Line operators on $S^1 \times R^3$

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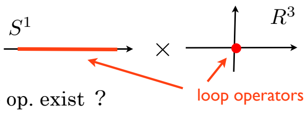
arXiv: 1111.4221 YI, T.Okuda, M.Taki

1. Motivation



What kinds of bound states with line op. exist ? in IR theory. [Gaiotto Moore Neitzke 2010]

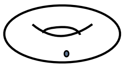
Check of AGT relation

2. AGT correspondence

$4D$ $\mathcal{N}=2$ $SU(2)$ gauge theories with 1 adj. hyp. mass m ⟷ $2D$ CFT Liouville theory

$$S=\frac{1}{4\pi}\int d^2z(g^{ab}\partial_a\phi\partial_b\phi+(b+\frac{1}{b})R\phi+4\pi\mu e^{2b\phi})$$

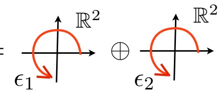
on torus with 1 puncture



instanton part. func. on $\mathbb{R}^4$ ⟷ **Conformal block**

with Ω background ϵ_1,ϵ_2 — with primary op. $V_m=e^{2m\phi}$

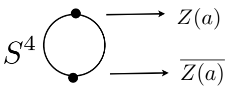
$\mathbb{R}^4 = \mathbb{R}^2 \oplus \mathbb{R}^2$

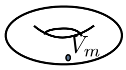


$$\sum_{\mathbf{n},\mathbf{n}'}\frac{\langle\alpha|L_{\mathbf{n}'}\,V_m\,L_{-\mathbf{n}}|\alpha\rangle}{\langle\alpha|V_m|\alpha\rangle}\langle\alpha|L_{\mathbf{n}}\,L_{-\mathbf{n}'}|\alpha\rangle^{-1}$$

$$Z_{\text{inst.}}(\vec{a},m,\epsilon_1,\epsilon_2) \;\overset{\epsilon_1/\epsilon_2=b^2}{=}\; \mathcal{F}_m(\alpha)$$

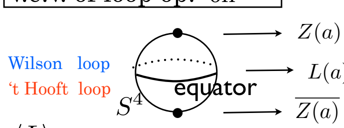
Full part. func. on S^4 ⟷ **Correlation func.**

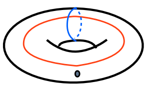




$$Z_{S^4} = \int d\vec{a}\,\overline{Z(\vec{a},\epsilon_{1,2}=1)}\,Z(\vec{a},\epsilon_{1,2}=1) \;\overset{b=1}{\longleftarrow}\; \langle V_m\rangle = \int d\alpha\,\langle\alpha|V_m|\alpha\rangle\,\overline{\mathcal{F}_m(\alpha)}\,\mathcal{F}_m(\alpha)$$

v.e.v. of loop op. on S^4 ⟷ **Verlinde loop op.**



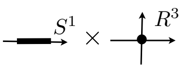


Verlinde op. acting on conformal blocks

$$\langle L\rangle = \int d\vec{a}\,\overline{Z(a)}\sum_{a'}L(a,a')Z(a') \;\overset{b=1}{\longleftarrow}\; \langle\mathcal{L}\rangle = \int d\alpha\,\overline{\mathcal{F}_m(\alpha)}\left(\mathcal{L}\cdot\mathcal{F}_m(\alpha)\right) = \int d\alpha\,\overline{\mathcal{F}_m(\alpha)}\left(\sum_{\alpha'}L(\alpha,\alpha',b)\mathcal{F}_m(\alpha')\right)$$

v.e.v. of loop op. on $S^1\times R^3$ ⟷ **Verlinde loop op.**

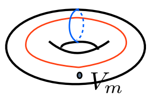
S^1 (τ) $\times R^3$ ($x^{1,2,3}$)



with twisted bdy. cond. around S^1

$$\phi(\tau+2\pi R,\varphi)=\phi(\tau,\varphi+2\pi\lambda)$$

radius of S^1, angle around x^3



$$\langle L\rangle_\lambda \;\overset{\lambda=b^2}{=}\; \mathcal{L}$$

3. Set up of 4D gauge th.

4D Euclidean $\mathcal{N}=2$ (τ = 4, $\mu=1,2,3,4$) — 10D Euclidean $\mathcal{N}=1$

gauge boson A_μ ← ($M=1,2,3,4$) A_M ($M=1,\cdots,9,0$)

Weyl fermions $\lambda\quad\psi$ — Ψ 16 component Majorana-Weyl spinor

cpx. scalar $\phi\equiv\phi_0+i\phi_1 = A_0+iA_9$ ← ($M=9,0$)

vector multiplet

SUSY transformation

$$\begin{cases} Q_\epsilon\cdot A_M=\epsilon^T\,\Gamma_M\Psi \\ Q_\epsilon\cdot\Psi=\frac{1}{2}F_{MN}\Gamma^{MN}\epsilon \end{cases}$$

Weyl fermion ψ_q

cpx. scalars $q\quad\tilde{q}^\dagger$ ← ($M=5,6,7,8$) $q=A_5+iA_6$, $\tilde{q}=A_7+iA_8$

Weyl fermion $\psi_{\tilde{q}}^\dagger$

hyper multiplet — doublets of $SU(2)_R$

$\Gamma_{5678}\epsilon=-\epsilon$ → 4D $\mathcal{N}=2$

We get the mass of hyp. by Scherk-Schwarz mechanism. $D_0A_i\to D_0A_i+M_{ij}A_j$

<u>Wilson loop in rep. j</u>

$$\langle W_j\rangle \equiv \langle \mathrm{Tr}_j P\exp\oint_{S^1}dx^4(-iA_4+\phi_0)\rangle$$

R^3 ($x^{1,2,3}$) $\times S^1$ (x^4) with radius R

preserves SUSY s.t. $(\Gamma_4+i\Gamma_0)\epsilon=0$

scalar v.e.v. $\alpha\equiv R\langle A_4+i\phi_0\rangle$

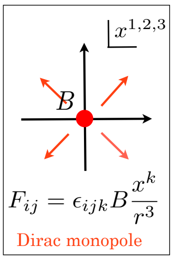
e.g.) $SU(2)$ $j=\frac{1}{2}$ $\alpha=\begin{pmatrix}\alpha & \\ & -\alpha\end{pmatrix}$

$$\langle W_{1/2}\rangle=e^{2\pi i\alpha}+e^{-2\pi i\alpha}$$

<u>'t Hooft loop in rep. j</u>

$$\langle T_j\rangle=\sum_B e^{iB\cdot\Theta}\int \mathcal{D}A\mathcal{D}\Psi e^{-S}$$

B: magnetic charge; background with magnetic charge B →



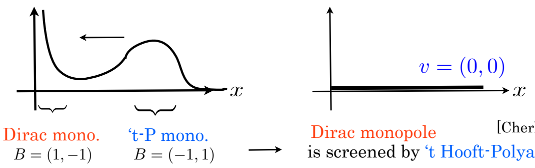
Θ chemical pot. of B

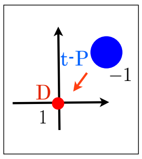
$$\beta=\frac{\Theta}{2\pi}-\frac{4\pi iR}{g^2}\langle\phi_1\rangle \qquad \beta=\begin{pmatrix}\beta & \\ & -\beta\end{pmatrix}$$

e.g.) $SU(2)$

$j=\frac{1}{2}$ $B=(\frac{1}{2},-\frac{1}{2})$, $B=(-\frac{1}{2},\frac{1}{2})$ ⟶ $\langle T_{1/2}\rangle|_{class.}=e^{2\pi i\beta}+e^{-2\pi i\beta}$

$j=1$ $B=(1,-1)$, $v=(0,0)$, $B=(-1,1)$





→ Dirac monopole is screened by 't Hooft-Polyakov monopole [Cherkis-Durcan '07]

$$\langle T_1\rangle=e^{4\pi i\beta}Z_{1\text{-loop}}(B=(1,-1)) \qquad B=(1,-1)$$
$$+Z_{1\text{-loop}}(v=(0,0))\,Z_{\text{mono.}}(B=(1,-1),\,v=(0,0)) \qquad v=(0,0)$$
$$+e^{-4\pi i\beta}Z_{1\text{-loop}}(B=(-1,1)) \qquad B=(-1,1)$$

<u>Loop operators v.e.v.</u>
<u>with twisted bdy. condition</u> on R^3 ($x^{1,2,3}$) $\times S^1$ (τ) with radius

$$\phi(\tau+2\pi R,\varphi)=\phi(\tau,\varphi+2\pi\lambda)$$

radius of S^1, angle around x^3

$$\langle L\rangle=\mathrm{Tr}_{\mathcal{H}_L}(-1)^Fe^{-2\pi RH}e^{2\pi i\,\lambda\,(J_3+I_3)}$$

λ: parameter; $\mathcal{H}_L$: Hilbert space on R^3 with line op. L; J_3: rotation around x^3; I_3: R-symmetry trans.

$$\partial_\tau\to\partial_\tau-\frac{i}{R}\lambda(J_3+I_3)$$

4. Localization

$$Z\equiv\int\mathcal{D}A\mathcal{D}\Psi\;e^{-S} \quad = \quad Z(t)\equiv\int\mathcal{D}A\mathcal{D}\Psi\;e^{-S-tQ_\epsilon V}$$

$$(\because)\quad \frac{\partial Z(t)}{\partial t}=-\int\mathcal{D}A\mathcal{D}\Psi\,(Q\cdot V)\,e^{-S-tQ\cdot V}=-\int\mathcal{D}A\mathcal{D}\Psi(Q\cdot(Ve^{-S-tQ\cdot V}))=0$$

if $Q\cdot S=0,\ \ Q^2\cdot V=0$

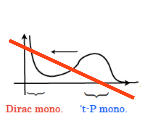
we choose $V=\langle\Psi,\overline{Q\cdot\Psi}\rangle$

For $Q^2\cdot V=0$ to hold at off-shell, we introduce auxiliary fields K_i $(i=1\sim7)$

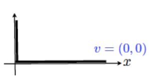
$$Z(t=\infty)=\int\mathcal{D}A\mathcal{D}\Psi\;e^{-S-tQ\cdot V}$$

<u>saddle points</u>

$Q\cdot\Psi=0$ →
- $*_3F=D\phi_1$ Bogomolny eq.
- field config. is inv. under <u>gauge trans.</u> and x^4 trans. with parameter α (Cartan valued)



→
- Dirac monopole $A=\frac{B}{2}\cos\theta d\phi$ $\phi_1=\frac{B}{2r}$
- ~~'tHooft-Polyakov monopole~~ not inv. under Cartan-valued gauge trans.
- Dirac mono. screened by 't-P mono.



ϕ_0, A_4 const. Cartan valued ⟶ $R(A_4+i\phi_0)=\alpha=\begin{pmatrix}\alpha & \\ & -\alpha\end{pmatrix}$

$q,\ \tilde{q}=0$ (hyp. scalar)

<u>fluctuation (perturbative)</u>

$t\to\infty$ infinitely weak coupling

we can get **exact** result by considering only **one-loop determinant** of $Q\cdot V$

One-loop determinant

$X_0^B \equiv (\tilde{A}_M)_{M=1}^9 \quad X_1^F \equiv (\Psi_\alpha, c, \tilde{c})_{\alpha=10}^{16}$

$\downarrow \hat{Q} \qquad \downarrow \hat{Q}$

$X_0^F \qquad X_1^B$

$\downarrow \hat{Q} \qquad \downarrow \hat{Q}$

$R_0 \cdot X_0^B \qquad R_1 \cdot X_1^F$

$\tilde{A}_M := A_M - A_M^{(Dirac)}$

$\tilde{A}_M = (\tilde{A}_0, \tilde{A}_1, \cdots, \tilde{A}_9)$

Q_ϵ

$\Psi_\alpha = (\Psi_1, \cdots, \Psi_{10}, \cdots, \Psi_{16})$

$$R := \hat{Q}^2 = -\partial_4 + \frac{i}{R}\lambda(J_3 + I_3) - \frac{i}{R}[\alpha, \cdot] - iM$$

$$\hat{V} = \langle \Psi, \overline{Q \cdot \Psi} \rangle + \int \tilde{c}\, D^M A_M$$

$$\hat{Q} \cdot \hat{V}|_{2次} = \hat{Q} \cdot \left\langle (X_0^F, X_1^F), \begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix} \begin{pmatrix} X_0^B \\ X_1^B \end{pmatrix} \right\rangle$$

$$= \left\langle (X_0^B, X_1^B) \begin{pmatrix} R_0 & \\ & 1 \end{pmatrix}, \begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix} \begin{pmatrix} X_0^B \\ X_1^B \end{pmatrix} \right\rangle$$

$$+ \left\langle (X_0^F, X_1^F), \begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix} \begin{pmatrix} 1 & \\ & R_1 \end{pmatrix} \begin{pmatrix} X_0^F \\ X_1^F \end{pmatrix} \right\rangle$$

$$\int DX^B DX^F e^{-\hat{Q}\cdot\hat{V}} \longrightarrow \frac{\det^{1/2}\left[\begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix}\begin{pmatrix} 1 & \\ & R_1 \end{pmatrix}\right]}{\det^{1/2}\left[\begin{pmatrix} R_0 & \\ & 1 \end{pmatrix}\begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix}\right]} = \left(\frac{\det_{X_1^F} R_1}{\det_{X_0^B} R_0}\right)^{\frac{1}{2}}$$

Monopole –(smooth field config. of 4D 1-form) correspondence

config. with Dirac singul. on 3D

$A_i(\vec{x}) = \tilde{A}_i(\vec{x}) + A_i^{(Dirac)}(\vec{x})$

$\phi_1(\vec{x}) = \tilde{\phi}_1(\vec{x}) + \phi_1^{(Dirac)}(\vec{x})$

$(i = 1, 2, 3) \quad \vec{x} = (x^1, x^2, x^3)$

,where $A_i^{(Dirac)} dx^i = -\frac{B}{2}\cos\theta d\varphi$

$\phi_1^{(Dirac)} = \frac{B}{2r}$

$\longleftrightarrow$ smooth config. on 4D

$\mathcal{A}^{(4D)}(z_1, z_2) \qquad \vec{x} = (z_1, \overline{z_2})\vec{\sigma}\begin{pmatrix} \overline{z_1} \\ z_2 \end{pmatrix}$

$$= g\left[A_i(\vec{x})dx^i + 2r(d\psi + \tfrac{1}{2}\cos\theta d\varphi)\phi_1(\vec{x})\right]g^{-1} - igdg^{-1}$$

$(g = e^{iB\psi})$

$$\begin{cases} \psi \to \psi + \nu \\ (z_1, z_2) \to (e^{-2\pi i\nu} z_1, e^{2\pi i\nu} z_2) \end{cases}$$

Config. of $A_i(\vec{x})dx^i, \ \phi_1(\vec{x})$ with Dirac singul. that satisfy $\star_3 F = D\phi_1$	$\longleftrightarrow$ **one-to-one corresp.**	Config. of $\mathcal{A}^{(4D)}(z_1, z_2)$ 4D 1-form that is inv. under $\begin{cases}(z_1, z_2) \to (e^{-i\nu} z_1, e^{i\nu} z_2) \\ \text{gauge trans.} \quad g = e^{iB\nu}\end{cases}$ and satisfy $\star_4 F = -F$

One-loop det. of vector multiplet

$X_0^B = (\tilde{A}_{1,2,3}(\vec{x}), \tilde{\phi}_1(\vec{x}), \tilde{A}_4)$

$X_1^F = (\Psi_{10,11,12}(\vec{x}), \ c, \tilde{c})$

$\longleftrightarrow$ **one-to-one corresp.**

$X_0^{B,\text{smooth}} = (\mathcal{A}^{(4D)}(z_1, z_2), \ A_4(z_1, z_2))$ (1- form; scalar)

$X_1^{F,\text{smooth}} = (\Psi_{10,11,12}(z_1, z_2), \ c, \tilde{c})$ (anti-self-dual 2- form; scalar)

$\bar{\psi}^{\mu\nu} = \bar{\sigma}^{\mu\nu}{}_{\dot{\alpha}A}\bar{\psi}^{A,\dot{\alpha}}$

that are inv. under $\psi \to \psi + \nu$

$$R = \cdots + \boxed{\tfrac{i}{R}\lambda(J_3 + I_3)} + \cdots$$

$(x_1 + ix_2) \to e^{2\pi i\lambda}(x_1 + ix_2) \qquad (z_1, z_2) \to (t_1 z_1, t_2 z_2) \quad t_{1,2} = e^{\pi i\lambda}$

$$\frac{\det_{X_0^{B,\text{smooth}}} R}{\det_{X_1^{F,\text{smooth}}} R} \longrightarrow \mathrm{Tr}_{X_0^{B,\text{smooth}}} e^{2\pi iR} - \mathrm{Tr}_{X_1^{F,\text{smooth}}} e^{2\pi iR}$$

scalar $A_4(z_1, z_2) = \sum_{k,\bar{l},m,\bar{n}\geq 0} c_{k,\bar{l},m,\bar{n}}\, z_1^k \bar{z}_1^{\bar{l}} z_2^m \bar{z}_2^{\bar{n}}$	$\xrightarrow{\mathrm{Tr}\, e^{2\pi iR}}$	$\sum_{k,\bar{l},m,\bar{n}\geq 0} t_1^k t_1^{-\bar{l}} t_2^m t_2^{-\bar{n}}$
1- form $\mathcal{A}^{(4D)}(z_1, z_2) = \sum_{k,\bar{l},m,\bar{n}\geq 0} c_{k,\bar{l},m,\bar{n}}\, z_1^k \bar{z}_1^{\bar{l}} z_2^m \bar{z}_2^{\bar{n}} \times \{dz_{1,2}, d\bar{z}_{1,2}\}$	$\xrightarrow{\mathrm{Tr}\, e^{2\pi iR}}$	$(t_1 + t_1^{-1} + t_2 + t_2^{-1}) \times \sum_{k,\bar{l},m,\bar{n}\geq 0} t_1^k t_1^{-\bar{l}} t_2^m t_2^{-\bar{n}}$
ASD 2- form $\Psi_{10,11,12} = \sum_{k,\bar{l},m,\bar{n}\geq 0} c_{k,\bar{l},m,\bar{n}}\, z_1^k \bar{z}_1^{\bar{l}} z_2^m \bar{z}_2^{\bar{n}} \times \{dz_1 d\bar{z}_1 + dz_2 d\bar{z}_2, \ dz_1 dz_2, \ d\bar{z}_1 d\bar{z}_2\}$	$\xrightarrow{\mathrm{Tr}\, e^{2\pi iR}}$	$(1 + t_1 t_2 + t_1^{-1} t_2^{-1}) \times \sum_{k,\bar{l},m,\bar{n}\geq 0} t_1^k t_1^{-\bar{l}} t_2^m t_2^{-\bar{n}}$

$$= \frac{(t_1 + t_1^{-1} + t_2 + t_2^{-1} + 1) - (1 + t_1 t_2 + t_1^{-1} t_2^{-1} + 1 + 1)}{(1 - t_1)(1 - t_2)(1 - t_1^{-1})(1 - t_2^{-1})}$$

$\downarrow \quad R = i\lambda(J_3 + I_3) \longrightarrow R = i\lambda(J_3 + I_3) - \boxed{i[\alpha, \cdot]}$

$$\frac{(t_1 + t_1^{-1} + t_2 + t_2^{-1} + 1) - (1 + t_1 t_2 + t_1^{-1} t_2^{-1} + 1 + 1)}{(1 - t_1)(1 - t_2)(1 - t_1^{-1})(1 - t_2^{-1})} \boxed{\sum_{\omega \in \text{root}} e^{2\pi i\omega\cdot\alpha}}$$

extract config. inv. under $\psi \to \psi + \nu \longrightarrow \int_0^1 d\nu \Big[\qquad \Big] \Big|_{\substack{t_1 = e^{\pi i\lambda}e^{-2\pi i\nu} \quad t_2 = e^{\pi i\lambda}e^{2\pi i\nu} \\ \alpha \to \alpha + B\nu}}$

$$\frac{\det_{X_1^F} R_1}{\det_{X_0^B} R_0} \longleftarrow \mathrm{Tr}_{X_0^B} e^{2\pi iR} - \mathrm{Tr}_{X_1^F} e^{2\pi iR}$$

Non perturbative part (effect of 't Hooft-Polyakov monopole)

To get $Z_{\text{mono.}}(B = 1, v = 0)$

extract the factors inv. under $\begin{cases} \epsilon_1 \to \epsilon_1 - \nu \\ \epsilon_2 \to \epsilon_2 + \nu \end{cases}$ from $Z_{1-\text{inst}}(\epsilon_1, \epsilon_2)$

$$Z_{1-\text{inst}}(\epsilon_1, \epsilon_2) = \sum_{s=\pm} \frac{(2s\alpha + \frac{\epsilon_+}{2} - m)(2s\alpha + \frac{\epsilon_+}{2} + m)(\frac{\epsilon_-}{2} - m)(-\frac{\epsilon_-}{2} - m)}{(2s\alpha)(2s\alpha + \epsilon_+)\epsilon_1\epsilon_2}$$

$$\times \sum_{n\in\mathbb{Z}} e^{2\pi in}, \ \epsilon_{1,2} = \frac{\lambda}{2} \longrightarrow Z_{\text{mono.}}(B = 1, v = 0) = \sum_{s=\pm} \frac{\prod_\pm \sin\pi(2\alpha \pm m + s\lambda/2)}{\sin(2\pi\alpha)\sin\pi(2\alpha + s\lambda)}$$

S-duality

Wilson loop $\quad \langle W_j \rangle \equiv \langle \mathrm{Tr}_j P \exp \oint_{S^1} dx^4(-iA_4 + A_0) \rangle$

$$\langle W_{1/2} \rangle = e^{2\pi i\alpha} + e^{-2\pi i\alpha}$$

$$\langle (W_{1/2})^2 \rangle = \langle W_1 \rangle + \langle W_0 \rangle = (e^{4\pi i\alpha} + 1 + e^{-4\pi i\alpha}) + 1 = \langle W_{1/2} \rangle^2$$

't Hooft loop

$$\langle T_{1/2} \rangle = e^{2\pi i\beta} Z_{1-\text{loop}}(B = \tfrac{1}{2}) + e^{-2\pi i\beta} Z_{1-\text{loop}}(B = -\tfrac{1}{2})$$

$$= (e^{2\pi i\beta} + e^{-2\pi i\beta}) \left(\frac{\sin(2\pi\alpha + \pi m)\sin(2\pi\alpha - \pi m)}{\sin(2\pi\alpha + \frac{\pi}{2}\lambda)\sin(2\pi\alpha - \frac{\pi}{2}\lambda)}\right)^{1/2}$$

$$\langle (T_{1/2})^2 \rangle = \langle T_1 \rangle$$

$$= e^{4\pi i\beta} Z_{1-\text{loop}}(B = 1) + Z_{\text{mono.}}(B = 1 \to 0) + e^{-4\pi i\beta} Z_{1-\text{loop}}(B = -1)$$

$$= (e^{4\pi i\beta} + e^{-4\pi i\beta}) \left(\frac{\prod_{s_1, s_2 = \pm 1} \sin^{1/2}(2\pi\alpha + s_1\pi m + s_2\frac{\pi}{2}\lambda)}{\sin^{1/2}(2\pi\alpha + \pi\lambda)\sin^{1/2}(2\pi\alpha - \pi\lambda)\sin(2\pi\alpha)}\right)$$

$$+ \sum_{s=\pm} \frac{\prod_\pm \sin\pi(2\alpha \pm m + s\lambda/2)}{\sin(2\pi\alpha)\sin\pi(2\alpha + s\lambda)}$$

$= \langle T_{1/2} \rangle^2 \longrightarrow$ S-duality $\quad \boxed{\langle L^2 \rangle = \langle L \rangle * \langle L \rangle \equiv e^{i\frac{\lambda}{4\pi}(\partial_\beta\partial_{\alpha'} - \partial_\alpha\partial_{\beta'})} \langle L \rangle \langle L \rangle' |_{\alpha'=\alpha, \beta'=\beta}}$

if $[\alpha, \beta] = i\frac{\lambda}{4\pi}$

Origin of Noncommutativity

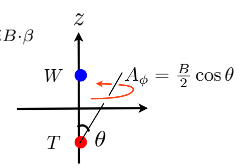
$$\langle W \cdot T \rangle \sim \left(e^{-iq\oint_{S^1}(A_\tau + i\Phi_0)d\tau}\Big|_{A = \frac{B}{2}\cos\theta d\phi}\right) \cdot e^{2\pi iB\cdot\beta}$$

$z_W > z_T$

$(\tau + 2\pi R, \phi) \sim (\tau, \phi + 2\pi\lambda)$

$\to (\tau' + 2\pi R, \ \phi') \sim (\tau', \phi')$ periodic. bdy. cond.

$(\tau', \phi') = (\tau, \phi + \frac{\lambda}{R}\tau) \qquad A_{\tau'} = A_\tau - \frac{\lambda}{R}A_\phi$



$q = 1, B = 1$

$$\langle W \cdot T \rangle \sim \left(e^{-i\oint_{S^1}(A_{\tau'} + i\Phi_0)d\tau'}\Big|_{A = \frac{1}{2}\cos\theta d\phi}\right) \cdot e^{2\pi i\beta}$$

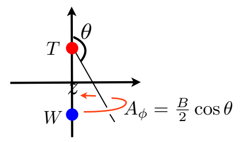
$z_W > z_T$

$$\overset{\theta = 0}{=} \ e^{-2\pi i(\alpha - \frac{1}{2}\lambda)} e^{2\pi i\beta}$$

$$\langle T \cdot W \rangle \sim e^{2\pi i\beta} \cdot \left(e^{-i\oint_{S^1}(A_{\tau'} + i\Phi_0)d\tau'}\Big|_{A = \frac{1}{2}\cos\theta d\phi}\right)$$

$z_T > z_W$

$$\overset{\theta = \pi}{=} \ e^{2\pi i\beta} e^{-2\pi i(\alpha + \frac{1}{2}\lambda)}$$



$\langle W \rangle = e^{2\pi i\alpha}$

$\langle T \rangle = e^{2\pi i\beta}$

$\longrightarrow$ consistenet with $\langle L_1 \cdot L_2 \rangle = \langle L_1 \rangle * \langle L_2 \rangle$

AGT correspondence

4D $\mathcal{N} = 2$ $SU(2)$ gauge theories with 1 adj. hyp. $\longleftrightarrow$ 2D CFT Liouville theory on torus with 1 vtx. op.

v.e.v. of loop. op. on S^4 — Verlinde loop op.

$$\langle L \rangle = \int d\vec{a}\, \overline{Z(a)} \sum_{a'} L(a, a') Z(a')$$

$b = 1$

$$\langle \mathcal{L} \rangle = \int d\alpha\, \overline{\mathcal{F}_m(\alpha)} \left(\mathcal{L} \cdot \mathcal{F}_m(\alpha)\right) = \int d\alpha\, \overline{\mathcal{F}_m(\alpha)} \left(\sum_{\alpha'} L(\alpha, \alpha', b)\mathcal{F}_m(\alpha')\right)$$

v.e.v. of loop op. on $S^1 \times R^3$ with twisted bdy. cond. around S^1 λ — Verlinde loop op.

$$\langle L \rangle = \sum_B e^{2\pi iB\cdot\beta} L(\alpha, B, \lambda) \ \overset{\lambda = b^2}{\underset{\beta \sim \partial_\alpha}{=\!=}} \ \sum_{\alpha'} L(\alpha, \alpha', b)\, e^{(\alpha' - \alpha)\partial_\alpha}$$

4D $\mathcal{N} = 2$ $SU(2)$ gauge theories on $R^3 \times S^1$ with twist

Wilson loop $\longleftrightarrow$ 't Hooft loop

holonomies of $SL(2, \mathbb{C})$ flat connection on Riemann surface = geodesic length on Riemann surface with negative const. curvature

$X := 2\cosh(\frac{1}{2}\text{length}(\gamma_W)) = X(\alpha) \quad =: 4\pi i\alpha$

$Y := 2\cosh(\frac{1}{2}\text{length}(\gamma_T)) = Y(\alpha, \beta)$

$$X = 2\cosh\left(\frac{\text{length}(\gamma_W)}{2}\right) = e^{2\pi i\alpha} + e^{-2\pi i\alpha}$$

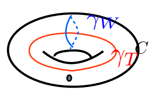
$$Y = 2\cosh\left(\frac{\text{length}(\gamma_T)}{2}\right)$$

$$= 2\cosh(2\pi i\beta)\frac{\sqrt{\sinh(2\pi i\alpha + \pi im)\sinh(2\pi i\alpha - \pi im)}}{\sinh(2\pi i\alpha)}$$

$$= \sum_\pm \frac{1}{\sqrt{\sinh(2\pi i\alpha)}} e^{\pm\pi i\beta} \sqrt{\sinh(2\pi i\alpha + \pi im)\sinh(2\pi i\alpha - \pi im)}\, e^{\pm\pi i\beta} \frac{1}{\sqrt{\sinh(2\pi i\alpha)}}$$

$$[\alpha, \beta] = i\frac{\lambda}{4\pi} \longrightarrow \sum_\pm e^{\pm\pi i\beta} \left(\frac{\sinh(2\pi i\alpha + \pi im)\sinh(2\pi i\alpha - \pi im)}{\sinh(2\pi i\alpha + \frac{\pi i}{2}\lambda)\sinh(2\pi i\alpha - \frac{\pi i}{2}\lambda)}\right)^{1/2} e^{\pm\pi i\beta}$$

$$\overset{!}{=} T_{1/2}$$



β twist para.