Momentum relation and classical limit in the future-not-included complex action theory

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Introduction

Complex action theory (CAT)

- coupling parameters are complex
- dynamical variables such as q and p are fundamentally real but can be complex at the saddle points (asymptotic values are real).

Possible extension of quantum theory

Expected to give falsifiable predictions

Intensively studied by H. B. Nielsen and M. Ninomiya

Complex coordinate formalism

KN, H.B.Nielsen, PTP126 (2011)102

Non-Hermitian operators $\hat{q}_{new}$ and $\hat{p}_{new}$:

$$\hat{q}^{\dagger}_{new}|q\rangle_{new} = q|q\rangle_{new} \text{ for complex } q,$$
$$\hat{p}^{\dagger}_{new}|p\rangle_{new} = p|p\rangle_{new} \text{ for complex } p,$$
$$[\hat{q}_{new}, \hat{p}_{new}] = i\hbar.$$

Our proposal is to replace the usual Hermitian operators $\hat{q}$, $\hat{p}$, and their eigenstates $|q\rangle$ and $|p\rangle$, which obey $\hat{q}|q\rangle = q|q\rangle$, $\hat{p}|p\rangle = p|p\rangle$, and $[\hat{q}, \hat{p}] = i\hbar$ for real q and p, with $\hat{q}^{\dagger}_{new}$, $\hat{p}^{\dagger}_{new}$, $|q\rangle_{new}$ and $|p\rangle_{new}$.

$$\hat{q}_{new} \equiv \frac{1}{\sqrt{1-\epsilon\epsilon'}}\left(\hat{q} - i\epsilon\hat{p}\right), \hat{p}_{new} \equiv \frac{1}{\sqrt{1-\epsilon\epsilon'}}\left(\hat{p} + i\epsilon'\hat{q}\right),$$

$$|q\rangle_{new} \equiv \left(\frac{1-\epsilon\epsilon'}{4\pi\hbar\epsilon}\right)^{\frac{1}{4}} e^{-\frac{1}{4\hbar\epsilon}(1-\epsilon\epsilon')q^2} |\sqrt{\frac{1-\epsilon\epsilon'}{2\hbar\epsilon}}q\rangle_{coh},$$

$$|p\rangle_{new} \equiv \left(\frac{1-\epsilon\epsilon'}{4\pi\hbar\epsilon'}\right)^{\frac{1}{4}} e^{-\frac{1}{4\hbar\epsilon'}(1-\epsilon\epsilon')p^2} |i\sqrt{\frac{1-\epsilon\epsilon'}{2\hbar\epsilon'}}p\rangle_{coh'}.$$

$|\lambda\rangle_{coh} \equiv e^{\lambda a^\dagger}|0\rangle$ satisfies $a|\lambda\rangle_{coh} = \lambda|\lambda\rangle_{coh}$, where $a = \sqrt{\frac{1}{2\hbar\epsilon}}\left(\hat{q} + i\epsilon\hat{p}\right)$.

$|\lambda\rangle_{coh'} \equiv e^{\lambda a'^\dagger}|0\rangle$, where $a'^\dagger = \sqrt{\frac{\epsilon'}{2\hbar}}\left(\hat{q} - i\frac{\hat{p}}{\epsilon'}\right)$, is another coherent state defined similarly.

Modified complex conjugate $*_{\{\}}$:
ex.) for $f(q,p) = aq^2 + bp^2$,

$$f(q,p)^{*_{q,p}} = f^*(q,p) = a^* q^2 + b^* p^2,$$

Modified bra ${}_m\langle\ |$, ${}_{\{\}}\langle\ |$:
Modified hermitian conjugate $\dagger_m$, $\dagger_{\{\}}$:

$$ {}_m\langle\lambda| = \langle\lambda^*| = (|\lambda\rangle)^{\dagger_m}.$$
$$ (|\ \rangle)^{\dagger_{\{\}}} = {}_{\{\}}\langle\ |.$$

For example, a wave function :
$\psi(q) = \langle q|\psi\rangle \to \psi(q) = {}_m\langle_{new}\, q|\psi\rangle$

We decompose some function f as

$$f = \text{Re}_{\{\}} f + i\text{Im}_{\{\}} f,$$

where $\text{Re}_{\{\}} f$ and $\text{Im}_{\{\}} f$ are the "{}-real" and "{}-imaginary" parts of f defined by $\text{Re}_{\{\}} f \equiv \frac{f+f^{*\{\}}}{2}$ and $\text{Im}_{\{\}} f \equiv \frac{f-f^{*\{\}}}{2i}$.

ex) for $f = kq^2$, $\text{Re}_q f = \text{Re}(k)q^2$, $\text{Im}_q f = \text{Im}(k)q^2$.

If f satisfies $f^{*\{\}} = f$, we say f is {}-real, while if f obeys $f^{*\{\}} = -f$, we call f purely {}-imaginary.

Theorem on matrix elements

${}_m\langle_{new}\, q'$ or $p'|\mathcal{O}(\hat{q}_{new}, \hat{q}^{\dagger}_{new}, \hat{p}_{new}, \hat{p}^{\dagger}_{new})|q''$ or $p''\rangle_{new}$,
where $\mathcal{O}$ is a Taylor-expandable function, can be evaluated as if inside $\mathcal{O}$ we had the hermiticity conditions $\hat{q}_{new} \simeq \hat{q}^{\dagger}_{new} \simeq \hat{q}$ and $\hat{p}_{new} \simeq \hat{p}^{\dagger}_{new} \simeq \hat{p}$ for q', q'', p', p'' such that the resulting quantities are well defined in the sense of distribution.

$\rightarrow$ We do not have to worry about the anti-Hermitian terms in $\hat{q}_{new}$, $\hat{q}^{\dagger}_{new}$, $\hat{p}_{new}$ and $\hat{p}^{\dagger}_{new}$, provided that we are satisfied with the result in the distribution sense.

Deriving the momentum relation via FPI

KN, H.B.Nielsen, IJMPA27 (2012)1250076

Lagrangian in a system with a single d.o.f.:

$$L(q(t), \dot{q}(t)) = \frac{1}{2} m \dot{q}^2 - V(q),$$

$V(q) = \sum_{n=2}^{\infty} b_n q^n$, $V = V_R + iV_I$, $L = L_R + iL_I$, where
$V_R \equiv \mathrm{Re}_q(V) = \sum_{n=2}^{\infty} \mathrm{Re} b_n \; q^n$,
$V_I \equiv \mathrm{Im}_q(V) = \sum_{n=2}^{\infty} \mathrm{Im} b_n \; q^n$,
$L_R \equiv \mathrm{Re}_q(L) = \frac{1}{2} m_R \dot{q}^2 - V_R(q)$,
$L_I \equiv \mathrm{Im}_q(L) = \frac{1}{2} m_I \dot{q}^2 - V_I(q)$. $m = m_R + i m_I$.

$$ {}_m\langle new\ q_{t+dt}|\psi(t+dt)\rangle = \int_C e^{\frac{i}{\hbar}\Delta t L(q,\dot{q})} {}_m\langle new\ q_t|\psi(t)\rangle dq_t. $$

We consider ${}_m\langle new\ q_t|\xi\rangle$ which obeys

$$ \begin{aligned} {}_m\langle new\ q_t|\hat{p}_{new}|\xi\rangle &= \frac{\hbar}{i}\frac{\partial}{\partial q_t}\ {}_m\langle new\ q_t|\xi\rangle \\ &= \frac{\partial L}{\partial \dot{q}}\left(q_t, \frac{\xi - q_t}{dt}\right) {}_m\langle new\ q_t|\xi\rangle. \end{aligned} $$

Introducing a dual basis ${}_m\langle \text{anti}\ \xi|$, we have

$$ \begin{aligned} {}_m\langle new\ q_t|\psi(t)\rangle &\simeq \int_C d\xi\ {}_m\langle new\ q_t|\xi\rangle\ {}_m\langle \text{anti}\ \xi|\psi(t)\rangle \\ &= \int_C d\xi\ {}_m\langle new\ q_t|\psi(t)\rangle|_\xi. \end{aligned} $$

Then, we obtain

$$
\begin{aligned}
& {}_m\langle new\ q_{t+dt}|\psi(t+dt)\rangle|_\xi \\
= \quad & \sqrt{\frac{2\pi\hbar dt}{m}}\ {}_m\langle \text{anti } \xi|\psi(t)\rangle \exp\left[\frac{im}{2\hbar dt}(q_{t+dt}^2 - \xi^2)\right] \\
& \times \Bigg\{\delta_c(\xi - q_{t+dt}) \\
& - \sum_{n=2}\left(\frac{\hbar dt}{m}\right)^n (-i)^n \frac{idt}{\hbar} b_n \frac{\partial^n \delta_c(\xi - q_{t+dt})}{\partial \xi^n}\Bigg\}.
\end{aligned}
$$

→ Only the component with $\xi = q_{t+dt}$ contributes to ${}_m\langle new\ q_{t+dt}|\psi(t+dt)\rangle$.
Thus, we have obtained the momentum relation :

$$p = \frac{\partial L}{\partial \dot{q}} = m\dot{q}.$$

Properties of the future-included theory

KN, H.B.Nielsen, PTEP(2013) 023B04

Nielsen and Ninomiya, Proc. Bled 2006, p87.

$$\langle q|A(t)\rangle = \int_{\text{path}(t)=q} e^{\frac{i}{\hbar}S_{T_A=-\infty \text{ to } t}} \mathcal{D}\text{path},$$

$$\langle B(t)|q\rangle \equiv \int_{\text{path}(t)=q} e^{\frac{i}{\hbar}S_{t \text{ to } T_B=\infty}} \mathcal{D}\text{path},$$

$|A(t)\rangle$ and $|B(t)\rangle$ time-develop according to
$i\hbar\frac{d}{dt}|A(t)\rangle = \hat{H}|A(t)\rangle$, $i\hbar\frac{d}{dt}|B(t)\rangle = \hat{H}_B|B(t)\rangle$, where
$\hat{H}_B = \hat{H}^\dagger$.

$$\langle\mathcal{O}\rangle^{BA} \equiv \frac{\langle B(t)|\mathcal{O}|A(t)\rangle}{\langle B(t)|A(t)\rangle}$$

Utilizing $\frac{d}{dt}\langle \mathcal{O} \rangle^{BA} = \langle \frac{i}{\hbar}[\hat{H}, \mathcal{O}] \rangle^{BA}$, we obtain

- Heisenberg equation
- Ehrenfest's theorem:

$$\frac{d}{dt}\langle \hat{q}_{new} \rangle^{BA} = \frac{1}{m}\langle \hat{p}_{new} \rangle^{BA},$$
$$\frac{d}{dt}\langle \hat{p}_{new} \rangle^{BA} = -\langle V'(\hat{q}_{new}) \rangle^{BA}.$$

* momentum relation $p = m\dot{q}$

KN, H.B.Nielsen, IJMPA27 (2012)1250076

- Conserved probability current density

Properties of the future-not-included theory

KN, H.B.Nielsen, PTEP(2013) 073A03

$$i\hbar\frac{d}{dt}\langle\hat{O}\rangle^{AA} = \langle[\hat{O},\hat{H}_h]\rangle^{AA} + \left\{\hat{O} - \langle\hat{O}\rangle^{AA}, \hat{H}_a\right\},$$
$$\simeq \langle[\hat{O},\hat{H}_h]\rangle^{A(t)A(t)},$$

where $\langle\hat{O}\rangle^{AA} \equiv \frac{\langle A(t)|\hat{O}|A(t)\rangle}{\langle A(t)|A(t)\rangle}$. Thus, we obtain

$$\frac{d}{dt}\langle\hat{q}_{new}\rangle^{AA} \simeq \frac{1}{m_{\rm eff}}\langle\hat{p}_{new}\rangle^{AA},$$
$$\frac{d}{dt}\langle\hat{p}_{new}\rangle^{AA} \simeq -\langle V_R'(\hat{q}_{new})\rangle^{AA},$$

where $m_{\rm eff} \equiv m_R + \frac{m_I^2}{m_R}$. $\rightarrow p = m_{\rm eff}\dot{q}$.

We show that the method works also in FNIT.

They give Ehrenfest's theorem:

$$m_{\text{eff}} \frac{d^2}{dt^2} \langle \hat{q}_{new} \rangle^{AA} \simeq -\langle V_R'(\hat{q}_{new}) \rangle^{AA}.$$

This suggests that the classical theory of FNIT is described not by a full action S, but S_{eff}:

$$\begin{aligned} S_{\text{eff}} &\equiv \int_{T_A}^{t} dt L_{\text{eff}}, \\ L_{\text{eff}}(\dot{q}, q) &\equiv \frac{1}{2} m_{\text{eff}} \dot{q}^2 - V_R(q) \neq L_R. \end{aligned}$$

Thus, we claim that in FNIT the classical theory is described by $\delta S_{\text{eff}} = 0$, and $p = m_{\text{eff}} \dot{q} = \frac{\partial L_{\text{eff}}}{\partial \dot{q}}$.
This is quite in contrast to the classical theory of FIT, which would be described by $\delta S = 0$, where $S = \int_{T_A}^{T_B} dt L$, and $p = m\dot{q}$.

Table: Comparison between FIT and FNIT

	FIT	FNIT
action	$S = \int_{T_A}^{T_B} dtL$	$S = \int_{T_A}^{t} dtL$
“exp. value”	$\langle\hat{O}\rangle^{BA} = \frac{\langle B(t)\vert\hat{O}\vert A(t)\rangle}{\langle B(t)\vert A(t)\rangle}$	$\langle\hat{O}\rangle^{AA} = \frac{\langle A(t)\vert\hat{O}\vert A(t)\rangle}{\langle A(t)\vert A(t)\rangle}$
time development	$i\hbar\frac{d}{dt}\langle\hat{O}\rangle^{BA} = \langle[\hat{O},\hat{H}]\rangle^{BA}$	$i\hbar\frac{d}{dt}\langle\hat{O}\rangle^{AA} \simeq \langle[\hat{O},\hat{H}_h]\rangle^{AA}$
classical theory	$\delta S = 0$	$\delta S_{\text{eff}} = 0$, $S_{\text{eff}} = \int_{T_A}^{t} dtL_{\text{eff}}$
momentum relation	$p = m\dot{q}$	$p = m_{\text{eff}}\dot{q}$

Reconsideration of the method in FNIT

In the method we looked at a transition amplitude from t_i to t_f, which is similar to that in FIT: $\langle B(t)|A(t)\rangle = \langle B(T_B)|e^{-\frac{i}{\hbar}\hat{H}(T_B-T_A)}|A(T_A)\rangle$.

In FNIT :

$$
\begin{aligned}
I &\equiv \langle A(t)|A(t)\rangle \\
&= \langle A(T_A)|e^{\frac{i}{\hbar}\hat{H}^\dagger(t-T_A)}e^{-\frac{i}{\hbar}\hat{H}(t-T_A)}|A(T_A)\rangle \\
&= \int_C \mathcal{D}q \int_{C'} \mathcal{D}q' e^{-\frac{i}{\hbar}S_{T_A \text{ to } t}(q)^{*q}} e^{\frac{i}{\hbar}S_{T_A \text{ to } t}(q')} \\
&\quad \times \psi_A(q_{T_A}, T_A)^{*q_{T_A}} \psi_A(q'_{T_A}, T_A).
\end{aligned}
$$

$\rightarrow$ a path from T_A to t, and that from t to T_A.

We formally rewrite $\langle A(t)|A(t)\rangle$ into another expression similar to $\langle B(t)|A(t)\rangle$ by inverting the time direction of the transition amplitude from T_A to t, and introduce L_{formal}.

$$
\begin{aligned}
& S_{T_A \text{ to } t}(q)^{*_q} \\
= & \int_{T_A}^{t} dt' L(q(t'), \dot{q}(t'))^{*_q} \\
= & \int_{t}^{-T_A+2t} dt'' L(q_{\text{formal}}(t'', t), -\partial_{t''} q_{\text{formal}}(t'', t))^{*_{q_{\text{formal}}}},
\end{aligned}
$$

where $t'' = -t' + 2t$,
$q_{\text{formal}}(t'', t) \equiv q(-t'' + 2t) = q(t')$.

Then I is written as

$$\begin{aligned} I &= \int_{C'} \mathcal{D}q' \int_{C''} \mathcal{D}q_{\text{formal}} e^{\frac{i}{\hbar}\int_{T_A}^{t} dt' L(q'(t'),\dot{q}'(t'))} \\ &\quad \times e^{-\frac{i}{\hbar}\int_{t}^{T_B} dt'' L(q_{\text{formal}}(t'',t),-\partial_{t''} q_{\text{formal}}(t'',t))^{*q_{\text{formal}}}} J\psi_A(q'_{T_A}, T_A), \end{aligned}$$

where C'' is a contour of $q_{\text{formal}}(t'', t)$, and

$$\begin{aligned} J &= \int_{C'''} \mathcal{D}q'_{\text{formal}} e^{-\frac{i}{\hbar}\int_{T_B}^{-T_A+2t} dt'' L(q'_{\text{formal}}(t'',t),-\partial_{t''} q'_{\text{formal}}(t'',t))^{*q'_{\text{formal}}}} \\ &\quad \times \psi_A(q'_{\text{formal}}(-T_A + 2t, t), T_A)^{*q'_{\text{formal}}} \\ &= \langle A(2t - T_B) | q'_{\text{formal}}(T_B, t) \rangle \\ &= \psi_A(q'_{\text{formal}}(T_B, t), 2t - T_B)^{*q'_{\text{formal}}}. \end{aligned}$$

Expressing $q'(t')$ for $T_A \le t' \le t$ as $q_{\text{formal}}(t', t)$, we can rewrite I as

$$
\begin{aligned}
I \;\simeq\; & \int \mathcal{D}q_{\text{formal}} e^{\frac{i}{\hbar}\int_{T_A}^{T_B} dt' \{-\epsilon(t'-t)\} L_{\text{formal}}(q_{\text{formal}}(t',t),\partial_{t'} q_{\text{formal}}(t',t),t'-t)} \\
& \times \psi_A(q_{\text{formal}}(T_B, t), 2t - T_B)^{*q_{\text{formal}}} \psi_A(q_{\text{formal}}(T_A, t), T_A),
\end{aligned}
$$

where $\epsilon(t)$ is 1 for $t > 0$ and -1 for $t < 0$, and

$$
\begin{aligned}
& L_{\text{formal}}(q_{\text{formal}}(t', t), \partial_{t'} q_{\text{formal}}(t', t), t' - t) \\
= \;& \frac{1}{2} m_{\text{formal}}(t' - t) \left(\partial_{t'} q_{\text{formal}}(t', t)\right)^2 \\
& - V_{\text{formal}}(q_{\text{formal}}(t', t), t' - t),
\end{aligned}
$$

$$
\begin{aligned}
m_{\text{formal}}(t' - t) &\equiv m_R - i\epsilon(t' - t) m_I, \\
V_{\text{formal}}(q_{\text{formal}}(t', t), t' - t) &\equiv V_R(q_{\text{formal}}(t', t)) \\
&\quad - i\epsilon(t' - t) V_I(q_{\text{formal}}(t', t)).
\end{aligned}
$$

Replacing L with L_{formal} in the method, we obtain

$$\begin{aligned} p_{\text{formal}}(t',t) &= \frac{\partial L_{\text{formal}}(q_{\text{formal}}(t',t), \partial_{t'} q_{\text{formal}}(t',t), t'-t)}{\partial(\partial_{t'} q_{\text{formal}}(t',t))} \\ &= m_{\text{formal}}(t'-t)\partial_{t'} q_{\text{formal}}(t',t). \end{aligned}$$

We take the time average of $\partial_{t'} q_{\text{formal}}$ around $t' = t$.

$$\begin{aligned} \frac{d}{dt}q(t) &\simeq \left\{ \frac{\partial}{\partial t'} q_{\text{formal}}(t',t) \right\} |_{t'=t} \\ &\simeq \frac{1}{2\Delta t} \int_{t-\Delta t}^{t+\Delta t} dt' \partial_{t'} q_{\text{formal}}(t',t) \\ &= \frac{1}{2\Delta t} \int_{t-\Delta t}^{t+\Delta t} dt' \frac{p_{\text{formal}}(t',t)}{m_{\text{formal}}(t'-t)} \simeq \frac{1}{m_{\text{eff}}} p(t), \end{aligned}$$

where $p(t) \equiv p_{\text{formal}}(t,t)$.

Thus, we have reproduced $p = m_{\text{eff}} \dot{q}$.

Summary

In our previous paper we derived the momentum relation $p = m\dot{q}$ by considering a transition amplitude from some initial time to final time, which is similar to that in FIT.

In this paper we provided a way to properly apply the method to FNIT by rewriting the transition amplitude in FNIT into another expression similar to that in FIT, and by introducing L_{formal}.

We explicitly derived the momentum relation $p = m_{\text{eff}}\dot{q}$ in FNIT via this method.

In FNIT

- classical physics is described not by a full action S but a certain real action S_{eff} $(\neq S_R)$: $S_{\text{eff}} = \int_{-\infty}^{t} L_{\text{eff}}$, where $L_{\text{eff}} = \frac{1}{2}m_{\text{eff}}\dot{q}^2 - V_R(q)$.
- momentum relation is given by $\langle \hat{p}_{new} \rangle^{AA} = m_{\text{eff}} \frac{d}{dt} \langle \hat{q}_{new} \rangle^{AA}$, $p = m_{\text{eff}}\dot{q}$, where $m_{\text{eff}} = m_R + \frac{m_I^2}{m_R}$.

$\rightarrow$ quite different from those in FIT.

In FIT

- classical theory is described by a full action S.
- momentum relation is given by $\langle \hat{p}_{new} \rangle^{BA} = m \frac{d}{dt} \langle \hat{q}_{new} \rangle^{BA}$, $p = m\dot{q}$.

Outlook

- It is interesting to see the dynamics of the CAT in a simple model such as a harmonic oscillator.
- The potential of the slow roll inflation is extremely flat. The imaginary part might help us to have more natural potential.