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Inflation - Lecture 1

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Inflation

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contents

1. Inflationary Universe

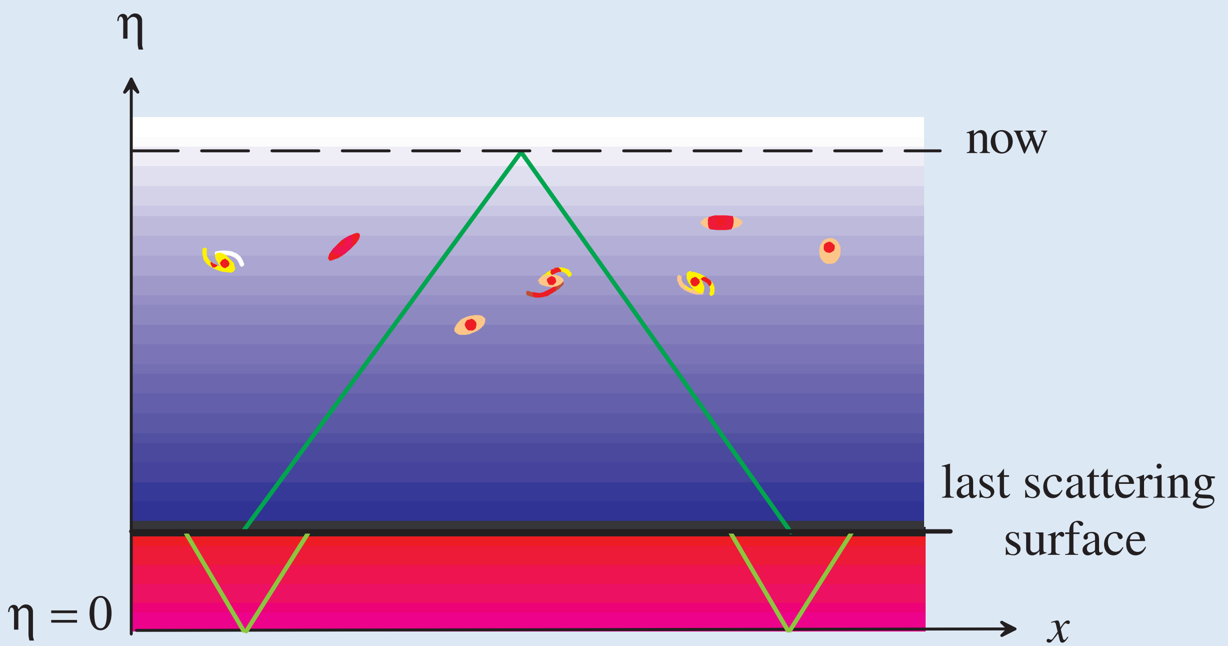
- horizon problem

$$ds^2 = -dt^2 + a^2(t)d\sigma^2_{(3)} \qquad \ddot{a} = -\frac{4\pi G}{3}(\rho + 3P)a < 0 \ \text{ for } \ P > -\frac{\rho}{3}$$

if $a \propto t^n, \ n < 1$

gravity=attractive

$$ds^2 = a^2(\eta)(-d\eta^2 + d\sigma^2_{(3)}) \qquad d\eta = \frac{dt}{a(t)} : \ \text{conformal time}$$



$$\int_{t_0 \to 0}^{t} \frac{dt}{a(t)} = \text{finite}$$

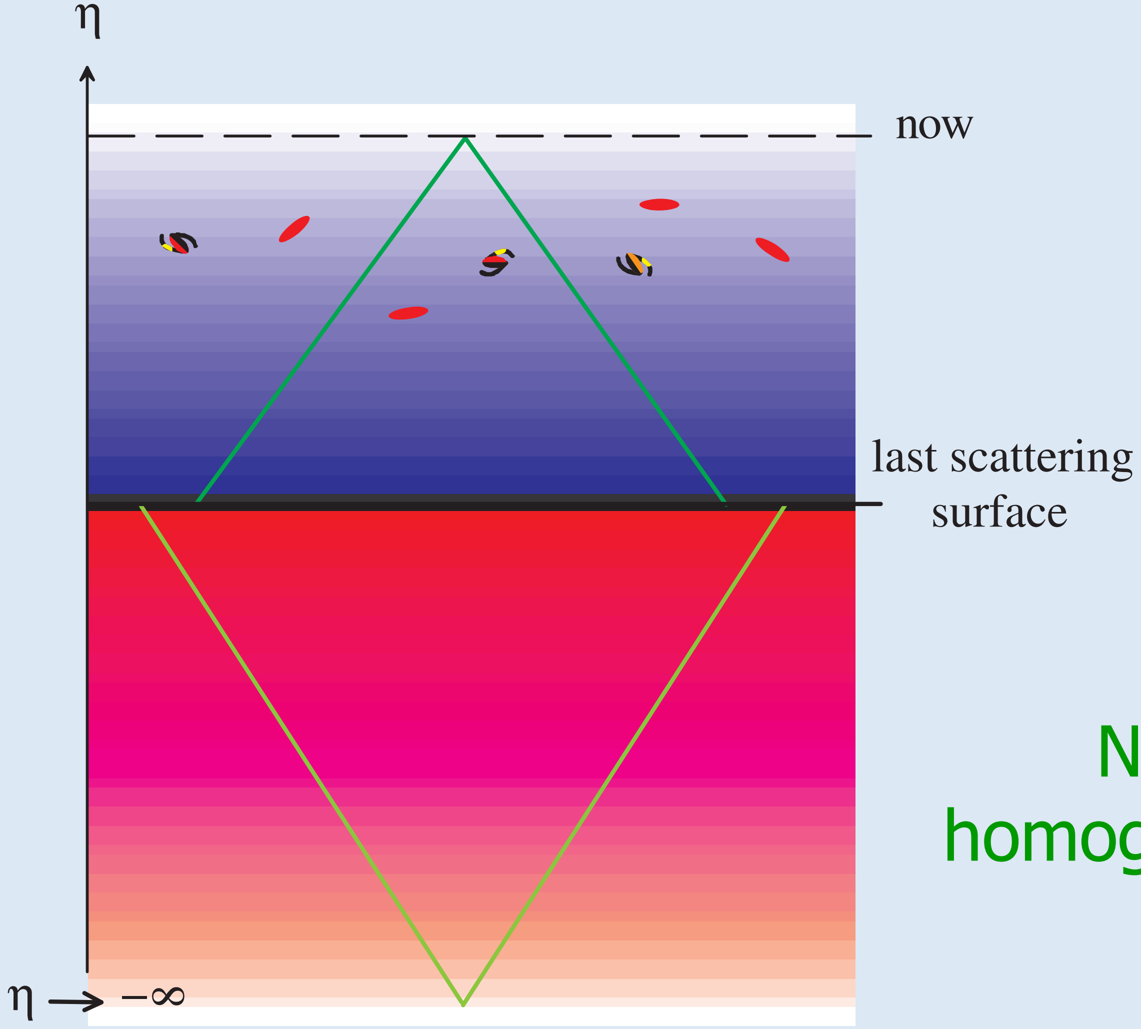
conformal time is bounded from below

∃ particle horizon

- solution to the horizon problem

$$\ddot{a} = -\frac{4\pi G}{3}(\rho + 3P)a > 0$$

for a sufficient lapse of time in the early universe



$$\eta - \eta_0 = \int_{t_0}^{t} \frac{dt}{a(t)} \xrightarrow[t_0 \to 0]{} \infty$$

or large enough to cover the present horizon size

NB: horizon problem ≠ homogeneity & isotropy problem

- flatness problem (= entropy problem)

$$H^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2}; \qquad -\infty < K < +\infty$$

if $\rho \propto a^{-4}$, $\rho \gg \frac{|K|}{a^2}$ in the early universe.

conversely if $\rho \approx |K|/a^2$ at an epoch in the early universe, the universe must have either collapsed (if $K>0$) or become completely empty (if $K<0$) by now.

alternatively, the problem is the existence of huge entropy within the curvature radius of the universe

$$S = T^3\left(\frac{a}{\sqrt{|K|}}\right)^3 \approx T_0^3\left(\frac{a_0}{\sqrt{|K|}}\right)^3 > T_0^3 H_0^{-3} \approx 10^{87}$$

(# of states = exp[S])

solution to horizon & flatness problems

spatially homogeneous scalar field:

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi), \quad P = \frac{1}{2}\dot{\phi}^2 - V(\phi)$$

$\Rightarrow$ $\rho + 3P = 2\left(\dot{\phi}^2 - V(\phi)\right) < 0$ if $\dot{\phi}^2 < V(\phi)$

$\Rightarrow$ $\rho \approx -P \approx V(\phi)$ if $\dot{\phi}^2 \ll V(\phi)$ potential dominated

V ~ cosmological const./vacuum energy

$\rho \approx const.$ $\frac{K}{a^2}$ decreases rapidly

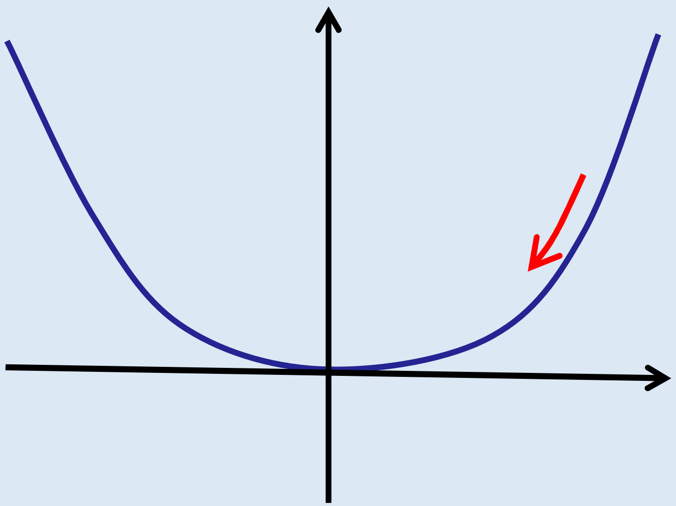
$\Rightarrow$ $H^2 \approx \frac{8\pi G}{3}\rho \approx const.$ inflation

"vacuum energy" converted to radiation
after sufficient lapse of time

solves horizon & flatness problems simultaneously

slow-roll inflation

- single-field slow-roll inflation Linde '82, ...



metric: $ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j$

field eq.:

$$\cancel{\ddot{\phi}} + 3H\dot{\phi} + V'(\phi) = 0 \;\Rightarrow\; \dot{\phi} = -\frac{V'(\phi)}{3H}$$

$$\left(\frac{\dot{a}}{a}\right)^2 \equiv H^2 = \frac{8\pi G}{3}\left[\cancel{\frac{1}{2}\dot{\phi}^2} + V(\phi)\right]$$

$$\Rightarrow \quad -\frac{\dot{H}}{H^2} = \frac{\frac{3}{2}\dot{\phi}^2}{\frac{1}{2}\dot{\phi}^2 + V} \approx \frac{3}{2}\frac{\dot{\phi}^2}{V} \ll 1$$

… slow variation of H

$$a \sim e^{Ht}$$

inflation!

slow-roll conditions

$$\varepsilon \equiv -\frac{\dot{H}}{H^2} = \frac{\frac{3}{2}\dot{\phi}^2}{\frac{1}{2}\dot{\phi}^2 + V} \approx \frac{3}{2}\frac{\dot{\phi}^2}{V} = \frac{M_P^2 V'^2}{2V^2} \ll 1$$

condition for quasi-de Sitter (inflationary) expansion

$$\delta \equiv \frac{\ddot{\phi}}{H\dot{\phi}} = \varepsilon + \frac{\dot{\varepsilon}}{2H\varepsilon} \approx \varepsilon - \frac{M_P^2 V''}{V}; \quad |\delta| \ll 1$$

condition for friction-dominated (over-damped) evolution

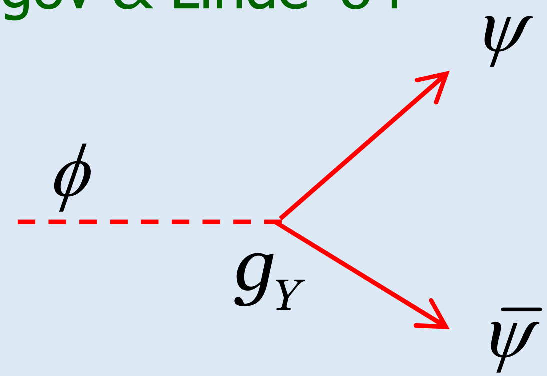
sufficient condition on potential:

$$\varepsilon_V \equiv \frac{M_P^2 V'^2}{2V^2} \ll 1, \;\; \eta_\mathrm{v} \equiv \frac{M_P^2 V''}{V}; \;\; |\eta_\mathrm{v}| \ll 1$$

reheating

Abbott & Wise ’84, Dolgov & Linde ‘84

- standard scenario



e.g. $L_{\text{int}} \sim g_Y \phi \bar{\psi} \psi$

decay rate: $\Gamma \sim g_Y^2 m_\phi; \quad m_\phi \gg m_\psi$

effective equation of motion: $V(\phi) = \frac{1}{2} m_\phi^2 \phi^2 + \cdots$

when $m_\phi \gg H > \Gamma$, damped oscillation:

$$\ddot{\phi} + 3H\dot{\phi} + m_\phi^2 \phi = 0 \quad \Rightarrow \quad \phi \propto a^{-3/2} \cos(m_\phi t + \alpha)$$

effect of Γ $\Rightarrow$ $\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = -\Gamma\dot{\phi}$

$$\Leftrightarrow \frac{d}{dt}\left(\frac{1}{2}\dot{\phi}^2 + V\right) = -(3H + \Gamma)\langle \dot{\phi}^2 \rangle$$

$$\Rightarrow \dot{\rho}_\phi + 3H\rho_\phi = -\Gamma\rho_\phi$$

energy conservation eqns

$$\dot{\rho}_\phi + 3H\rho_\phi = -\Gamma\rho_\phi$$

$$\dot{\rho}_r + 4H\rho_r = \Gamma\rho_\phi$$ ρ_r: produced radiation

- $\Gamma < H \sim t^{-1}$

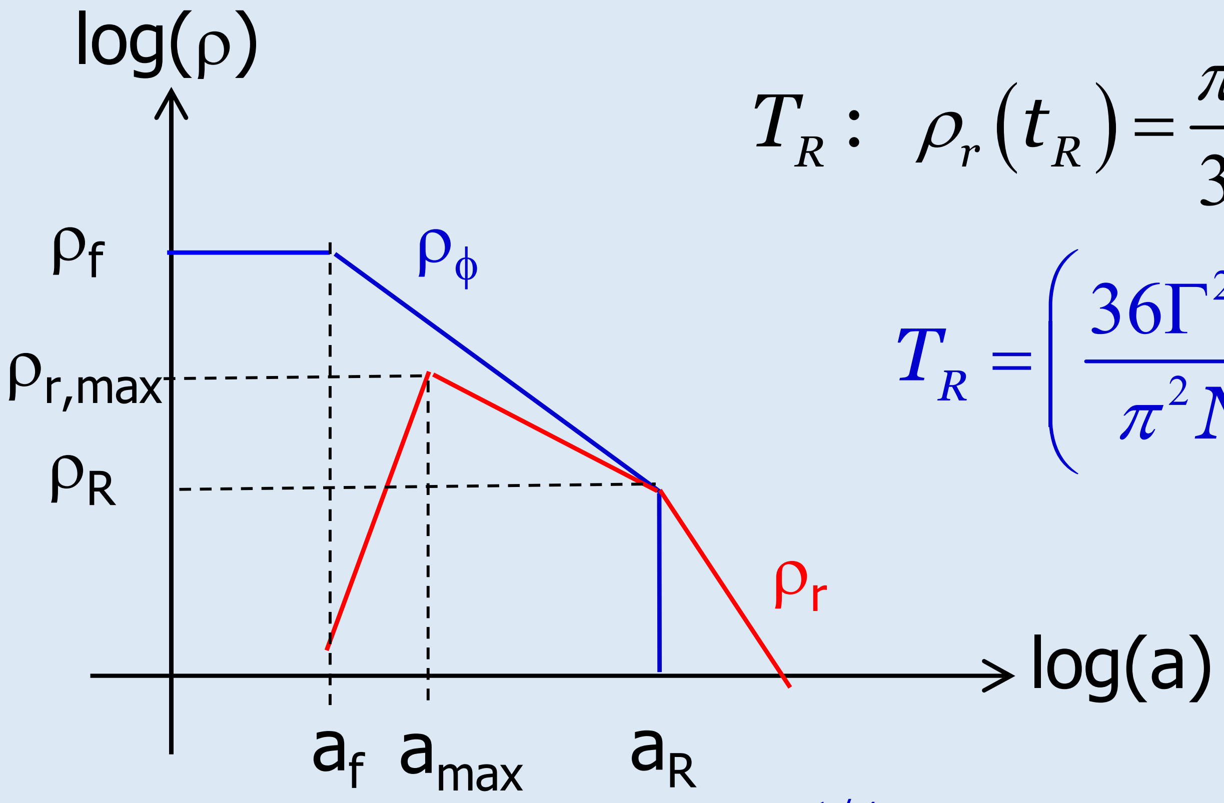
$$\rho_\phi = \rho_{\phi_f}\left(\frac{a}{a_f}\right)^{-3}, \quad \rho_r = \frac{2\Gamma}{5H_f}\rho_{\phi_f}\left(\frac{a}{a_f}\right)^{-4}\left(\left(\frac{a}{a_f}\right)^{5/2} - 1\right)$$

$\rho_r =$ max at $\frac{a}{a_f} = \left(\frac{8}{3}\right)^{2/5} \approx 1.48$

- $\Gamma > H \sim t^{-1}$

$$\rho_\phi = 0, \quad \rho_r = \rho_r(t_R)\left(\frac{a}{a_R}\right)^{-4}$$ t_R:def by $H(t_R) = \Gamma$

reheating temperature & max temperature



$$T_R: \quad \rho_r(t_R) = \frac{\pi^2}{30} N_{eff} T_R^4$$

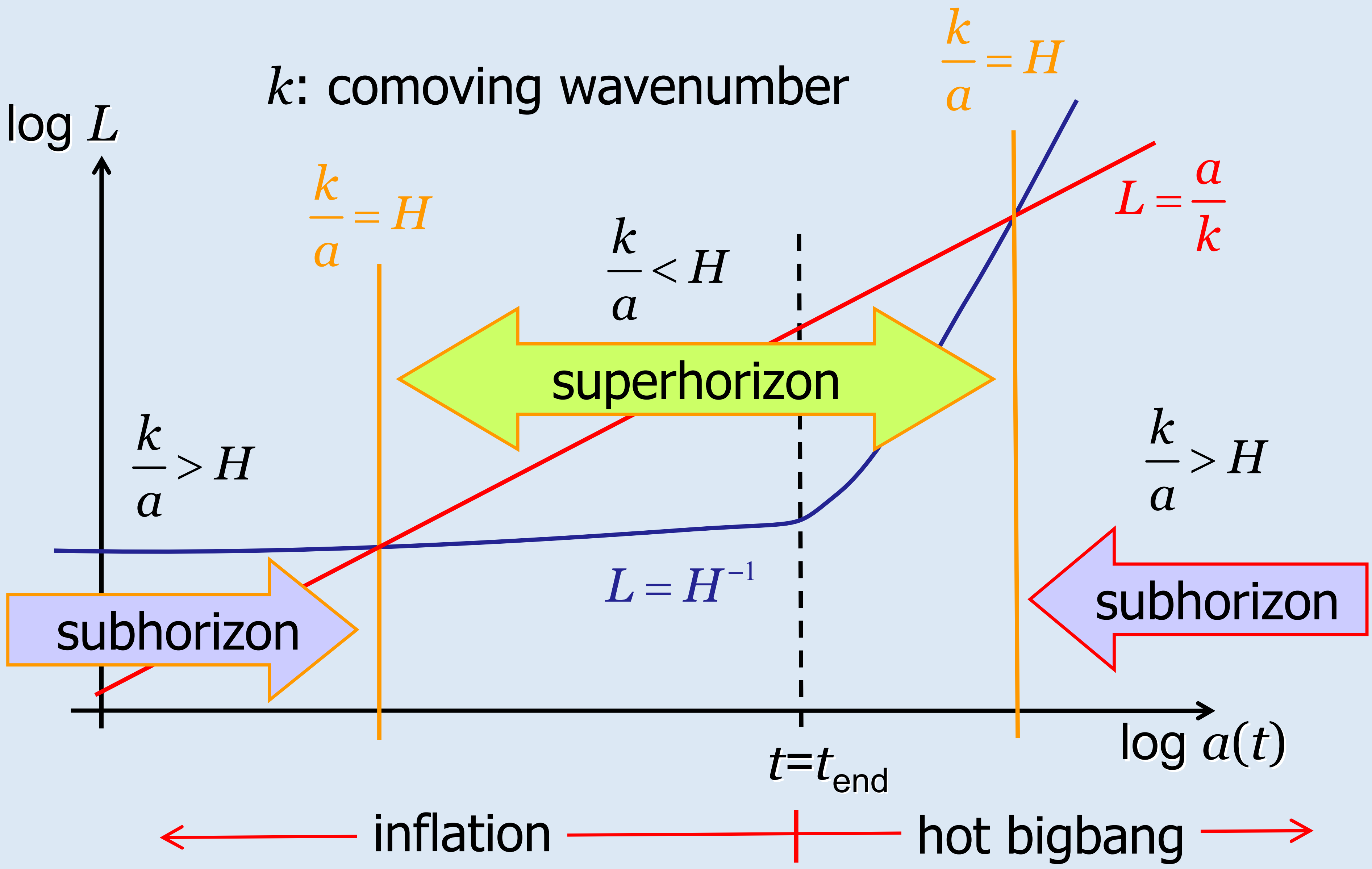
$$T_R = \left(\frac{36\Gamma^2 M_P^2}{\pi^2 N_{eff}} \right)^{1/4} \sim \frac{(\Gamma M_P)^{1/2}}{N_{eff}^{1/4}}$$

⇧ indep. of ρ_f

$$T_{max} \sim \left(\frac{H_f \Gamma}{M_P^2} \right)^{1/4} M_P \sim \left(\frac{H_f}{\Gamma} \right)^{1/4} T_R$$ ⇦ dep. on ρ_f

T_{max} is important for thermal history
T_R is important for horizon problem

comoving scale vs Hubble horizon radius



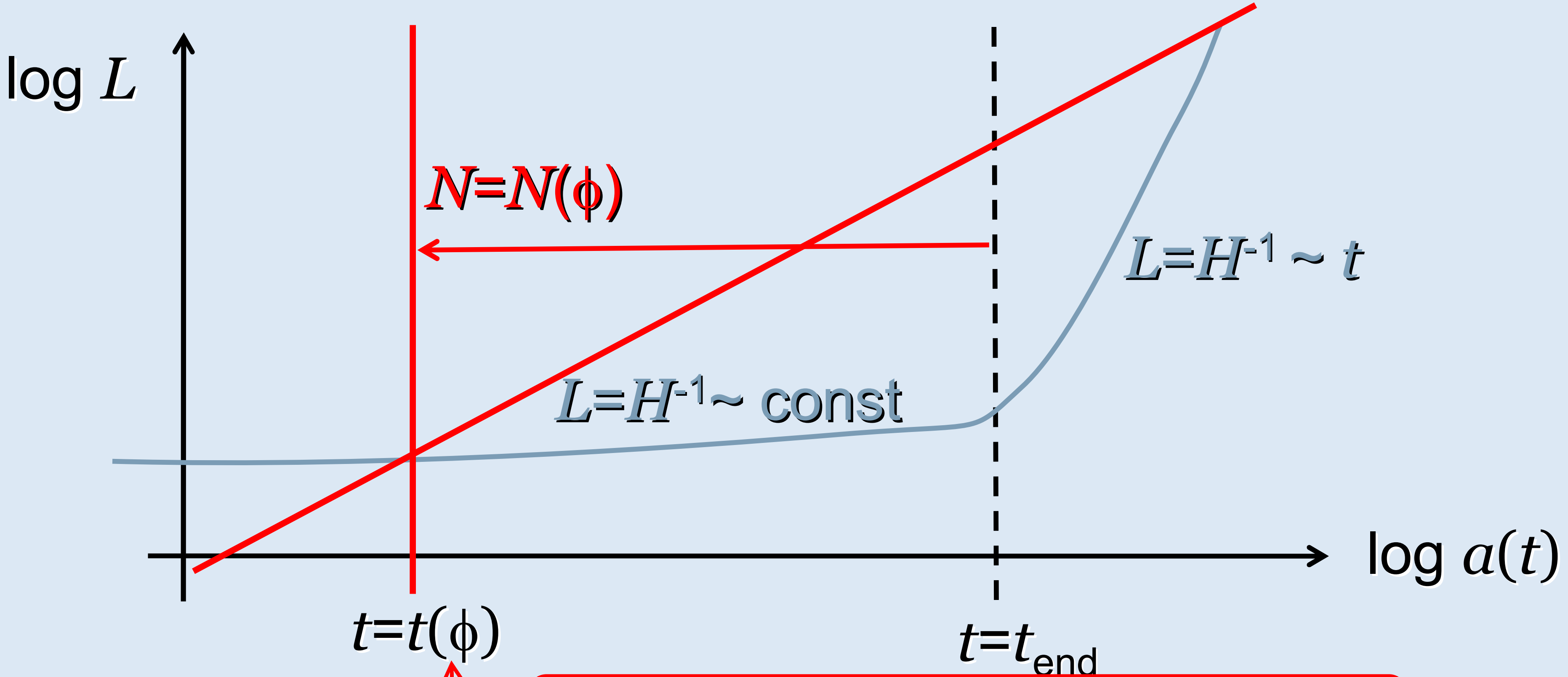
Hubble horizon=causal horizon for local physics

e-folding number: N

of e-folds from $\phi=\phi(t)$
until the end of inflation

redshift

$$\frac{a(t_{\rm end})}{a(t)} = \exp[N(t \to t_{\rm end})] \;\Rightarrow\; N = N(\phi) = \int_{t(\phi)}^{t_{\rm end}} H dt \sim \ln[1+z(\phi)]$$



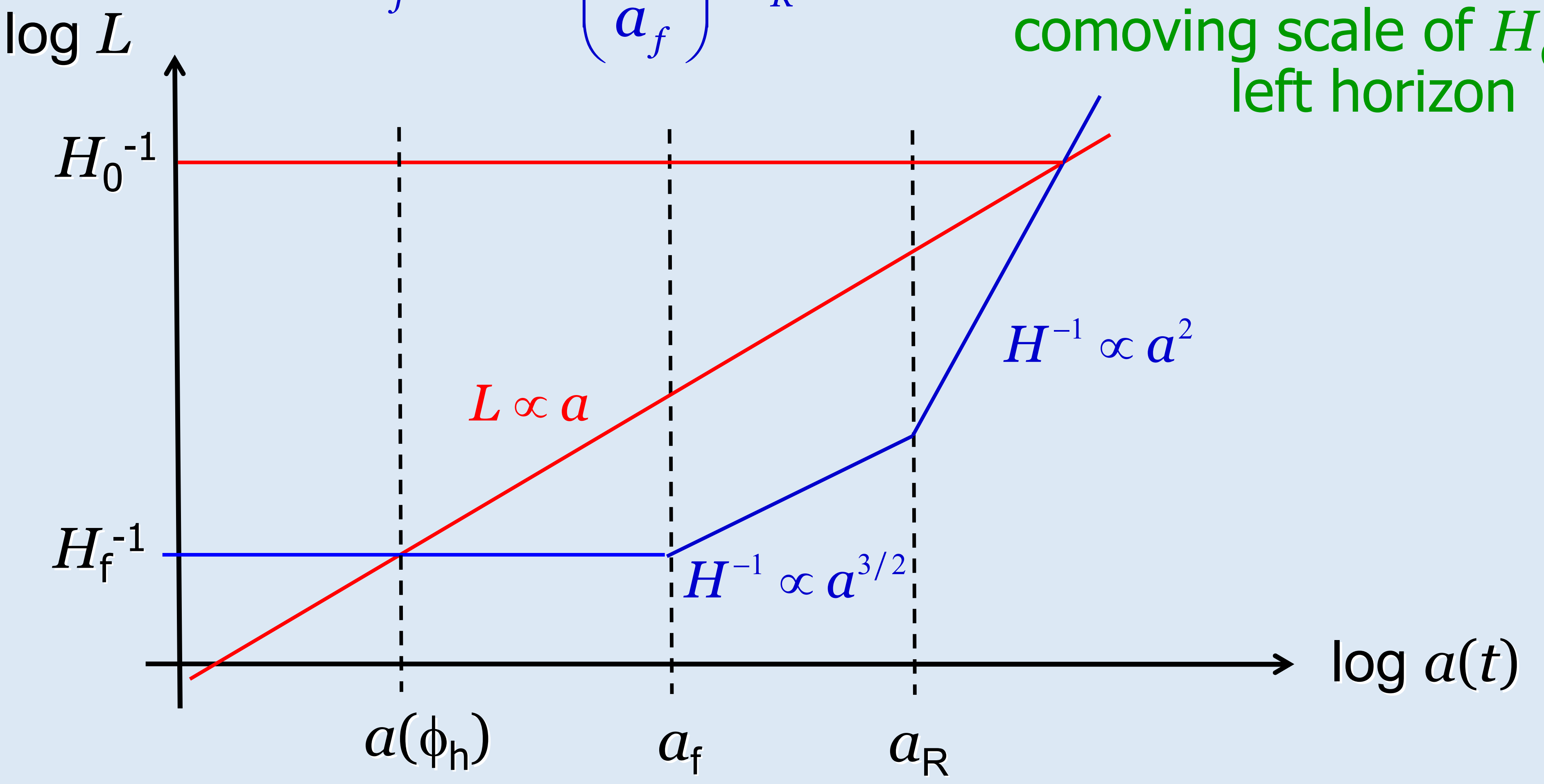
ϕ determines comoving scale k

condition on e-folding number

ignore variation of H during inflation.
entropy generated within present Hubble volume:

$$S = H_f^{-3} e^{3N(\phi_h)} \left(\frac{a_R}{a_f} \right)^3 T_R^3$$

$\phi_h \equiv$ value of ϕ at which comoving scale of H_0^{-1} left horizon



$$S = H_f^{-3} e^{3N(\phi_h)} \left(\frac{a_R}{a_f}\right)^3 T_R^3 \sim \left(\frac{\rho_f}{M_P^2}\right)^{-3/2} e^{3N(\phi_h)} \left(\frac{\rho_f}{T_R^4}\right) T_R^3$$

$$\approx \left(\frac{M_P^3}{T_R \rho_f^{1/2}}\right) e^{3N(\phi_h)} > \left(\frac{T_0}{H_0}\right)^3 \sim 10^{87}$$

$$\Rightarrow \quad N(\phi_h) > 53 + \frac{2}{3}\ln\left[\frac{\rho_f^{1/4}}{10^{15}\,\mathrm{GeV}}\right] + \frac{1}{3}\ln\left[\frac{T_R}{10^{10}\,\mathrm{GeV}}\right]$$

- N>50-60 solves horizon & flatness problems
- changing T_R by one order (by 10) changes N by 1

Q1. Show that conformal time η_h at $\phi=\phi_h$ satisfies $|\eta_h|>\eta_0$, where η_0 is the conformal time today.

preheating

Kofman, Linde & Starobinsky '94

If ϕ couples to other light scalar (bose) fields

e.g. $L_{\text{int}} \sim g\phi^2\chi^2, \quad m_\chi^2 \ll m_\phi^2$

catastrophic χ- particle creation can occur

$$\ddot{\chi}_k + 3H\dot{\chi}_k + \left((k/a)^2 + g\phi^2\right)\chi_k = 0$$

$$\phi^2 = \phi_f^2 (a_f/a)^3 \sin^2 m_\phi t$$

oscillating potential

for $m_\phi \Delta t \gg 1 \gtrsim H\Delta t$

$$\ddot{\chi}_k + \left((k/a)^2 + g\phi^2 \sin^2 m_\phi t\right)\chi_k = 0$$

$$\Leftrightarrow \quad \ddot{\chi}_k + \left(a - 2b\cos 2m_\phi t\right)\chi_k = 0$$ Mathiew eqn

possible parametric amplification of χ_k

instability bands

$$a = \left(\frac{k}{a m_\phi}\right)^2 + 2b$$

$$b = \left(\frac{g\phi^2}{4m_\phi^2}\right)$$

For $\dfrac{k}{a} \ll m_\phi,\ a = 2b$

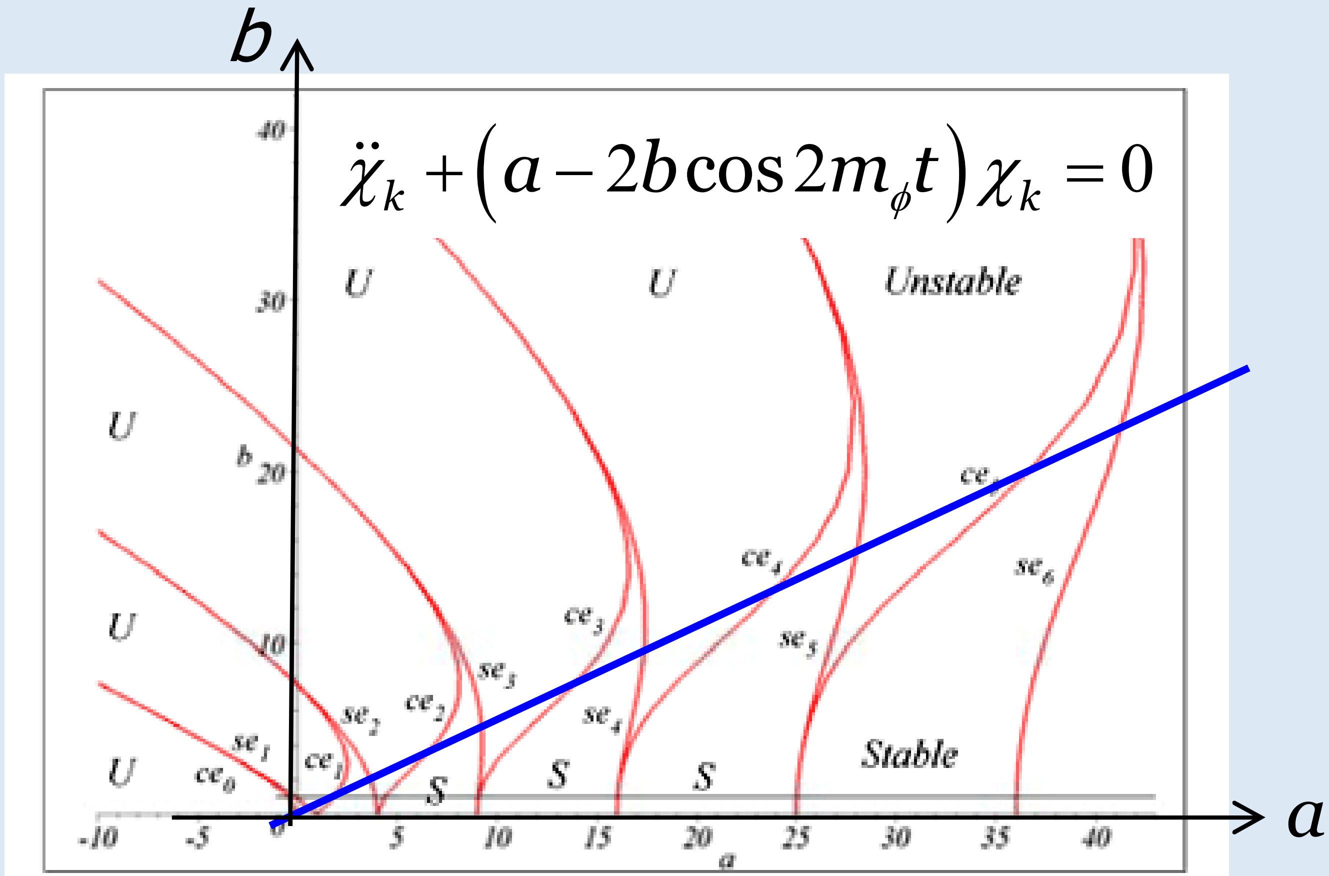


Figure 1. Mathieu stability chart based on the numerical values, generated by (McLachlan, 1947).

if b>1 initially, evolutionary path passes through unstable region

instantaneous reheating

2. Cosmological Perturbations from Inflation

- curvature perturbation: intuitive derivation

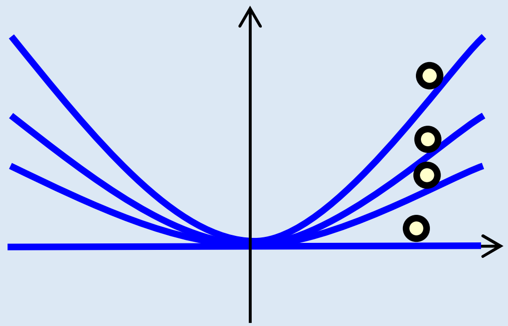
zero-point (vacuum) fluctuations of ϕ : $\delta\phi = \sum_k \delta\phi_k(t) e^{ik\cdot x}$

$$\delta\ddot{\phi}_k + 3H\delta\dot{\phi}_k + \omega^2(t)\delta\phi_k = 0 \; ; \; \omega^2(t) = \frac{k^2}{a^2(t)} \equiv \left(\frac{2\pi c}{\lambda(t)}\right)^2$$

physical wavelength

$\lambda(t) \sim a(t)$

harmonic oscillator with friction term and time-dependent ω



$\delta\phi_k \rightarrow$ const.

··· frozen when $\lambda > c\,H^{-1}$ (on superhorizon scales)

gravitational wave modes also satisfy the same eq.

- fluctuation amplitude (vacuum fluctuations=Gaussian)

$$\left|\left\langle \phi \middle| \vec{k} \right\rangle\right|^2 = |\varphi_k|^2, \quad \varphi_k \sim \frac{1}{a^{3/2}\sqrt{2w_k}} e^{-iw_k t} \ ; \quad w_k = \frac{k}{a} \gg H$$

$$\varphi_k(a \gg k/H) \approx \varphi_k(a = k/H) \approx \frac{H}{\sqrt{2k^3}} \quad \Rightarrow \quad \left\langle \delta\phi_k^2 \right\rangle = \left(\frac{H}{2\pi}\right)^2_{k/a \sim H}$$

⇧

frozen at $a = k/H$

In the above, metric perturbations δg are ignored

~ a gauge in which δg is minimized

= hypersurface on which $\delta R^{(3)}=0$: "flat" slice

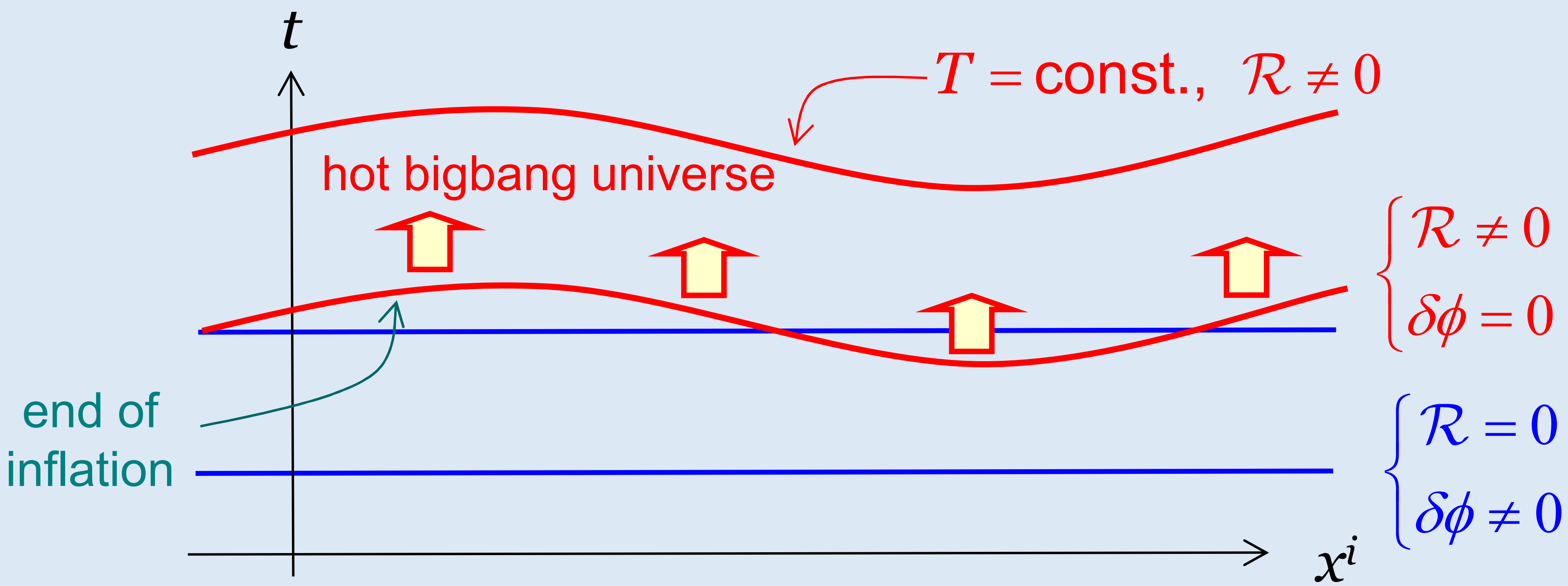
$$R^{(3)} = \frac{K}{6a^2}, \quad \delta R^{(3)} = \frac{4k^2}{a^2}\mathcal{R} \ \Rightarrow \ \delta K = \frac{2k^2}{3}\mathcal{R}$$

⇧

$\mathcal{R}$: called curvature perturbation

generation of "comoving" curvature perturbation

- $\delta\phi$ is frozen on "flat" ($\mathcal{R}$=0) 3-surface (t =const. hypersurface)
- Inflation ends/damped osc starts on ϕ =const. 3-surface.



ϕ =const. 3-surface is called "comoving" slice.

- curvature perturbation on comoving slices:

gauge transf. ⟹ $\mathcal{R}_c = -\frac{H}{\dot{\phi}}\delta\phi$ ← evaluated on flat slice

conservation of comoving curvature perturbation

Kodama & MS '84

- eom

$$\mathcal{R}_C'' + \frac{(z^2)'}{z^2}\mathcal{R}_C' + k^2\mathcal{R}_C = 0; \qquad z^2 \equiv \frac{a^2\dot{\phi}^2}{H^2} = 2\varepsilon a^2 M_P^2; \qquad ' = \frac{d}{d\eta} = a\frac{d}{dt}$$

$$\Big\Downarrow \; k^2 \to 0 \qquad\qquad \varepsilon = -\frac{\dot{H}}{H^2} = \frac{3}{2}(1+w), \qquad w = \frac{P}{\rho}$$

ε : slow-roll parameter

$$\mathcal{R}_C'' + \frac{(z^2)'}{z^2}\mathcal{R}_C' = 0$$

$\mathcal{R}_C' \propto \dfrac{1}{z^2}$: decaying mode

$\mathcal{R}_C = const.$: "growing" mode

if $\mathcal{R}_C$ becomes const., "adiabatic" limit is reached

$$\mathcal{R}_C(k \ll aH) \approx \mathcal{R}_C(k = aH) = -\left(\frac{H}{\dot{\phi}}\delta\phi\right)(k = aH)$$

Curvature perturbation spectrum

- spectrum $$P_{\mathcal{R}}(k) = \left(\frac{H^2}{2\pi\dot{\phi}} \right)^2_{k=aH} = \frac{1}{2}\left(\frac{H}{2\pi M_P \varepsilon^{1/2}} \right)^2_{k=aH}$$

- spectral index

$$P_{\mathcal{R}}(k) = Ak^{n_S - 1} \ ; \quad n_S - 1 = M_P^2 \left(2\frac{V''}{V} - 3\frac{V'^2}{V^2} \right) = 2\eta_V - 6\varepsilon_V$$

Liddle & Lyth ('92)

spectrum derived by 1st principle calculation

Mukhanov ('85), MS ('86)

more elegantly derived a la Faddeev-Jackiw method

Garriga, Montes, MS & Tanaka ('98)

generalized to k-inflation: $L = P(X, \phi); \quad X = -g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$

Garriga & Mukhanov ('99)

- generalized action for $\mathcal{R}_C$ Garriga & Mukhanov ('99)

$$S = \int d\eta d^3x \frac{z^2}{2c_s^2}\left[\mathcal{R}_C'^2 - c_s^2 k^2 \mathcal{R}_C^2\right]; \qquad z^2 = 3(1+w)a^2 M_P^2$$

$c_s =$ sound velocity (=1 for canonical case)

canonical quantization:

$$\pi_R = \frac{\delta S}{\delta \mathcal{R}_C'} = z^2 c_s^{-2} \mathcal{R}_C' \qquad \left[\mathcal{R}_C, \pi_R\right] = i\hbar$$

positive freq fcn

$$\mathcal{R}_C = a_{\vec{k}} r_k(\eta) + a_{-\vec{k}}^\dagger r_k^*(\eta)\,; \quad r_k \to \frac{1}{\sqrt{2c_s k}} \frac{c_s}{z} e^{-ic_s k\eta} \quad (\eta \to -\infty)$$

$$\Rightarrow \quad P_{\mathcal{R}}(k) = \frac{4\pi k^3}{(2\pi)^3} |r_k|^2_{c_s k|\eta|=1} = \frac{1}{3c_s(1+w)} \left(\frac{H}{2\pi M_P}\right)^2_{c_s k = aH}$$

Q2. Derive the above spectrum by performing canonical quantization as outlined above.

- δN - formula   Starobinsky ('85)

$$N(\phi) = \int_{t(\phi)}^{t_{end}} H dt = \int_{\phi}^{\phi_{end}} \frac{H}{\dot{\phi}} d\phi$$

$$\Rightarrow \quad \delta N(\phi) = \left[\frac{\partial N}{\partial \phi} \delta\phi \right]_{k=aH} = \left[-\frac{H}{\dot{\phi}} \delta\phi \right]_{k=aH} = \mathcal{R}_c$$

$$P_{\mathcal{R}}(k) = \left(\frac{H^2}{2\pi\dot{\phi}} \right)^2_{k=aH} = \left(\frac{\partial N}{\partial \phi} \right)^2 |\varphi_k|^2_{k|\eta|=1} ; \qquad |\varphi_k|^2 = \langle \delta\phi_k^2 \rangle = \left(\frac{H}{2\pi} \right)^2_{k=aH}$$

geometrical justification $\delta N = \sum_A \frac{\partial N}{\partial \phi^A} \delta\phi^A$ MS & Stewart ('96)

NL generalization Lyth, Malik & MS ('04)

only knowledge of background evolution is necessary

Tensor Perturbation

$$\partial^i h_{ij}^{TT} = \delta^{ij} h_{ij}^{TT} = 0$$: transverse-traceless

Starobinsky ('79)

- canonically normalized tensor field

$$S \sim \int d^4x \sqrt{-g}\,\frac{1}{2}\left(\frac{\partial \phi_{ij}}{\partial t}\right)^2 + \cdots$$

$$\phi_{ij} \equiv \frac{1}{\sqrt{32\pi G}} h_{ij}^{TT} = \frac{M_P}{2} h_{ij}^{TT}; \quad M_P \equiv \frac{1}{\sqrt{8\pi G}}$$

$$\phi_{ij}(k;t) = \sum_{\sigma=+,\times} a_k^\sigma P_{ij}^\sigma(k)\varphi_k(t) + h.c.$$

$\varphi_k(t)$: same as massless scalar

- tensor spectrum

$$\sum_\sigma \left|\left\langle h_{ij}^{TT} \middle| \vec{k},\sigma\right\rangle\right|^2 = \frac{4}{M_P^2}\sum_\sigma \left|\left\langle \phi_{ij} \middle| \vec{k},\sigma\right\rangle\right|^2 = \frac{8|\varphi_k|^2}{M_P^2} = 8\left(\frac{H}{2\pi M_P}\right)^2$$

Tensor-to-scalar ratio

$$\|\nabla N\|^2 \equiv H^{ab}(\phi)\frac{\partial N}{\partial \phi^a}\frac{\partial N}{\partial \phi^b}$$

- scalar spectrum: $$P_s(k)\frac{4\pi k^3}{(2\pi)^3} = \frac{H^2}{(2\pi)^2}\|\nabla N\|^2 \quad \propto k^{n_s-1}$$

- tensor spectrum: $$P_g(k)\frac{4\pi k^3}{(2\pi)^3} = \frac{8}{M_P^2}\frac{H^2}{(2\pi)^2} \quad \propto k^{n_g}$$

- tensor spectral index: $$-n_g = -\frac{2\dot{H}}{H^2} = \frac{\|\dot{\phi}\|^2}{M_P^2 H^2} = \frac{\|\dot{\phi}\|^2}{M_P^2\|\dot{\phi}\cdot\nabla N\|^2}$$

$$H = -\frac{dN}{dt} = -\dot{\phi}^a\nabla_a N$$

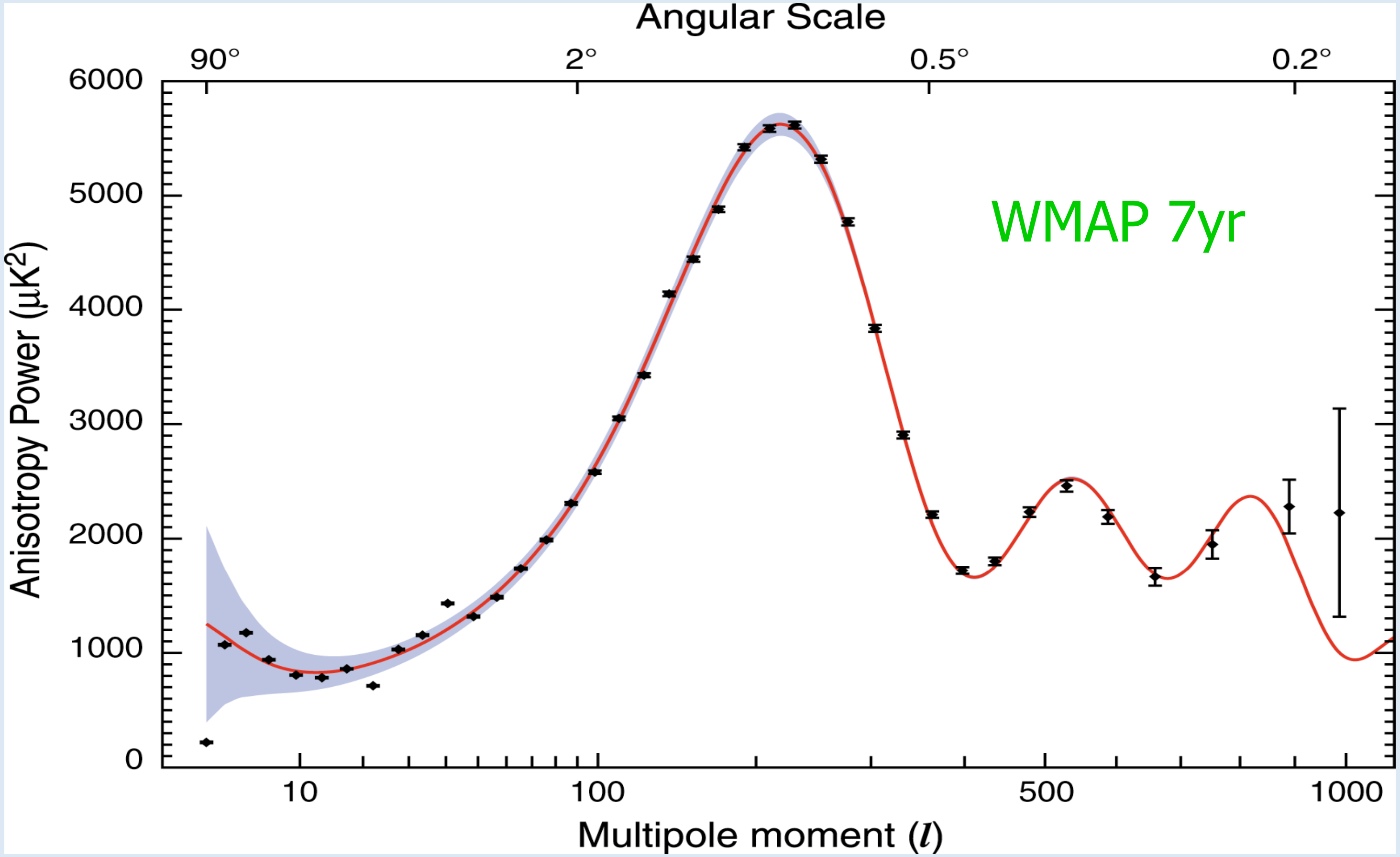
$$\geq \frac{1}{M_P^2\|\nabla N\|^2} = \frac{P_g}{8P_s}$$

$$\Rightarrow \quad r \equiv \frac{P_g}{P_s} \leq 8|n_g|$$

··· valid for all slow-roll models with canonical kinetic term

Comparison with observation

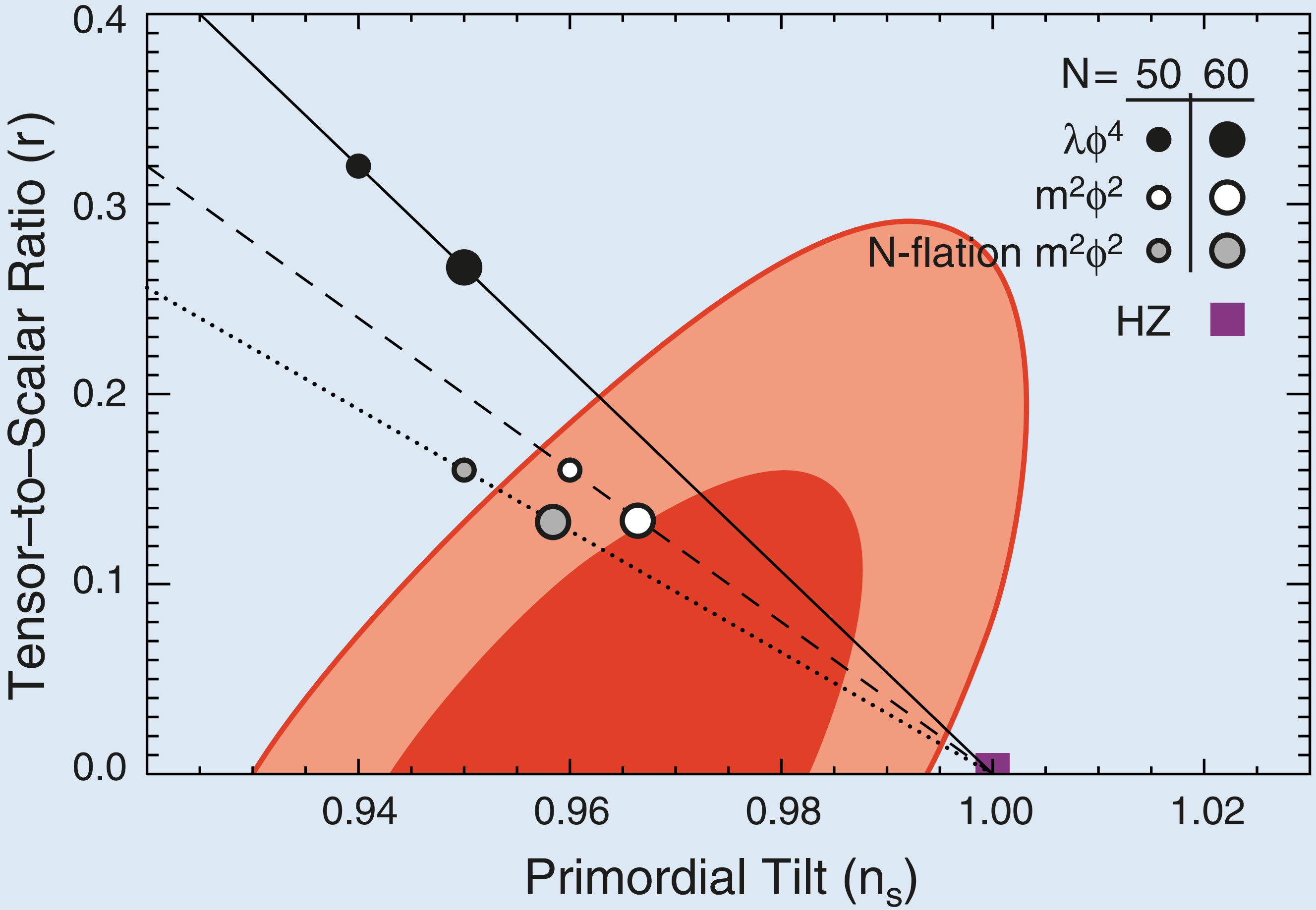
- Standard (single-field, slowroll) inflation predicts scale-invariant Gaussian curvature perturbations.



- CMB (WMAP) is consistent with the prediction.
- Linear perturbation theory seems to be valid.

CMB constraints on inflation

Komatsu et al. '10



- scalar spectral index: $n_s = 0.95 \sim 0.98$
- tensor-to-scalar ratio: $r < 0.15$

However,....

- Inflation may be non-standard

  multi-field, non-slowroll, DBI, extra-dim's, ...

- PLANCK, ... may detect Non-Gaussianity

  (comoving) curvature perturbation:

$$\mathcal{R}_{\mathrm{C}} = \mathcal{R}_{\mathrm{gauss}} + \frac{3}{5} f_{NL} \mathcal{R}_{\mathrm{gauss}}^2 + \cdots; \quad f_{NL} \gtrsim 5?$$

- B-mode (tensor) may or may not be detected.

  energy scale of inflation $H^2 \gtrless 10^{-10} M_{\mathrm{Planck}}^2$?

  modified (quantum) gravity? NG signature?

Quantifying NL/NG effects is important