On the calculation of entanglement entropy in quantum field theory

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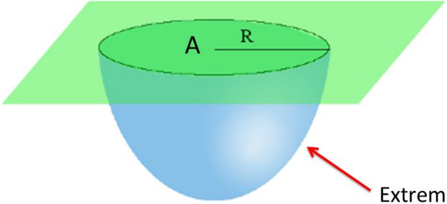
Before I begin...

How I got interested in entanglement entropy (and also in quantum information)

- I originally work on superstring theory, in particular exact solutions of supergravity models.
- Like many, I got interested in EE after the celebrated proposal of Ryu and Takayanagi.

Holographic Entanglement entropy

Ryu, Takayanagi
Hubeny, Rangamani, Takayanagi



$$S_A = \frac{\text{Area of M}}{4\text{G}_\text{N}}$$

Ryu-Takayanagi formula and AdS/CFT

- It's about EE in AdS/CFT (or gravity/gauge) correspondence, and suggests that EE in the gravity *dual* is given by minimal area surface which is homologous to the entangling surface on the boundary theory.
- After RT formula was shown to pass various requirements, people began to apply it to various supergravity (i.e. superstring) backgrounds whose dual gauge theories are known.

Ryu-Takayanagi formula and AdS/CFT

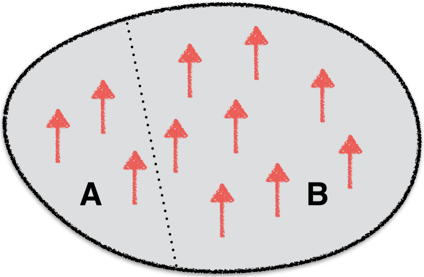
- I know some string theorists were initially rather dubious about RT formula, because it had no string theory origin.
- But conversely this property made EE and RT formula a more versatile probe of AdS/CFT, in particular for bottom-up approach of AdS/CMT.
- Today I am going to talk about the calculation of EE in (free) QFT.

Entanglement entropy

- A very useful order parameter of a quantum system

$$\rho_A = \mathrm{tr}_B\, \rho_{total}$$

$$S_A = -\mathrm{tr}_A\, \rho_A \log \rho_A$$



Properties of EE

- If $B = A^C$, $S_A = S_B$ (for any *pure* state)
  - broken at finite temperature
- If A is divided into A_1, A_2, $S_{A_1} + S_{A_2} \geq S_A$ (subadditivity)
- Due to the first property, EE cannot be an extensive quantity (or follow volume law) and it can only depend on the boundary.

Area law

- So we may naturally expect EE should exhibit area law for ground state at leading order, and for free scalar field theory it was explicitly checked numerically by Srednicki (1993).
- At first it was expected to explain the origin of black hole entropy, but it turned out that the coefficient of area law is not universal.
- To compute EE one employs techniques like replica trick or heat kernel method. (Casini, Huerta, Fursaev, Dowker, ...)

EE for CFT

- When the entangling surface is sphere, time evolution of the ball can be mapped to $R \times H^{d-1}$ through conformal transformation.
- And EE is the thermal entropy on hyperbolic space.

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_{d-2}^2$$

$$t = R\frac{\sinh(\tau/R)}{\cosh u + \cosh(\tau/R)}, \quad r = R\frac{\sinh u}{\cosh u + \cosh(\tau/R)}$$

$$ds^2 = \Omega^2(u,\tau)[-d\tau^2 + R^2(du^2 + \sinh^2 u d\Omega_{d-2}^2)]$$

- $t = 0, r = R$ is $u \to \infty$, and $t = \pm R, r = 0$ is $\tau \to \pm\infty$.
- The causal diamond in original coordinate is mapped to the thermal entropy of the entire hyperbolic space, with curvature radius R.

Log term in EE and anomaly

- In even dimensions there are log-correction terms in addition to area law.
- The coefficient of log is related to conformal anomaly. (see e.g. Casini, Huerta, Myers 1102.0440)

Rényi entropy

- A one-parameter generalization of EE,

$$S_q = \frac{1}{1-q} \log \mathrm{tr} \rho^q$$

- For a free scalar field theory in 4d,

$$S = \alpha \left(\frac{R}{\epsilon}\right)^2 - \frac{(1+q)(1+q^2)}{360 q^3} \log(R/\epsilon)$$

(Casini, Huerta 2010)

Numerical evaluation of EE for field theory

- Initiated in Srednicki 1993 and confirmed the area law.
- The coeff of log term for 4d scalar, $-1/90$ is numerically checked by Lohmayer, Neuberger, Schwimmer, and Theisen 2010.
- I extended the previous works to computation of Rényi entropy for $q \neq 1$. (Phys. Lett. B 2014)

Setup for scalar field

- Hamiltonian

$$H = \frac{1}{2}\int d^3x\,(\pi^2(x) + (\nabla\phi)^2)$$

- In terms of partial wave, $H = \sum H_{lm}$

$$H_{lm} = \frac{1}{2}\int dx \left\{ \pi_{lm}^2 + x^2 \left[\frac{d}{dx}\left(\frac{\phi_{lm}}{x}\right)\right]^2 + \frac{l(l+1)}{x^2}\phi_{lm}^2 \right\}$$

- Discretize it and we get a system of coupled harmonic oscillators.

$$H_{lm} = \frac{1}{2a}\sum_{j=1}^{N}\left[\pi_{lm,j}^2 + (j+\frac{1}{2})^2\left(\frac{\phi_{lm,j}}{j} - \frac{\phi_{lm,j+1}}{j+1}\right)^2 + \frac{l(l+1)}{j^2}\phi_{lm,j}^2\right]$$

Srednicki's prescription

- Then from the mass matrix K,

$$H = \frac{1}{2a} \sum_{i,j} (\delta_{ij} \pi_i \pi_j + \phi_i K_{ij} \phi_j)$$

- Calculate square root $\Omega = \sqrt{K}$, and express ($A : n \times n$, $B : (N-n) \times (N-n)$).

$$\Omega = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$$

- Compute step by step

$$\beta = \frac{1}{2} B^T A^{-1} B, \quad \beta' = \frac{1}{\sqrt{C-\beta}} \beta \frac{1}{\sqrt{C-\beta}}, \quad \Xi = \frac{\beta'}{1+\sqrt{1-\beta'^2}}$$

Srednicki's prescription

- EE for ℓ-th partial wave

$$S(l,n,N) = -\mathrm{tr}\left[\log(1-\Xi) + \frac{\Xi}{1-\Xi}\log\Xi\right]$$

and the final answer

$$S(n,N) = \sum_{l=0}^{\infty}(2l+1)S(l,n,N)$$

For Rényi entropy, we use instead

$$S_q(l,n,N) = \frac{1}{1-q}\mathrm{tr}\left[q\log(1-\Xi) - \log(1-\Xi^q)\right]$$

Procedure

- For given (l, n, N) one computes $S(l, n, N)$ and repeat it for different N. Converges to a particular value for $N \to \infty$.
- Repeat it for different l. For $l \gg N > n$ it is shown

$$S(l, n) = \xi(l, n)(-\log \xi(l, n) + 1)$$

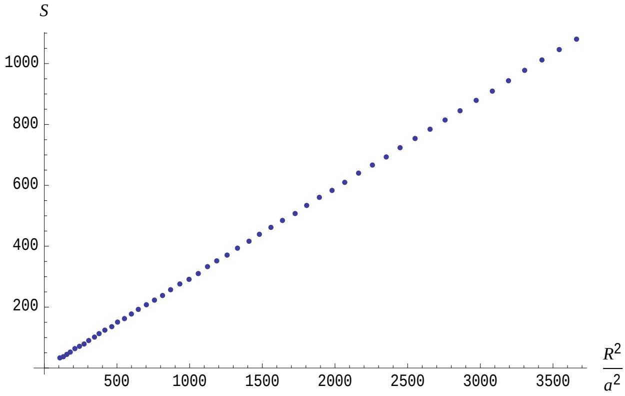
  with

$$\xi(l, n) = \frac{n(n+1)(2n+1)^2}{64l^2(l+1)^2} + \mathcal{O}(l^{-6})$$

  so we compute the sum over l until $S(l, n)$ gets sufficiently small, and approximate the rest of the series by integration.
- Similarly for Rényi entropy

EE vs. Area - á la Lohmayer, Neuberger, Schwimmer, Theisen

- Subsystem size $10 \leq n \leq 60$



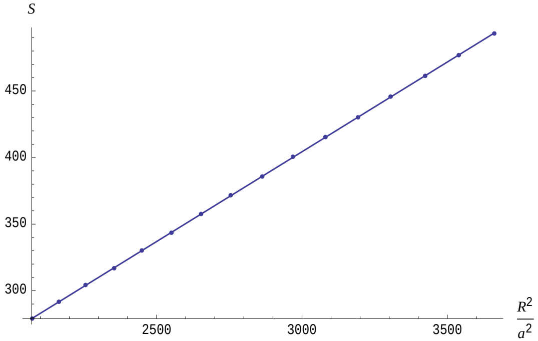
- $S = s(R/a)^2 + c' \log(R/a)^2 + d$, and χ^2-fitting gives $2c' = -0.0110731$ which is 3.4% off from $1/90$.

Rényi entropy

- For $q > 1$, $S(l)$ decays faster for large l so the calculation is easier.
- $S(l) \sim 1/l^{4q-1}$

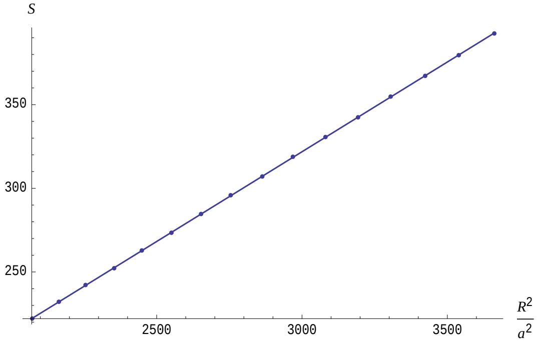
S vs. Area: $q = 1.5$

- Fitting for $45 \leq n \leq 60$.
- Linear fitting gives $0.134793x - 0.0401857$. Inclusion of log gives $0.134794x - 0.00331911 \log x - 0.0172047$.
- Compare to $-(1+q)(1+q^2)/360/q^3/2 = -0.00334362$. Matches within 1%.



S vs. Area: $q = 1.8$

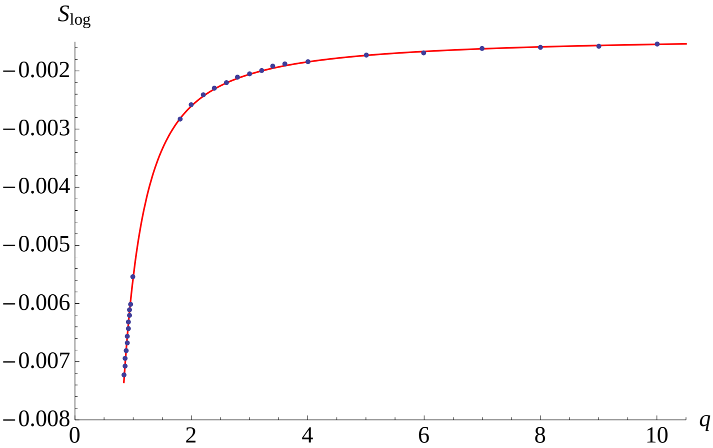
- Fitting for $45 \leq n \leq 60$.
- Linear fitting gives $0.107294x - 0.0336493$. Inclusion of log gives $0.107295x - 0.00282556 \log x - 0.0140856$.
- Compare to $-(1+q)(1+q^2)/360/q^3/2 = -0.00282731$. Matches within 0.1%.



For $q < 1$

- Convergence is slow, and it takes much longer to do the numerical computation.
- Done for $q = 0.85, 0.9, 0.95$.
- Mismatch with $-(1+q)(1+q^2)/360/q^3/2$ is $.22, .024, .08\%$.

Fitting the data



Comments on the result so far

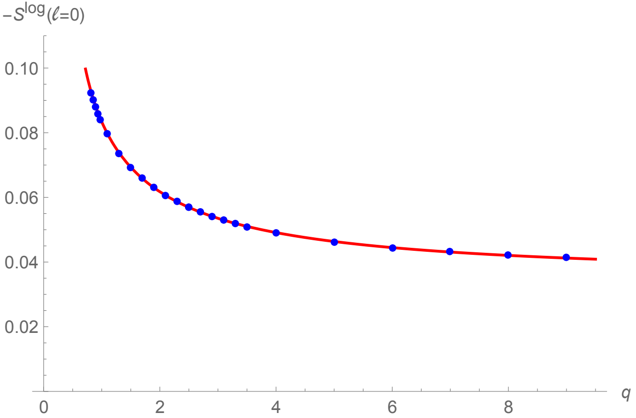
- Numerical result agrees nicely with analytic computation (central charges) - for Renyi entropy of scalar field
- The same method can be applied to non-CFT (inclusion of mass etc.), interacting theories in principle.
- One can also consider fields with spin, other dimensions etc.
  - $g^{scalar}_{6d} = (q+1)(q^2+3)(2q^2+3)/q^5/30240$ (Casini, Huerta 2010)
  - $g^{Weyl}_{4d} = -(q+1)(37q^2+7)/q^3/1440$ (Fursaev 2012)
  - $g^{vector}_{4d} = -(91q^3+31q^2+q+1)/q^3/180$ (Fursaev 2012) - But there's ambiguity!

The computation for Maxwell field

- The result of Fursaev (2012) is from trace anomaly coefficient. For EE ($q = 1$), $g = -31/45$.
- But more explicit computation of thermal entropy, done by Dowker (2010) gives $g = -16/45$.
- Casini and Huerta (2015) considered spherical entanglement surface and pointed out, following Srednicki's method, $H^{\text{vector}} = 2(H^{\text{scalar}} - H^{l=0})$, and obtained $-16/45$.
- Soni and Trivedi (2016) proposed extended Hilbert space and performed path integral in a gauge-invariant way, to get $g = -31/45$.

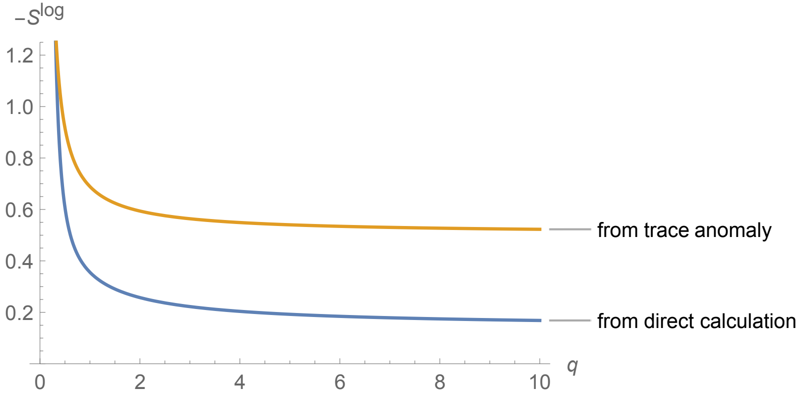
Renyi entropy for $l = 0$ mode

- Coefficient of Log term: Data vs. Fit to a cubic polynomial



Coefficient of log

- Direct comp. vs. Anomaly coefficient



Discussion

- EE or Renyi EE can be calculated by various methods.
- For scalar or spinor fields, they all match.
- But for gauge fields there's subtlety in the coefficient of Log-term, due to the apparent gauge symmetry breaking of the splitting spacetime into separate parts.
- For D=4 Maxwell field, using RT formula, trace anomaly consideration, gauge invariant prescription of Soni and Trivedi give -31/45, while heat kernel method and real-time method of Srednicki give -16/45.
- Soni and Trivedi argue this discrepancy is generic, and call -16/45 as the "extractable part", which is related to the number of Bell pairs one can obtain, thus more physical.