Cosmic Inversion

– reconstructing the primordial spectrum from CMB –

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based on

M. Matsumiya, MS & J. Yokoyama, PRD65, 083007 (2002).

M. Matsumiya, MS & J. Yokoyama, JCAP02, 003 (2003).

N. Kogo, M. Matsumiya, MS & J. Yokoyama, in preparation.

§1. Introduction

CMB observations:

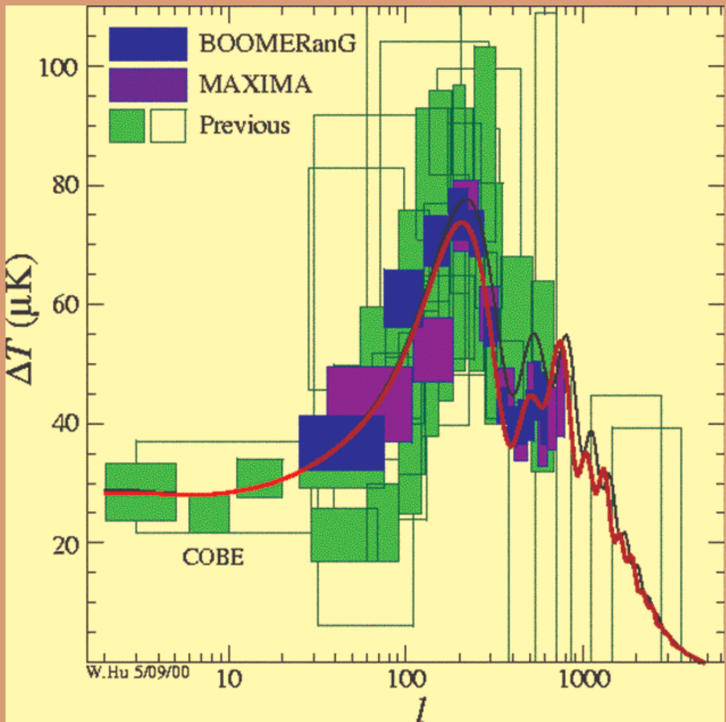
⋆ COBE confirmed the existence of super-horizon primordial perturbations.

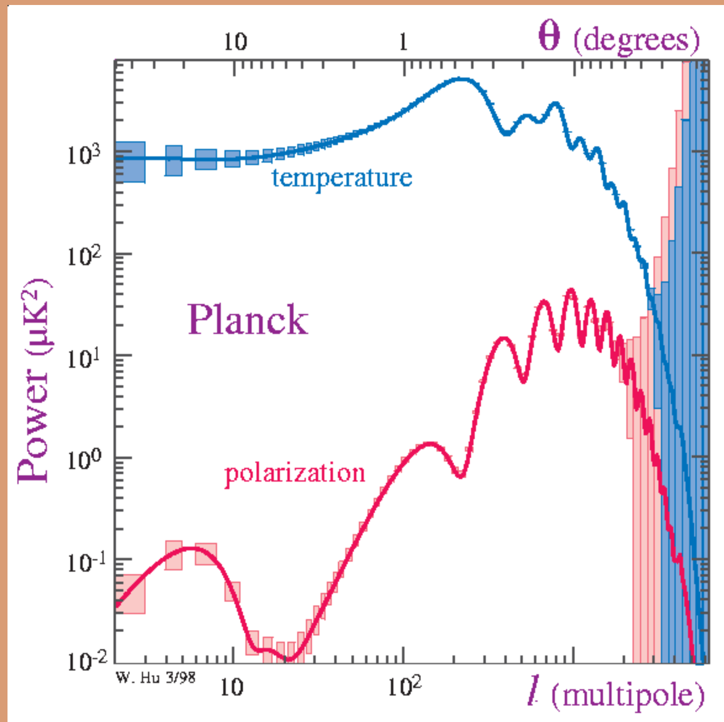
⋆ BOOMERanG and MAXIMA detected 1st and 2nd accoustic peaks.

  ⇒ strong support for the gravitational instability scenario.

⋆ MAP and PLANCK will accurately determine C_ℓ to $\ell \sim 1500$.

  ⇒ bringing up cosmology to precision physics.





What can we do? Or what should we do now?

So far,

- No truely realistic model of the early universe.
- No precise prediction on the primordial perturbation spectrum $P(k)$
- Testing various models of $P(k)$ by "likelihood analysis"

Can't we determine (or contrain) $P(k)$ directly from CMB data?

$$\Downarrow$$

Formulate the inverse problem of reconstructing $P(k)$ from C_ℓ

⋆ The angular correlation function:

$$C(\theta) = \langle \Theta(\vec{\gamma}_1)\Theta(\vec{\gamma}_2)\rangle = \sum_l \frac{2l+1}{4\pi} C_l \, P_l(\cos\theta) \, ;$$

⋆ Inverse problem:

$$C_\ell = \int dk \, K(\ell, k) P(k) \quad \Rightarrow \quad P(k) = \sum_\ell K^{-1}(k, \ell) C_\ell \quad ?$$

§2. CMB anisotropy and cosmic inversion

⋆ Multipole moment decomposition of $\Theta(\eta, k, \mu)$:

$$\Theta(\eta, k, \mu) = \sum_l (-i)^l \Theta_l(\eta, k) P_l(\mu)\,; \quad \mu = \frac{\vec{k}\cdot\vec{\gamma}}{k}$$ (we focus on a flat universe)

⋆ Boltzmann equation for $\Theta(\eta, \vec{k})$: (in Newton gauge)

$$\dot{\Theta} + ik\mu(\Theta + \Psi) = -\dot{\Phi} + an_e\sigma_T\left[\Theta_0 - \Theta - i\mu V_b - \frac{1}{10}\Theta_2 P_2(\mu)\right]; \quad {}^{\cdot} = \frac{\partial}{\partial\eta}$$

Ψ: Newton potential (lapse function) perturbation

Φ: Spatial curvature perturbation on Newton slice

⋆ Integral expression for $\Theta(\eta, k, \mu)$: (η_0 : conformal time today)

$$(\Theta + \Psi)(\eta_0, k, \mu) = \int_0^{\eta_0} \left\{\left[\Theta_0 + \Psi - i\mu V_b - \frac{1}{10}\Theta_2 P_2(\mu)\right]\mathcal{V}(\eta) + (\dot{\Psi} - \dot{\Phi})e^{-\tau(\eta)}\right\} e^{ik\mu(\eta-\eta_0)} d\eta\,,$$

where $\mathcal{V}(\eta)$ is called the visibility function:

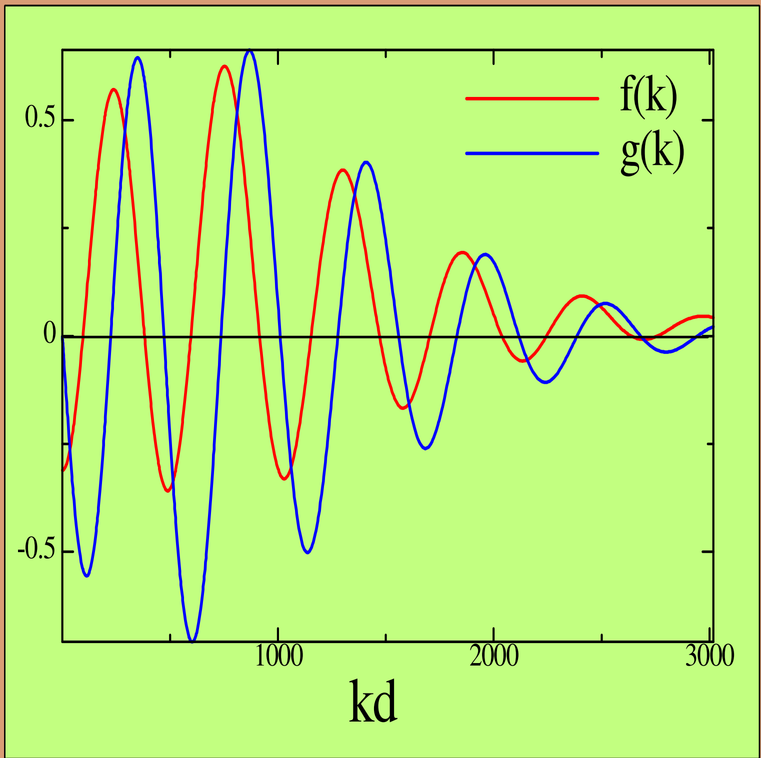
$$\mathcal{V}(\eta) = an_e\sigma_T e^{-\tau(\eta)}\,; \quad \tau(\eta) = \int_\eta^{\eta_0} an_e\sigma_T\, d\eta'.$$

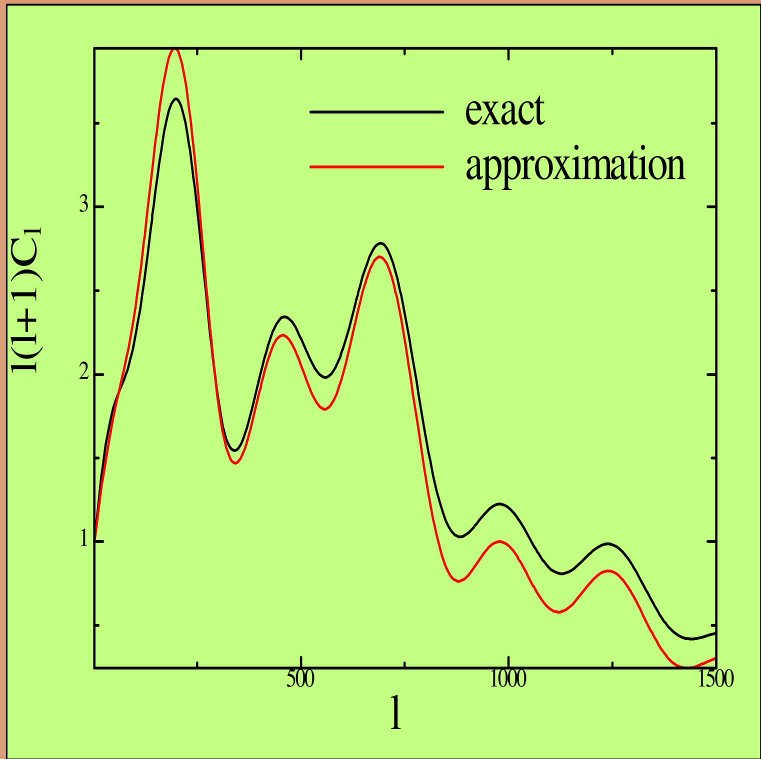
⋆ **Approximation for the last scattering surface (LSS) with a thin but non-zero width:**

$$(\Theta+\Psi)(\eta_0,k,\mu) \approx \int_0^{\eta_0} d\eta \left((\Theta_0+\Psi)\mathcal{V}(\eta) + (\dot{\Psi}-\dot{\Phi})e^{-\tau(\eta)} - i\mu V_b \mathcal{V}(\eta) \right) e^{ik\mu(\eta_*-\eta_0)} .$$

where η_* is the center of LSS, and the Θ_2 term is neglected. This gives

$$\Theta_l(\eta_0,k) \approx (2l+1)\left[\, f(k)\, j_l(kd) + g(k)\, j_l'(kd) \right] \; ; \quad d = \eta_0 - \eta_*(\approx \eta_0)$$





If the approximate formula were exact, we obtain

$$\begin{aligned}\tilde{C}(r) &\equiv 3rC(r) + r^2C'(r)\\ &= \frac{1}{2\pi^2}\int_0^\infty dk\,P(k)\left\{f^2(k)k^2r\cos kr + (2f^2(k)+g^2(k))k\sin kr\right\};\end{aligned}$$

$P(k)$: primordial spectrum of Ψ (Newton potential), $\quad r \equiv 2d\sin\frac{\theta}{2} \approx d\,\theta$

This leads to a first-order differential equation for $P(k)$:

$$-k^2f^2P'(k) + (-2kff' + g^2)kP(k) = 4\pi\int_0^\infty \tilde{C}(r)\sin kr dr \equiv S(k)\,.$$

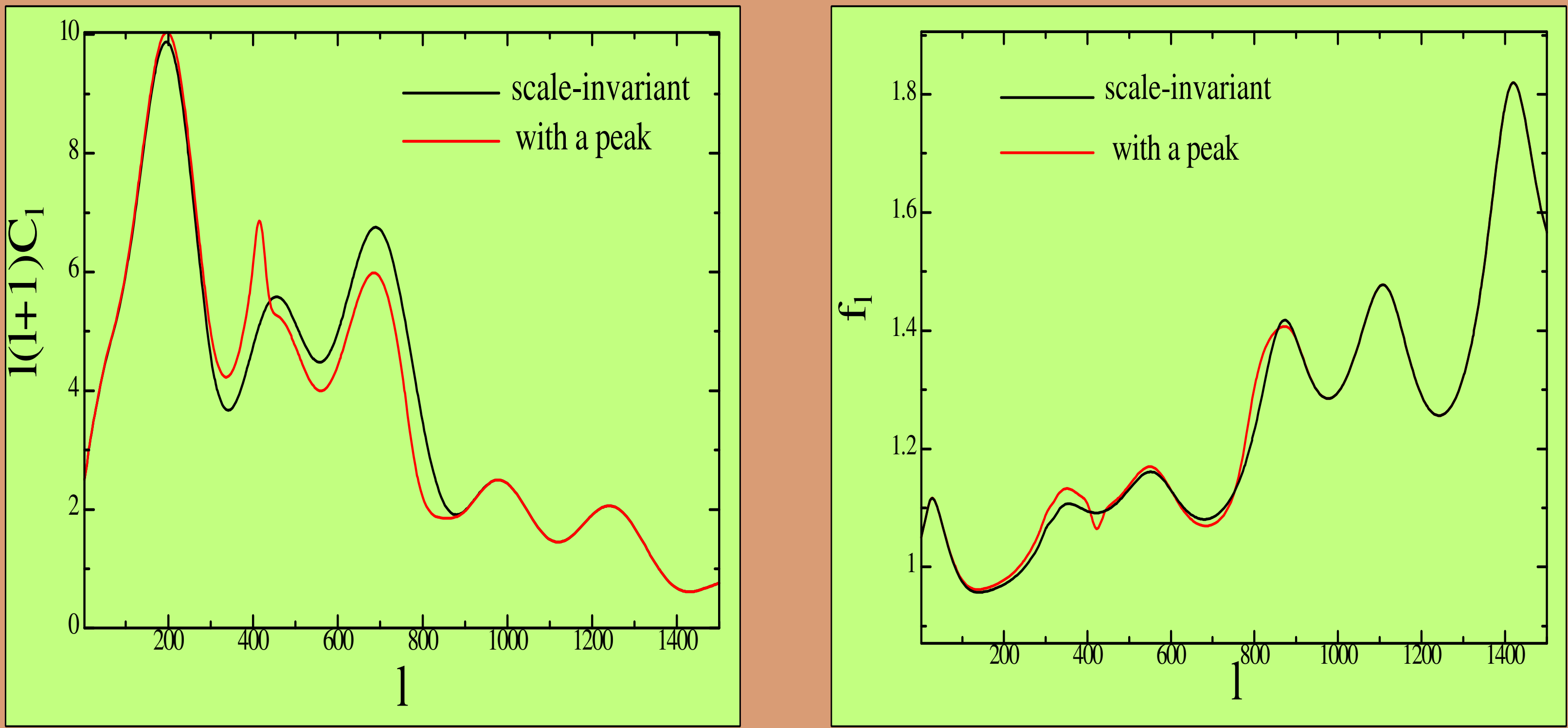
- ∗ The equation is singular at $f = 0$.
  Let $k = k_a$ ($a = 1, 2, \cdots$) be the singularities.
- ∗ At $k = k_a$, $P(k_a)$ is readily obtained as $P(k_a) = \dfrac{S(k_a)}{g^2(k_a)k_a}$
- ∗ $P(k)$ between the singularities can be easily solved as a boundary value problem.

In reality, however, the above equation holds only approximately.

ANY WAY-OUT?

Fortunately, we find empirically

$$f_\ell \equiv \frac{C_\ell^{\text{exact}}}{C_\ell^{\text{appx}}} \sim \text{ approximately independent of the shape of } P(k).$$



Comparisons of C_ℓ and f_ℓ for two very different spectra.

§3. Method

⋆ **Procedure of inversion:**

1. Pick up a fiducial spectrum $P^{(0)}(k)$, e.g., a scale-invariant spectrum.
2. Calculate C_ℓ and C_ℓ^{appx} for $P^{(0)}(k)$. Denote them by $C_\ell^{(0)}$ and $C_\ell^{\text{appx}(0)}$, respectively.
3. Estimate C_ℓ^{appx} of C_ℓ^{obs}, assuming $f_\ell = f_\ell^{(0)}$:

$$f_\ell^{(0)} = \frac{C_\ell}{C_\ell^{\text{appx}}} \quad \Rightarrow \quad C_\ell^{\text{appx}(1)} = \frac{C_\ell^{\text{obs}}}{f_\ell^{(0)}}$$

4. Reconstruct $P^{(1)}(k)$ from $C^{\text{appx}(1)}$. Calculate $C_\ell^{(1)}$ for $P^{(1)}(k)$, and repeat the procedure until it converges.

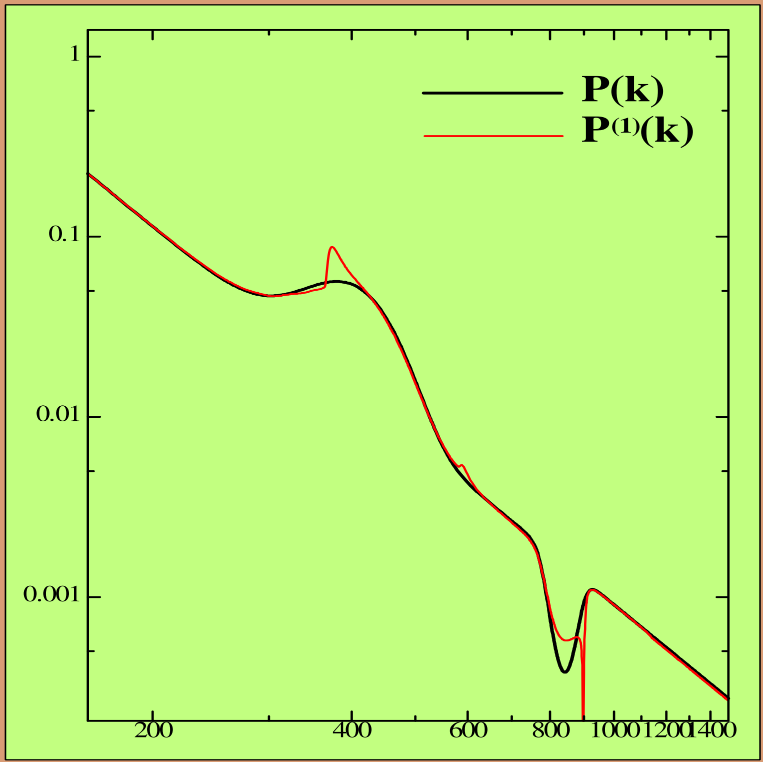
Schematically,

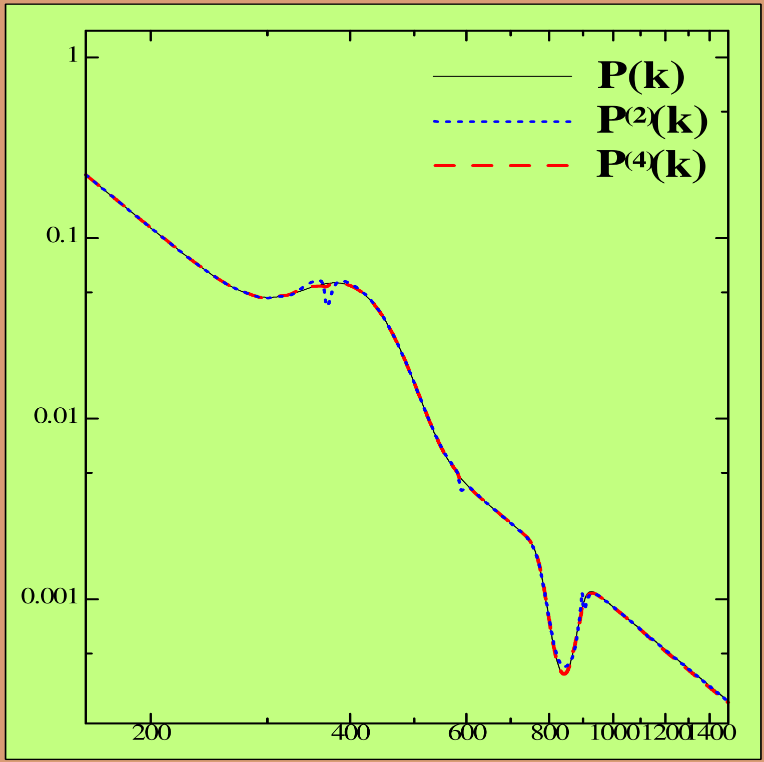
$$P^{(n)}(k) \Rightarrow f_\ell^{(n)} = \frac{C_\ell^{(n)}}{C_\ell^{\text{app}(n)}} \Rightarrow C_\ell^{\text{app}(n+1)} = \frac{C_\ell^{\text{obs}}}{f_\ell^{(n)}} \Rightarrow P^{(n+1)}(k) \Rightarrow \cdots$$

⋆ Test of the method

Example:

$P(k)$ with a peak and a dip superimposed on a scale-invariant spectrum



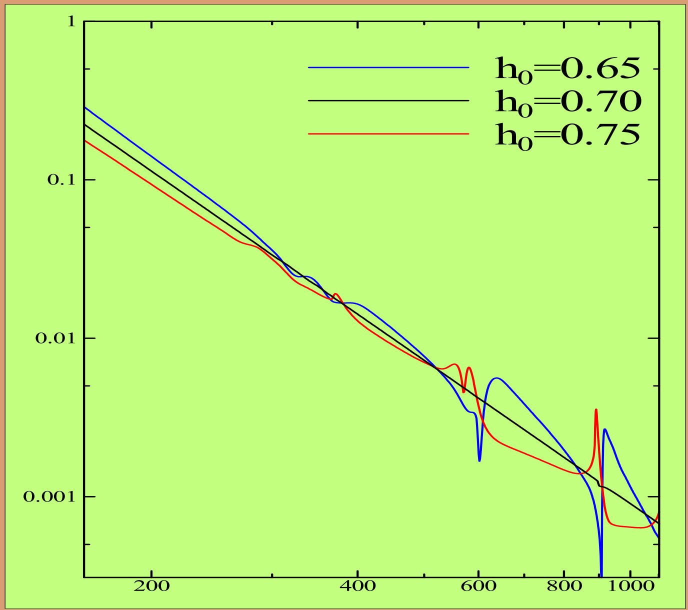


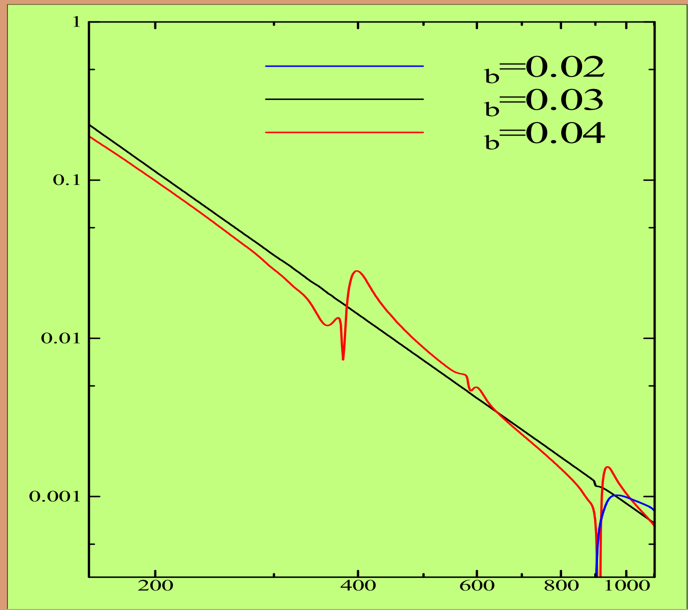
If the correct cosmological parameters are chosen, the convergence turns out to be very fast.

§4. Constraints on cosmological parameters

Assume the spectrum is scale-invariant and the cosmological parameters are

$$h_0 = 0.7\,, \quad \Omega_0 = \Omega_{CDM} = 1\,, \quad \Omega_b = 0.03\,.$$





* Spiky features appear at the singularities for different parameters. They do not disappear after iteration.
* For H_0 smaller than the 'real' value, the spectrum becomes negative near the singularities.
* In the case of Ω_b, $P(k)$ becomes negative in either direction of deviations.

§5. Application to WMAP data

- Reconstruction of $P(k)$ from binned (averaged) C_ℓ data

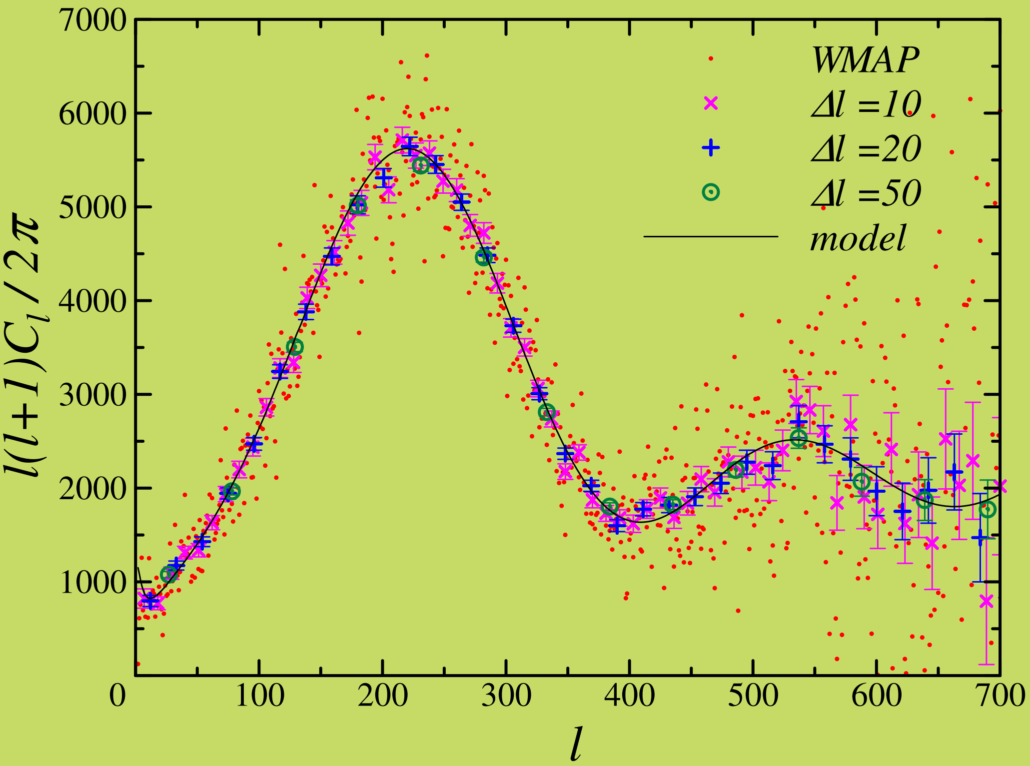
(We assume the cosmological parameters suggested by the WMAP team)

WMAP data: red dots (·)
scattered because of
cosmic variance

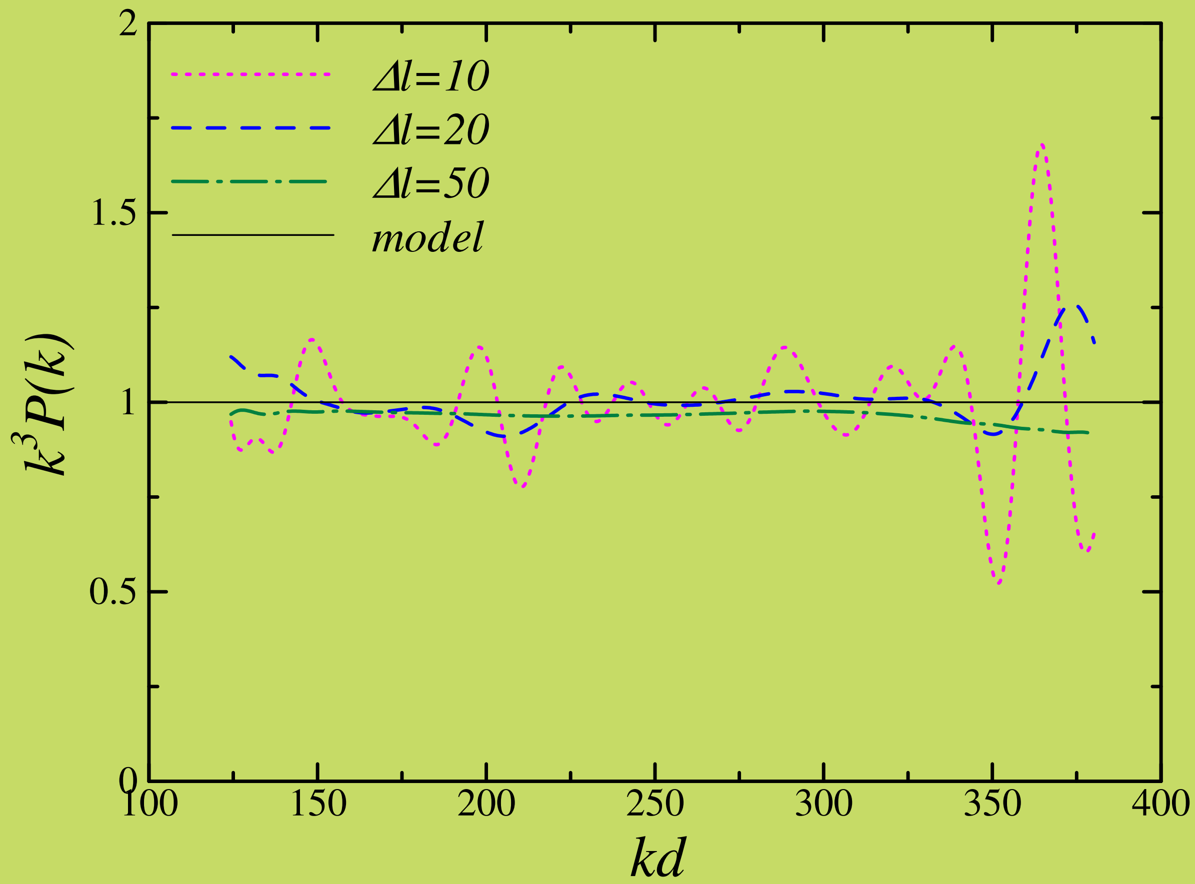
$\Delta\ell$ = binning size

(∼ best-fit) model:

scale-invariant spectrum
with
$h = 0.72$, $\Omega_{tot} = 1$,
$\Omega_b = 0.047$, $\Omega_\Lambda = 0.71$



- **Reconstructed spectra**



· **Oscillatory behavior appears for data with $\Delta = 10$ and 20.**

· **Is this real or acccidental (due to cosmic variance)?**

- χ^2-test of reconstructed spectra against WMAP data

	model	$\Delta\ell = 10$	$\Delta\ell = 20$	$\Delta\ell = 50$
χ^2	975	950	972	986
χ^2/d.o.f	1.090	1.091	1.104	1.110
Probability	3.05%	3.10%	1.69%	1.20%

- χ^2-test against simulated data from scale-invariant spectrum

	model	$\Delta\ell = 10$	$\Delta\ell = 20$	$\Delta\ell = 50$
χ^2	987	967	973	1002
χ^2/d.o.f	1.104	1.111	1.105	1.129
Probability	1.63%	1.22%	1.61%	0.437%

	model	$\Delta\ell = 10$	$\Delta\ell = 20$	$\Delta\ell = 50$
χ^2	930	912	921	970
χ^2/d.o.f	1.040	1.049	1.046	1.093
Probability	19.5%	15.5%	16.9%	2.74%

	model	$\Delta\ell = 10$	$\Delta\ell = 20$	$\Delta\ell = 50$
χ^2	918	905	923	944
χ^2/d.o.f	1.026	1.040	1.048	1.063
Probability	28.5%	19.8%	15.9%	9.31%

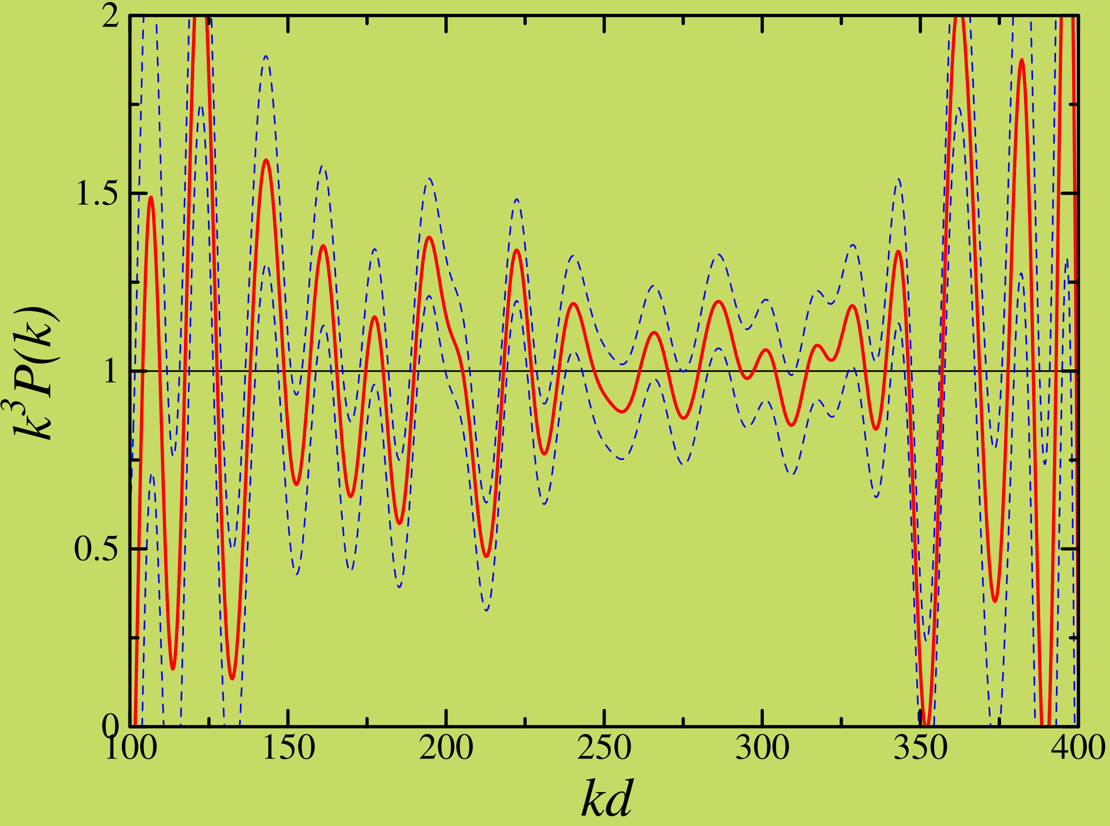
- **Another test $\cdots$ Reconstruction without binning C_ℓ**

Construct a statistical set of $P(k)$ by

$$C_\ell^{\rm WMAP} \quad \Rightarrow \quad \{C_\ell^1,\, C_\ell^2, \cdots\} \quad \Rightarrow \quad \{P_1(k),\, P_2(k), \cdots\}$$

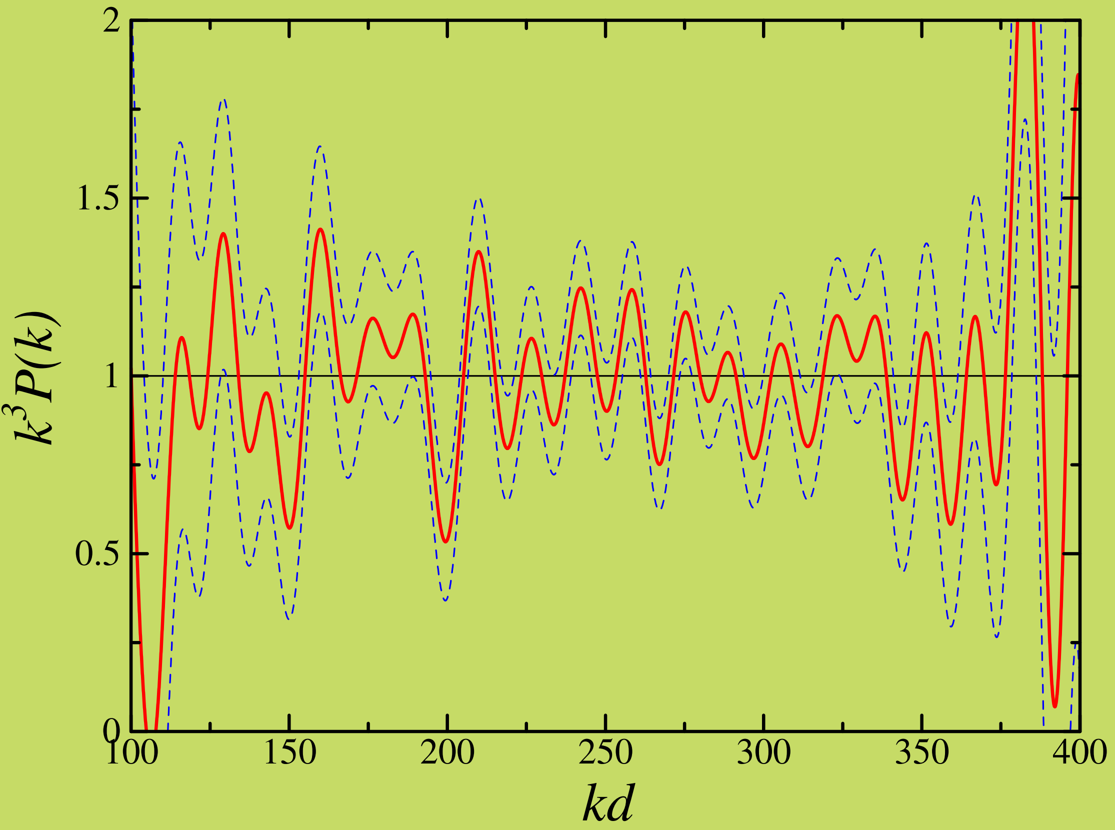
$\Uparrow$

add cosmic variance by Monte Carlo



$\overline{P(k)}$ and $\Delta P(k)$ for WMAP data

Do the same for $C_\ell^{\text{Simulated}}$ from scale-invariant $P^{\text{model}}(k)$



$\overline{P(k)}$ and $\Delta P(k)$ for Simulated data

- **Oscillations in WMAP case look a bit more prominent.**
- **But probably consistent with oscillations due to cosmic variance.**

Need more systematic statistical analysis

§6. Summary

- Cosmic inverse problem is formulated based on an approximate formula $C_\ell^{\rm appx}$ and an approximate independence of $C_\ell/C_\ell^{\rm appx}$ on $P(k)$.
  - For $\Omega_{tot} = 1$ universe, $P(k)$ can be reconstructed with good accuracy.
  - Our formalism is applicable also to ΛCDM models, with $h_0^2 \to h_0^2\Omega_{CDM}$.
- Our formalism provides an entirely new way to constrain cosmological parameters.
  - Models with smaller values of $h_0^2\Omega_{CDM}$ may be excluded with a high confidence level.
  - The baryon density parameter $h_0^2\Omega_b$ is severely constrained.
  - When applied to WMAP data, our method suggests an oscillatory $P(k)$.

⋆ Future issues:

- Extension to spatially curved cosmological models.
- Inclusion of the CMB polarization spectrum.
  - may be able to determine the tensor spectrum at the same time.
- Need to develop a more systematic method of statistical analysis.