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Brane-World Inflation Driven by a Bulk Scalar Field

— present status and future prospects —

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Y. Himemoto & MS, PRD63, 044015 (2001) [gr-qc/0010035].

N. Sago, Y. Himemoto & MS, PRD65, 024014 (2002) [gr-qc/0104033].

Y. Himemoto, T. Tanaka & MS, PRD65, 104020 (2002) [gr-qc/0112027].

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§1. Scenario

⋆ Randall-Sundrum's "default" parameters:

brane tension: $$\sigma = \sigma_c = \frac{3}{4\pi G_5 \ell}; \quad \ell = \left|\frac{6}{\Lambda_5}\right|^{1/2}.$$

$$ds^2 = dy^2 + e^{-2|y|/\ell}\eta_{\mu\nu}dx^\mu dx^\nu \quad \text{(Minkowski brane at } y = 0\text{)}$$

If $|\sigma| > \sigma_c$, then inflation occurs on the brane:

$$H^2 = \frac{1}{\ell^2}\left(\frac{\sigma^2}{\sigma_c^2} - 1\right) = \frac{|\Lambda_5|}{6}\left(\frac{\sigma^2}{\sigma_c^2} - 1\right)$$

$$\Updownarrow$$

If $|\sigma| = \sigma_c$ but $|\Lambda_{5,eff}| < |\Lambda_5|$, Inflation also occurs on the brane:

$$H^2 = \frac{|\Lambda_5 - \Lambda_{5,eff}|}{6}$$

$$\Downarrow$$

Brane-world inflation can be driven solely by bulk (gravitational) scalar fields.

§2. 5D Einstein-scalar system with a Z_2 brane

- **5D Einstein equations**

$$G_{ab} + \Lambda_5 \, g_{ab} = \kappa_5^2 \, T_{ab} \, ; \qquad \kappa_5^2 = 8\pi G_5$$

∗ (4 + 1)-decomposition (Gaussian Normal Coordinates)

$$g_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & q_{\mu\nu} \end{pmatrix} \; ; \quad q_{\mu\nu} \cdots \text{ 4D metric}$$

$$\Updownarrow$$

$$ds^2 = dr^2 + q_{\mu\nu} dx^\mu dx^\nu \; ; \quad r \cdots \text{ 5th dimension}$$

∗ Energy-momentum tensor

$$T_{ab} = \phi_{,a}\phi_{,b} - g_{ab}\left(\frac{1}{2} g^{cd} \phi_{,c}\phi_{,d} + V(\phi)\right) + S_{ab}\delta(r - r_0),$$

$$S_{ab} = -\sigma q_{ab}.$$

* Z_2-symmetry and RS brane tension

$$q_{ab}(r_0+y)=q_{ab}(r_0-y)\,,\quad \phi_{,r}(r_0)=0\,,$$

$$\sigma=\sigma_0=\frac{6}{\kappa_5^2\ell_0}\,,\quad \ell_0^2=\frac{6}{|\Lambda_5|}\,.$$

• 4D "Einstein-scalar" equations on the brane

$$G_{\mu\nu}=\kappa_4^2\,T^{(s)}_{\mu\nu}-E_{\mu\nu},\quad \kappa_4^2=\frac{\kappa_5^2}{\ell_0},$$

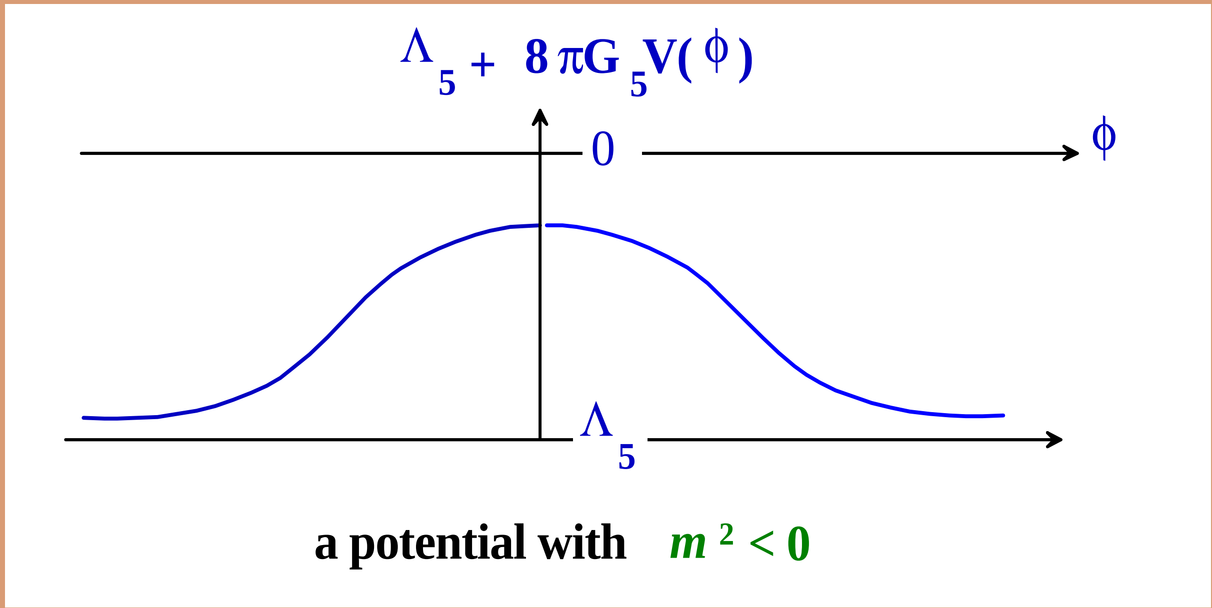
$$T^{(s)}_{\mu\nu}=\frac{\ell_0}{6}\left(4\phi_{,\mu}\phi_{,\nu}-\left(\frac{5}{2}q^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta}+3V(\phi)\right)q_{\mu\nu}\right),$$

$$E_{\mu\nu}={}^{(5)}C_{rbrd}\,q^b_\mu\,q^d_\nu\,.$$

$E_{\mu\nu}$ carries information of 5D bulk geometry.

§3. Quadratic Potential Model

$$V = V_0 + \frac{1}{2} m^2 \phi^2$$



When $|m^2|\phi^2 \ll V_0$, one can solve the field equations iteratively.

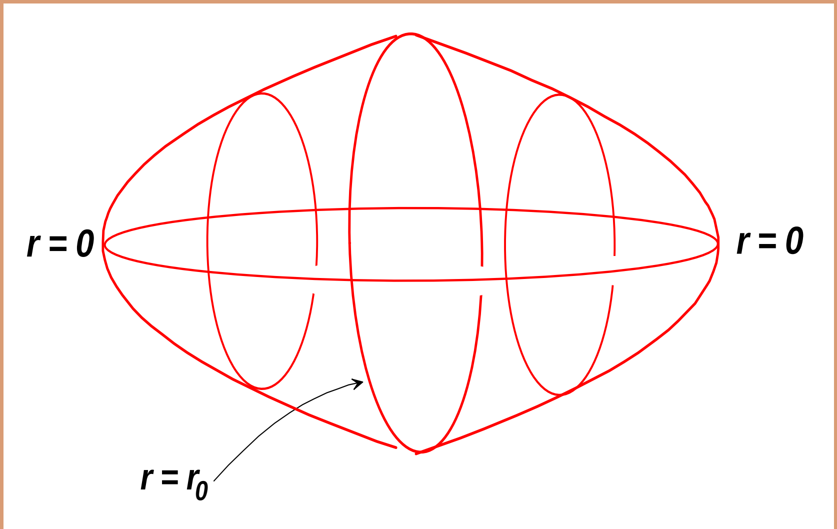
⋆ 0-th order:

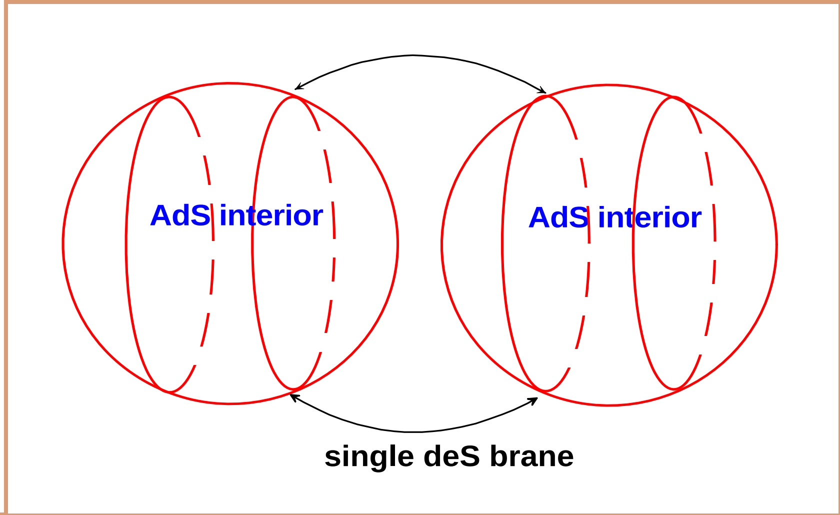
$$V = V_0\,, \quad \phi = 0\,,$$

$$ds^2 = dr^2 + (H\ell)^2\sinh^2(r/\ell)(-dt^2 + H^{-2}\cosh^2 Ht\, d\Omega^2_{(3)})$$

$$(r \le r_0)$$

$$\ell^2 = \frac{6}{|\Lambda_5 + \kappa_5^2 V_0|}\,, \quad H^2 = \frac{\kappa_5^2}{6} V_0$$





This is just an AdS_5-dS brane system with

a modified AdS curvature: $\ell > \ell_0$

⋆ 1st order:

$\phi = \psi(t)u(r)$ $\quad\cdots$ assumption

$$\psi(t) = e^{(\mu-3/2)Ht}, \quad u(r) = \frac{P^{-\mu}_{\nu-1/2}(\cosh(r/\ell))}{\sinh^{3/2}(r/\ell)},$$

$$\nu = \sqrt{m^2\ell^2+4}, \quad \mu \approx \sqrt{\frac{9}{4} - \frac{m_{\rm eff}^2}{H^2}} \approx \frac{3}{2} - \frac{m_{\rm eff}^2}{3H^2}.$$

$$m_{\rm eff}^2 \approx \begin{cases} \frac{1}{2}m^2 & \text{for } |m^2|\ell^2 \ll 1 \\ \frac{3}{5}m^2 & \text{for } |m^2|\ell^2 \gg 1 \end{cases}$$

⋆ This is the zero-mode (the lowest eigenvalue) solution.

⋆ For $|m^2| \ll H^2$, this gives slow-roll inflation on the brane.

⋆ In general,

$\phi = \psi(t)u(r) + \sum_n \psi_n(t)u_n(r)\,; \quad u_n(r)$ $\cdots$ Kaluza-Klein modes

But the zero-mode dominates at late times if $|m^2|\ell^2 \ll 1$.

⋆ 2nd order: (for $|m^2|\ell^2 \ll 1$)

$$3\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2}\right] \equiv 3H^2 = \kappa_4^2\,\rho_{\rm eff}\,, \qquad \left(\kappa_4^2 = \frac{\kappa_5^2}{\ell_0}\right)$$

$$\rho_{\rm eff} = \frac{\ell_0}{2}\left(\frac{\dot{\phi}^2}{2} + V(\phi)\right) - \frac{\ell_0}{\kappa_5^2}E_{tt}\,,$$

$$E_{tt} = \frac{\kappa_5^2}{2a^4}\int^t a^4\dot{\phi}(\partial_r^2\phi + \frac{\dot{a}}{a}\dot{\phi})\,dt = -\frac{\kappa_5^2}{4}\dot{\phi}^2 + \frac{C}{a^4} \longrightarrow -\frac{\kappa_5^2}{4}\dot{\phi}^2\,.$$

$$\Rightarrow \qquad \rho_{\rm eff} = \frac{\ell_0}{2}\left(\dot{\phi}^2 + V(\phi)\right)$$

For $V = V_0 + m^2\phi^2/2$, this means

$$\rho_{\rm eff} = \frac{\dot{\Phi}^2}{2} + U(\Phi)\,; \qquad \Phi = \sqrt{\ell_0}\,\phi\,,$$

$$U(\Phi) = \frac{\ell_0}{2}V(\Phi/\sqrt{\ell_0}) = \frac{\ell_0}{2}V_0 + \frac{1}{2}m_{\rm eff}^2\Phi^2, \qquad m_{\rm eff}^2 = \frac{m^2}{2}\,.$$

Consistent with the 1st order solution when $|m^2|\ell^2 \ll 1$. (small discrepancy when $|m^2|\ell^2 \gg 1$; non-negligible KK contributions)

What we need to work on now are:

⋆⋆ Effects of $O(m^2\ell^2, H^2\ell^2)$ corrections ⋆⋆

These include

- ⋆ **Quantifying KK corrections to the brane dynamics.**
- ⋆ **Quantum fluctuations and cosmological perturbations.**
  $\langle\phi^2\rangle$ has been calculated (Sago, Himemoto & MS (2002)).
  But this is not directly related to observables.

Need to evaluate $E_{\mu\nu}$ and $T^{(s)}_{\mu\nu}$.

- ⋆ **Initial condition of the brane universe.**

Need quantum cosmological considerations.

§4. Cosmological perturbations (work in progress)

- **Evaluation of $E_{\mu\nu}$**

· **Full background spacetime:**

$$ds^2 = dr^2 + b^2(r,t)(-dt^2 + a^2(r,t)d\Omega_{(3)}^2)\,, \quad \phi = \phi(r,t)\,.$$

where we have

$$b(r,t) = b(r) + O(\phi^2)\,, \quad a(r,t) = a(t) + O(\phi^2)\,;$$
$$b(r) = H\ell \sinh(r/\ell)\,, \quad a(t) = H^{-1}\cosh Ht$$

· **Lowest order background approximated by AdS$_5$:**

$$ds^2 = dr^2 + b^2(r)(-dt^2 + a^2(t)d\Omega_{(3)}^2)$$

· **$E_{\mu\nu}$ in the bulk satisfies an equation of the form**

$$\mathcal{L}\,E_{\mu\nu} = S_{\mu\nu};$$

$\mathcal{L}$ ··· d'Alembertian-like operator

$S_{\mu\nu}$ ··· source term quadratic in ϕ

with the boundary condition at the brane:

$\partial_r(b^2 E_{\mu\nu}) = \sigma_{\mu\nu}\,;$ $\quad\sigma_{\mu\nu}$ ~ energy momentum of ϕ on the brane

Strategy:

1. Solve $\delta\phi$ in the AdS bulk.
2. Take the perturbation of $\mathcal{L}E_{\mu\nu} = S_{\mu\nu}$:

$$\mathcal{L}\,\delta E_{\mu\nu} = \delta S_{\mu\nu}\,; \quad \delta S_{\mu\nu} \sim \phi(t,r)\delta\phi(t,r,x^i)$$

3. Solve $\mathcal{L}\,\delta E_{\mu\nu} = \delta S_{\mu\nu}$ by the Green function method:

$$\delta E(x) \sim \int_{\text{bulk}} d^5x' G(x,x')\delta S(x') + \int_{\text{brane}} d^4x' \partial_r(b^2\delta E(x'))G(x,x')$$

4. Analyze the late time behavior of $\delta E_{\mu\nu}$ on the brane.

$$\Downarrow$$

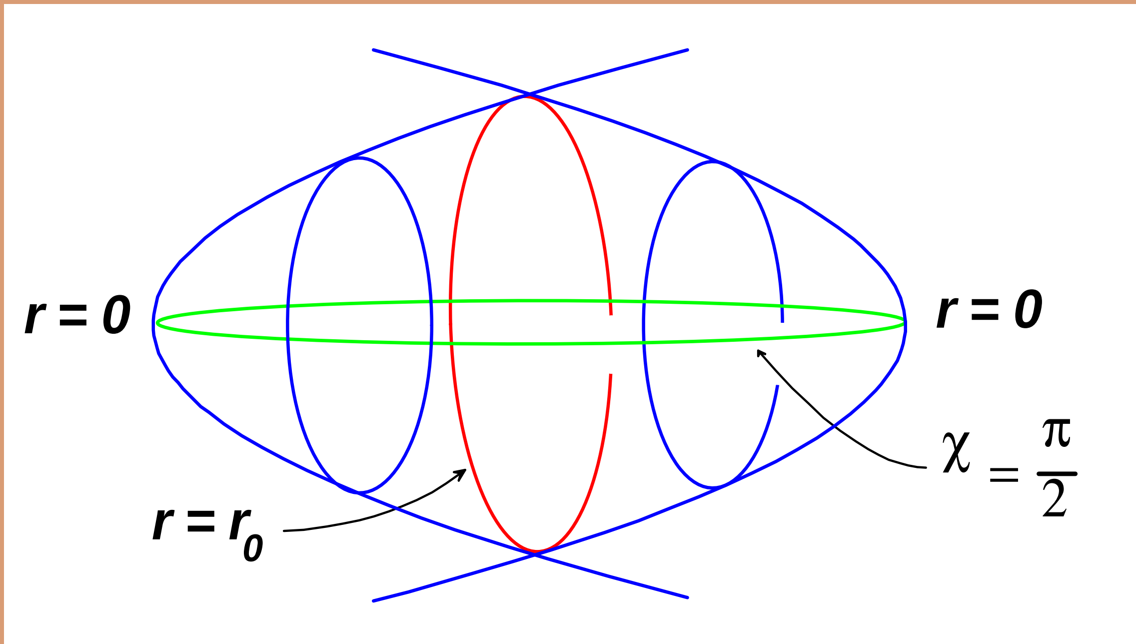
$$\delta G_{\mu\nu} = \kappa_4^2\,\delta T^{(s)}_{\mu\nu} - \delta E_{\mu\nu}\,.$$

Cosmological perturbation theory on the brane

§5. Quantum brane-cosmology (some remarks)

Geometry approximated by Euclidean AdS$_5$:

$$ds^2 = dr^2 + \ell^2 \sinh^2(r/\ell)(d\chi^2 + \sin^2\chi d\Omega^2_{(3)}) \; ; \quad r \leq r_0$$



Analytic continuation to an inflating brane at $\chi = \pi/2$:

$$\chi \to \frac{\pi}{2} + iHt$$

⋆ **Eulidean field equation for ϕ in AdS_5 $(r/\ell \to r)$**

$$\left[\frac{1}{\sinh^2 r\, \sin^3\chi}\frac{\partial}{\partial\chi}\sin^3\chi\frac{\partial}{\partial\chi}+\frac{1}{\sinh^4 r}\frac{\partial}{\partial r}\sinh^4 r\frac{\partial}{\partial r}-m^2\ell^2\right]\phi(r,\chi)=0$$

$$\text{b.c.:}\qquad \partial_r\phi(0,\chi)=\partial_r\phi(r_0,\chi)=0\,,$$

$$\partial_\chi\phi(r,0)=0\,,\qquad \partial_\chi\phi(r,\pi/2)=0\,(?)$$

- **Lowest order in $m^2\ell^2$:**

$$\phi=\phi_0=\text{const.}.$$

- **Effect of $m^2\ell^2\neq 0$:**

  · **Is $\phi=0$ (Hawking-Moss instanton) a unique solution when $H^2\ll|m^2|$?**

  · **Maybe $\partial_\chi\phi(r,\pi/2)=0$ should not be imposed.**
  (cf. 4D mini-superspace quantum cosmology)

§6. Summary

- ⋆ Brane-world inflation can be induced by dynamics of a bulk scalar field.
- ⋆ If $|m^2|\ell^2 \ll 1$, the zero-mode dominates the brane dynamics at late times.

  ($m_{\rm eff}^2 = m^2/2$ holds irrespective of the value of H^2/m^2)
- ⋆ If ϕ interacts with matter on the brane, reheating proceeds in the same way as in 4D models (Yokoyama & Himemoto '01)

The model is indistinguishable from a 4D theory at $O\left((m^2\ell^2)^0\right)$

$\Rightarrow$ Need to quantify the effect of $O(m^2\ell^2, H^2\ell^2)$.

- ⋆ The effect of quantum fluctuations of $T^{(s)}_{\mu\nu} - E_{\mu\nu}$.
- ⋆ Initial condition for the brane inflation

  Euclidean instanton that matches to an inflating brane.

quantum brane cosmology