[A^{-1} \bar{Q} F\right]_p$ $\qquad$ $\langle \varphi , \psi \rangle \equiv \sum_p \frac{1}{p^0} (p \cdot a_p) f^{\mathrm{eq}}_p \varphi_p \psi_p.$

The orthogonality condition due to the projection operator exactly corresponds to the constraints for the dissipative part of the energy-momentum tensor and the particle current!

(A) $a^\mu_p = u^\mu,$ i.e., Landau frame,

$$\boxed{\langle \varphi^\alpha_0 , \phi \rangle = 0} \Longrightarrow \sum_p \frac{1}{p^0} (p \cdot u) f^{\mathrm{eq}}_p \varphi^\alpha_p \phi_p = 0 \qquad p \bullet u = p^\mu u_\mu$$

$$\Updownarrow$$

Matching condition!

$$\begin{cases} u_\nu \, \delta J^{\mu\nu} = 0 \implies u_\mu u_\nu \, \delta J^{\mu\nu} = 0, \;\; \Delta_{\mu\rho} u_\nu \, \delta J^{\mu\nu} = 0, \\ u_\mu \, \delta J^{\mu 4} = 0, \end{cases}$$

(B) $a^\mu_p = \frac{m}{p \cdot u} u^\mu$, i.e., the Eckart frame, $\longrightarrow$ $(p \cdot a_p) = \mathrm{const.},$

$$\boxed{\langle \varphi^\alpha_0 , \phi \rangle = 0} \Longrightarrow \sum_p \frac{1}{p^0} m f^{\mathrm{eq}}_p \varphi^\alpha_p \phi_p = 0$$

$\alpha = 0, 1, 2, 3,$ $\qquad \boxed{\delta J^{\mu 4} = 0} \implies u_\mu \, \delta J$

$\alpha = 4,$ $\qquad \boxed{\delta J^\mu{}_\mu = 0}$ $\qquad m^2 =$ Eckart's constraints : $\begin{cases} 1. \; u_\mu u_\nu \, \delta T^{\mu\nu} = 0, \\ 2. \; u_\mu \, \delta N^\mu = 0, \\ 3. \; \Delta_{\mu\nu} \, \delta N^\nu = 0, \end{cases}$

(C) there exists no a^μ_p meeting the Eckart's constr..., ..., ...

Constraints 2, 3 $\Longrightarrow$ $(p \cdot a_p) = \mathrm{const.},$

Constraint 1 $\Longrightarrow$ $(p \cdot a_p) = \mathrm{const.} \times (p \cdot u)^2.$

Contradiction!

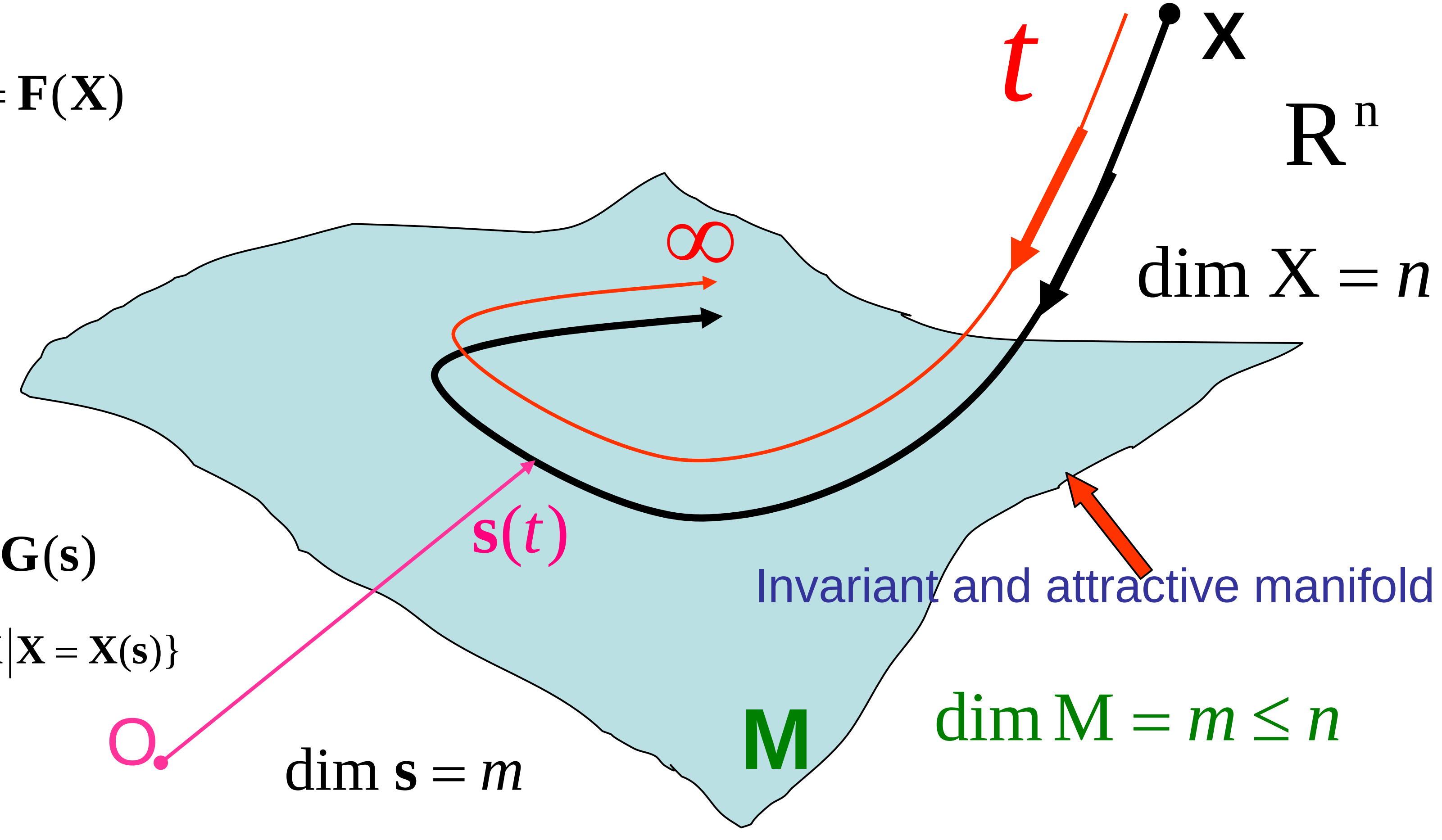
Israel-Stewart equations from Kinetic equation on the basis of the RG method

K. Tsumura and T.K., **arXiv:0906.0079[hep-ph]**

Geometrical image of reduction of dynamics

$$\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X})$$

$$\begin{cases} \dfrac{d\mathbf{s}}{dt} = \mathbf{G}(\mathbf{s}) \\ \mathrm{M}=\{\mathbf{X} \mid \mathbf{X} = \mathbf{X}(\mathbf{s})\} \end{cases}$$



eg.

$\mathbf{X} = f(\mathbf{r}, \mathbf{p})$; **distribution function in the phase space (infinite dimensions)**

$\mathbf{s} = \{u^{\mu}, T, n\}$; **the hydrodinamic quantities (5 dimensions), conserved quantities.**

$$\tilde{f}_p^{(0)}(\tau\,,\,\sigma\,;\,\tau_0) = \tilde{f}_p^{(00)}(\sigma\,;\,\tau_0) + \eta\,\tilde{f}_p^{(01)}(\tau\,,\,\sigma\,;\,\tau_0)$$

zero mode pseudo zero mode

$$\tilde{f}_p^{(00)}(\sigma\,;\,\tau_0) = (2\pi)^{-3}\,\exp\left[\frac{\mu(\sigma\,;\,\tau_0) - p^\mu\,u_\mu(\sigma\,;\,\tau_0)}{T(\sigma\,;\,\tau_0)}\right] \equiv f_p^{\mathrm{eq}}(\sigma\,;\,\tau_0)$$

Five integral const's; $T(\sigma\,;\,\tau_0),\ \mu(\sigma\,;\,\tau_0),\ u_\mu(\sigma\,;\,\tau_0)\ \left(u^\mu(\sigma\,;\,\tau_0)\,u_\mu(\sigma\,;\,\tau_0) = 1\right)$

Eq. governing the pseudo zero mode;

$$\frac{\partial}{\partial\tau}\tilde{f}_p^{(01)} = \frac{1}{p\cdot a_p}\sum_q \frac{\partial}{\partial f_q}C[f]_p\bigg|_{f=f^{\mathrm{eq}}}\tilde{f}_q^{(01)} \equiv \sum_q A_{pq}\,\tilde{f}_q^{(01)}$$

Lin. Operator; $L_{pq} = f_p^{eq-1}A_{pq}\,f_q^{eq}$

zero mode $\varphi_{0p}^\alpha = \begin{cases} p^\mu & \alpha = \mu, \\ m & \alpha = 4. \end{cases}$ $\Longleftrightarrow$ collision invariants

pseudo zero mode sol. $\left(f^{\mathrm{eq}\text{-}1}\,\tilde{f}^{(01)}\right)(\tau) = \mathrm{e}^{(\tau-\tau_0)L}\left(f^{\mathrm{eq}\text{-}1}\,\tilde{f}^{(01)}\right)(\tau_0)$

Init. valu $\left[f^{\mathrm{eq}\text{-}1}\,\tilde{f}^{(01)}\right]_p(\tau_0\,,\,\sigma\,;\,\tau_0) = \epsilon(\sigma\,;\,\tau_0) + p^\mu\,\epsilon_\mu(\sigma\,;\,\tau_0) + p^\mu\,p^\nu\,\epsilon_{\mu\nu}(\sigma\,;\,\tau_0) \equiv \phi_p(\sigma\,;\,\tau_0)$

Constraints; $\epsilon_{\mu\nu} = \epsilon_{\nu\mu}$ and $\epsilon^\mu{}_\mu = 0)$.

$$\langle\,\varphi_0^\alpha\,,\,\phi\,\rangle = 0\ \ \text{for}\ \ \alpha = 0\,,\,1\,,\,2\,,\,3\,,\,4.$$

Orthogonality condition with the zero modes

Thus,

$$\tilde{f}_p^{(0)}(\tau) = f_p^{\mathrm{eq}} \left(1 + \eta \left[\mathrm{e}^{(\tau-\tau_0)L}\,\phi\right]_p\right)$$

with the initial cond.; $f_p^{(0)}(\tau_0) = f_p^{\mathrm{eq}}\left(1 + \eta\,\phi_p\right)$

Def.

$$F_p \equiv -\frac{1}{p\cdot \boldsymbol{a}_p}\, p\cdot \boldsymbol{\nabla} f_p^{\mathrm{eq}},$$

$$K_p(\tau-\tau_0) \equiv -\frac{1}{p\cdot \boldsymbol{a}_p}\, p\cdot \boldsymbol{\nabla}\left\{ f_p^{\mathrm{eq}} \left[\mathrm{e}^{(\tau-\tau_0)L}\,\phi\right]_p\right\}$$

Projection to the pseudo zero modes ;

$$\left[P_1\,\psi\right]_p = \phi_p\,\langle\,\phi\,,\,\psi\,\rangle \Big/ \langle\,\phi\,,\,\phi\,\rangle, \qquad Q_1 \equiv Q_0 - P_1$$

Up to 1st order;

$$\bar{f}^{(1)}(\tau) = (\tau - \tau_0)\,\bar{P}_0\,F + \left(\mathrm{e}^{(\tau-\tau_0)A} - 1\right)A^{-1}\,\bar{P}_1\,F - A^{-1}\,\bar{Q}_1\,F + \eta\Bigg(\int_{\tau_0}^{\tau}\mathrm{d}s\,\bar{P}_0\,K(s-\tau_0)$$
$$+ \int_{\tau_0}^{\tau}\mathrm{d}s\,\mathrm{e}^{(\tau-s)A}\,\bar{P}_1\,K(s-\tau_0) + \int_{\tau_0}^{\tau}\mathrm{d}s\,\mathrm{e}^{(\tau-s)A}\,\bar{Q}_1\,K(s-\tau_0) - \mathrm{e}^{(\tau-\tau_0)A}\,A^{-1}\,\bar{Q}_1\,K(0)\Bigg)$$

Initial condition；
（Invariant manifold）

$$f^{(1)}(\tau_0) = -A^{-1}\,\bar{Q}_1\,F + \eta\left(-A^{-1}\,\bar{Q}_1\,K(0)\right) = -A^{-1}\,\bar{Q}_1\left(F + \eta\,K(0)\right)$$

RG/E equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau_0}\left(\bar{f}^{(0)}(\tau) + \varepsilon\,\bar{f}^{(1)}(\tau) + O(\varepsilon^2)\right)\Bigg|_{\tau_0=\tau} = 0$$

Slow dynamics (Hydro dynamics）

$$\sum_p \frac{1}{p^0}\,\varphi^{\alpha}_{0p}\,p^{\mu}\,\partial_{\mu}\left[f^{\mathrm{eq}}_p\left(1 + \eta\,\phi_p\right)\right] = 0$$
$$\sum_p \frac{1}{p^0}\,\phi_p\,p^{\mu}\,\partial_{\mu}\left[f^{\mathrm{eq}}_p\left(1 + \eta\,\phi_p\right)\right] = \eta\langle\phi\,,\,L\,\phi\rangle$$

Include relaxation equations

Explicitly;

$$J^{\mu} = \tilde{n}\, u^{\mu} + N^{\mu}$$

$$T^{\mu\nu} = \tilde{e}\, u^{\mu}\, u^{\nu} - \tilde{p}\, \Delta^{\mu\nu} + u^{\mu}\, Q^{\nu} + u^{\nu}\, Q^{\mu} + \pi^{\mu\nu}$$

$$\begin{aligned}
\tilde{n} &\equiv J_a\, u^a, \\
N^{\mu} &\equiv J_a\, \Delta^{a\mu}, \\
\tilde{e} &\equiv T_{ab}\, u^a\, u^b, \\
\tilde{p} &\equiv -1/3\, T_{ab}\, \Delta^{ab}, \\
Q^{\mu} &\equiv T_{ab}\, u^a\, \Delta^{b\mu}, \\
\pi^{\mu\nu} &\equiv T_{ab}\, \Delta^{ab\mu\nu}.
\end{aligned}$$

$$\begin{aligned}
\tilde{n} &= I_{10}\,(1+A) + I_{20}\, B + I_{30}\, D - I_{31}\, E, \\
N^{\mu} &= \frac{1}{3}\, I_{21}\, C^{\mu} + \frac{2}{3}\, I_{31}\, F^{\mu}, \\
\tilde{e} &= I_{20}\,(1+A) + I_{30}\, B + I_{40}\, D - I_{41}\, E, \\
\tilde{p} &= -\frac{1}{3}\left(I_{21}\,(1+A) + I_{31}\, B + I_{41}\, D - I_{42}\, E\right), \\
Q^{\mu} &= \frac{1}{3}\, I_{31}\, C^{\mu} + \frac{2}{3}\, I_{41}\, F^{\mu}, \\
\pi^{\mu\nu} &= \frac{2}{15}\, I_{42}\, G^{\mu\nu}.
\end{aligned}$$

Integrals given in terms of the distribution function

Specifically,

Def. $\delta n \equiv \tilde{n} - n, \quad \delta e \equiv \tilde{e} - e, \quad \Pi \equiv \tilde{p} - p,$

New!

$$\begin{aligned}
\delta e &= f_E(\theta)\, \Pi, \\
\delta n &= f_N(\theta)\, \Pi.
\end{aligned}$$

$$\begin{aligned}
f_E(\theta) &\equiv \frac{-3\, m^2\, Z_p\, \sin^2\theta}{\cos^2\theta - m^2\, Z_e\, \sin^2\theta + m\, Z_n\, \sin\theta\, \cos\theta}, \\
f_N(\theta) &\equiv \frac{3\, m\, Z_p\, \sin\theta\, \cos\theta}{\cos^2\theta - m^2\, Z_e\, \sin^2\theta + m\, Z_n\, \sin\theta\, \cos\theta}.
\end{aligned}$$

For the velocity field,

$$a_p^{\mu} = \frac{(p \cdot u)\, \cos\theta + m\, \sin\theta}{p \cdot u}\, u^{\mu}$$

$\theta = 0$; Landau, $\theta = \pi / 2$;Eckart

The viscocities ζ, λ, η are frame-independent, in accordance with Lin. Res. Theory.

However, the relaxation times and legths are frame-dependent.

The form is totally different from the previous ones like I-S's, And contains many additional terms.

$$u_\mu \tau^{\mu\nu} u_\nu = 0$$

$$\varphi^{\mu\nu}_{1p} \equiv \left[Q_0 \, \tilde{\varphi}^{\mu\nu}_1 \right]_p, \quad \tilde{\varphi}^{\mu\nu}_{1p} \equiv p^\mu \, p^\nu$$

contains a zero mode of the linearized collision operator. $p^\mu p_\mu = m^2$

$$\tau^\mu_\mu = 0$$

Conformal non-inv. gives the ambiguity.

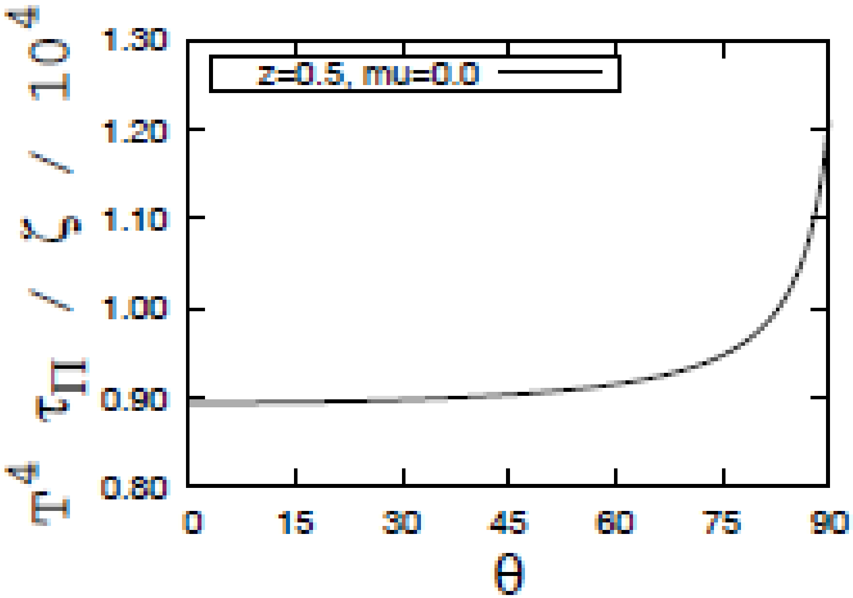
Energy frame]

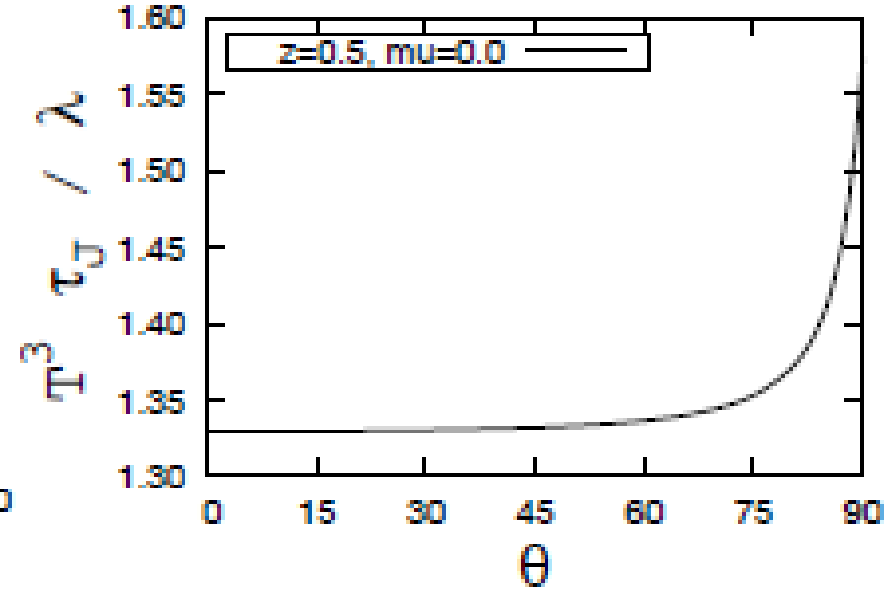
$$
\begin{aligned}
\Pi \;=\; & -\zeta\,\nabla^a u_a - \tau_\Pi\, D\Pi - \ell_{\Pi J}\,\nabla^a J_a \\
& -\frac{1}{2}\,\tau_\Pi \left\{ \kappa_\Pi\,\nabla^a u_a + \frac{\zeta\,T}{\tau_\Pi}\,\partial_a\!\left(\frac{\tau_\Pi}{\zeta\,T}\,u^a\right) \right\} \Pi \\
& -\frac{1}{2}\,\ell_{\Pi J} \left\{ \kappa^{(0)}_{\Pi J}\,\nabla^a \frac{\mu}{T} - \kappa^{(1)}_{\Pi J}\,Du^a + \frac{\zeta\,T}{\ell_{\Pi J}}\,\partial_b\!\left(\frac{\ell_{\Pi J}}{\zeta\,T}\,\Delta^{bc}\right)\Delta_c{}^a \right\} J_a \\
& -\frac{1}{2}\,\ell_{\Pi\pi} \left\{ -\kappa_{\Pi\pi}\,\Delta^{abcd}\,\nabla_c u_d \right\} \pi_{ab}, \\
J^\mu \;=\; & \lambda\,\hat{h}^{-2}\,\nabla^\mu \frac{\mu}{T} - \ell_{J\Pi}\,\nabla^\mu \Pi - \tau_J\,\Delta^{\mu a}\,DJ_a - \ell_{J\pi}\,\Delta^{\mu abc}\,\nabla_a \pi_{bc} \\
& -\frac{1}{2}\,\ell_{J\Pi} \left\{ \kappa^{(0)}_{J\Pi}\,\nabla^\mu \frac{\mu}{T} - \kappa^{(1)}_{J\Pi}\,Du^\mu + \frac{\lambda\,\hat{h}^{-2}}{\ell_{J\Pi}}\,\Delta^\mu{}_a\,\partial_b\!\left(\frac{\ell_{J\Pi}}{\lambda\,\hat{h}^{-2}}\,\Delta^{ab}\right) \right\} \Pi \\
& -\frac{1}{2}\,\tau_J \left\{ \Delta^{\mu a}\left[\kappa^{(0)}_J\,\nabla^b u_b + \frac{\lambda\,\hat{h}^{-2}}{\tau_J}\,\partial_b\!\left(\frac{\tau_J}{\lambda\,\hat{h}^{-2}}\,u^b\right)\right] - 2\,\kappa^{(1)}_J\,\Delta^{\mu abc}\,\nabla_b u_c - 2\,\omega^{\mu a} \right\} J_a \\
& -\frac{1}{2}\,\ell_{J\pi} \left\{ \Delta^{\mu cab}\left(\kappa^{(0)}_{J\pi}\,\nabla_c \frac{\mu}{T} - \kappa^{(1)}_{J\pi}\,Du_c\right) + \frac{\lambda\,\hat{h}^{-2}}{\ell_{J\pi}}\,\Delta^\mu{}_c\,\partial_d\!\left(\frac{\ell_{J\pi}}{\lambda\,\hat{h}^{-2}}\,\Delta^{cdef}\right)\Delta_{ef}{}^{ab} \right\} \pi_{ab}, \\
\pi^{\mu\nu} \;=\; & 2\,\eta\,\Delta^{\mu\nu ab}\,\nabla_a u_b - \ell_{\pi J}\,\Delta^{\mu\nu ab}\,\nabla_a J_b - \tau_\pi\,\Delta^{\mu\nu ab}\,D\pi_{ab} \\
& -\frac{1}{2}\,\ell_{\pi\Pi} \left\{ -\kappa_{\pi\Pi}\,\Delta^{\mu\nu ab}\,\nabla_a u_b \right\} \Pi \\
& -\frac{1}{2}\,\ell_{\pi J} \left\{ \Delta^{\mu\nu ba}\left(\kappa^{(0)}_{\pi J}\,\nabla_b \frac{\mu}{T} - \kappa^{(1)}_{\pi J}\,Du_b\right) + \frac{\eta\,T}{\ell_{\pi J}}\,\Delta^{\mu\nu}{}_{bc}\,\partial_d\!\left(\frac{\ell_{\pi J}}{\eta\,T}\,\Delta^{bcde}\right)\Delta_e{}^a \right\} J_a \\
& -\frac{1}{2}\,\tau_\pi \left\{ \Delta^{\mu\nu ab}\left[\kappa^{(0)}_\pi\,\nabla^c u_c + \frac{\eta\,T}{\tau_\pi}\,\partial_c\!\left(\frac{\tau_\pi}{\eta\,T}\,u^c\right)\right] - 4\,\kappa^{(1)}_\pi\,\Delta^{\mu\nu ce}\,\Delta_e{}^{dab}\,\Delta_{cd}{}^{fg}\,\nabla_f u_g - 4\,\Delta^{\mu\nu ce}\,\Delta_e{}^{dab}\,\omega_{cd} \right\} \pi_{ab}
\end{aligned}
$$

where $\omega^{\mu\nu} \equiv (\nabla^\mu u^\nu - \nabla^\nu u^\mu)/2$ is the vorticity.

Frame dependence of the relaxation times

$$a_p^\mu = \frac{1}{p \cdot u}\Big((p \cdot u)\cos\theta + m \sin\theta\Big)\, u^\mu$$





Calculated for relativistic ideal gas with $m/T = 0.5$, $\mu/T = 0.0$

τ_π ; frame independent

Summary

- The (dynamical) RG method is applied to derive generic second-order hydrodynamic equations, giving new constraints in the particle frame, consistent with a general phenomenological derivation.
- There are so many terms in the relaxation terms which are absent in the previous works, especially due to the conformal non-invariance, which gives rise to an ambiguity in the separation in the first order and the second order terms (matching condition)

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