On the Construction of Calorons with Cyclic Symmetry

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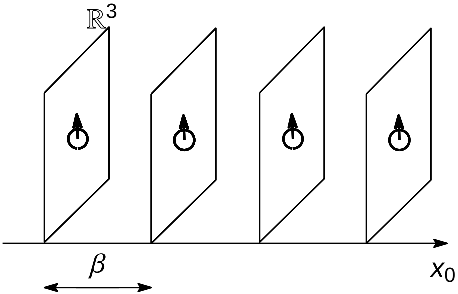
What is caloron?

Analytic solutions to (A)SDYM on $\mathbb{R}^3 \times S^1$

$=$ YM Instantons with periodicity



Topological soliton in finite temperature field theories

$\Rightarrow$Applications: *e.g. Seed of confinement in QCD*

♠ A few solution to the analytic description of calorons

(or their *dual* description = Nahm construction)

In this work

Attempt to find the caloron Nahm data with symmetry under the cyclic group of order N (C_N)

$\diamondsuit$

by applying

C_N symmetric ansatz for the Nahm data of monopoles by Sutcliffe

along with

Solutions to Periodic Toda lattice by elliptic theta functions

$\diamondsuit$

$\spadesuit$ $^{\exists}$ distinction: monopole Nahm data $\leftrightarrow$ caloron Nahm data leads to new type of solutions.

ADHM/Nahm construction-1

ASD Yang-Mills equations: First order PDE (4 variables) for A_μ

$$F_{\mu\nu} = -\tilde{F}_{\mu\nu}, \quad \tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\lambda\rho}F_{\lambda\rho} \quad (\mu,\nu = 0,1,2,3)$$

$$(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] \in g)$$

Hard to solve ASD! — however,

♠ On $\mathbb{R}^4$, $^\exists$ dual description: ADHM formalism
PDE $\longrightarrow$ Algebraic Eqns.

♣ On $\mathbb{R}^3 \times S^1$, $^\exists$ dual description: Nahm formalism
PDE $\longrightarrow$ ODE(Nahm eqns.) $\oplus$ Algebraic Eqns.

Note: if $\beta \to \infty$, then $S^1 \times \mathbb{R}^3 \to \mathbb{R}^4$
$^\exists$ "ADHM limit" of Nahm is expected, (but always true?)

ADHM/Nahm construction-2

cf. Bogomolnyi eqns. on $\mathbb{R}^3$: $B_j = -D_j\Phi$ (PDE for 3 variables)
Finite Energy solutions to BE: BPS Monopoles

♢

ASDYM → BE

$$F_{\mu\nu} = -\tilde{F}_{\mu\nu}\Big|\ A_0 := \Phi,\ \partial_0 = 0\ \Rightarrow\ B_j = -D_j\Phi$$

As is well known:
$^{\exists}$ dual description of BE: Nahm formalism (for monopoles)
PDE ⟶ Nahm eqns. (without algebraic Eqns.)

♠ We expect the picture:
Instantons ← Calorons → Monopoles
(but always true?)

The Nahm construction for calorons

Nahm data $\{T_j(u), W\}$: The dual of ASD gauge field A_μ on $\mathbb{R}^3 \times S^1$
(fixing $G = SU(2)$)

ASD $\Leftrightarrow$ Constraints on $T_j(u) \in M(N \times N), u \in [-\mu, \mu]$
and $W \in V(N) \otimes \mathbb{H}$:

1. $T_1{}' = i[T_2, T_3],\ T_2{}' = i[T_3, T_1],\ T_3{}' = i[T_1, T_2]$ (Nahm Eqns.)
2. $T_j(-\mu) - T_j(\mu) = \frac{1}{2}\mathrm{tr}_2\, \sigma_j W^\dagger W$ (Matching Conds.)
3. $T_j^\dagger(u) = T_j(u)$ (Hermiticity)
4. $T_j^t(u) = T_j(-u)$ (Reality)

From $\{T_j(u), W\} \to$ ASD A_μ (Nahm Transform)

We refer to $T_j(u)$: Bulk Data, W: Boundary data, resp..

The Nahm construction for monopoles

Nahm data $\{T_j(u)\}$: The dual of BPS monopole fields A_j, Φ on $\mathbb{R}^3$
(fixing $G = SU(2)$)

BPS Monopole $\Leftrightarrow$ Constraints on $T_j(u) \in M(N \times N), u \in [-\mu, \mu]$

Nahm Eqns., Hermiticity, Reality : commonly to calorons,

Matching Conds. $\to$ Boundary Conds.

$T_j(\pm\mu) \sim \dfrac{\rho_j}{u \pm \mu} + \ldots$, where ρ_j belongs to N-dim. irr. repr. of $su(2)$

From $\{T_j(u)\} \to$ BPS A_j, Φ (Nahm Transform)
N: Monopole charge

List on Analytic Nahm Data-Monopoles

(M-a) **1-monopole**: $N = 1$ Nahm eqns. are trivially solved.

(M-b) **2-monopoles**: Let $T_j = f_j\sigma_j$, then Nahm eqns. are

$$\dot{f}_1 = -2f_2f_3,\ \dot{f}_2 = -2f_3f_1,\ \dot{f}_3 = -2f_1f_2.$$

Solution enjoying the constraints:

$$f_1 = \frac{Kk'}{\mathrm{cn}2Ku},\ f_2 = -Kk'\frac{\mathrm{sn}2Ku}{\mathrm{cn}2Ku},\ f_3 = K\frac{\mathrm{dn}2Ku}{\mathrm{cn}2Ku}.$$

(M-c) **Symmetric monopoles**: Monopoles with Platonic symmetries by Hitchin, Manton and Murray (Nonlinearity 8(1995)661); Nahm data = Weierstrass elliptic functions.

(M-d) **Cyclic monopoles**: Ansatz for C_N symmetric monopoles, conjectured by P.M.Sutcliffe (NPB505(1997)517), and its uniqueness proved by H.W.Braden (CMP **308**(2011)303). No explicit solution.

List on Analytic Nahm Data-Calorons

- (C-a) **Harrington-Shepard 1-caloron**: Superposition of instantons (PRD17(1978)2122); The bulk Nahm data is trivial.
- (C-b) **Symmetric calorons**: Periodic counterparts of (M-b) and (M-c) by R.S.Ward (PLB582(2005)203); The bulk Nahm data are parallel to (M-b) and (M-c).
- (C-c) **1-caloron with non-trivial holonomy**: given by T.C.Kraan and P.van Baal (NPB533(1998)627), and K.Lee and C.Lu (PRD58(1998)025011); The bulk Nahm data is trivial. However, the Nahm transform is very complicated.
- (C-d) **2-calorons with non-trivial holonomy**: generalisation to $N=2$ are considered by F.Bruckmann, D.Nógrádi, P.van Baal (NPB698(2004)233), D.Harland (JMP48(2007)082905), A.N and J.Sakaguchi (JMP51(2010)043503). The bulk Nahm data are parallel to (M-a).

Ansatz for bulk data by Sutcliffe

The ansatz for bulk Nahm data with C_N-sym.

$$T_1=\frac{1}{2}\begin{bmatrix} & f_1 & & & f_0 \\ f_1 & & f_2 & & \\ & f_2 & & \ddots & \\ & & \ddots & & f_{N-1} \\ f_0 & & & f_{N-1} & \end{bmatrix},\; T_2=\frac{i}{2}\begin{bmatrix} & -f_1 & & & f_0 \\ f_1 & & -f_2 & & \\ & f_2 & & \ddots & \\ & & \ddots & & -f_{N-1} \\ -f_0 & & & f_{N-1} & \end{bmatrix}$$

$$T_3=\frac{1}{2}\,\mathrm{diag.}\,[p_1,p_2,\cdots,p_{N-1},p_0]$$

(Note: f_j, $p_j\in\mathbb{R}$ for the hermiticity)
This data has C_N-symmetry, *i.e.*,

$$R_N T_j R_N^{-1}=R_{jk}T_k,$$

R_N, R_{jk} are image of C_N in $U(N)$ and $SO(3)$, resp.

Reduced Nahm eqns.

The Nahm eqns. $\Rightarrow$ periodic Toda Lattice (of imaginary coupl.)

$$\begin{cases} f_j' = \frac{1}{2} f_j \left(p_{j+1} - p_j\right) \\ p_j' = f_{j-1}^2 - f_j^2 \end{cases} \Rightarrow \frac{d^2}{du^2} \log f_j^2 = -f_{j+1}^2 + 2f_j^2 - f_{j-1}^2$$

Introducing τ-function: $f_j^2 = -C^2 \frac{\tau_{j-1}\tau_{j+1}}{\tau_j^2}, \; p_j = \frac{d}{du}\left(\log \frac{\tau_j}{\tau_{j-1}}\right)$

$$\Rightarrow \frac{d^2}{du^2} \log \tau_j = C^2 \frac{\tau_{j-1}\tau_{j+1}}{\tau_j^2} \quad \text{(Nahm eqns. for } \tau\text{)}$$

Solution by Jacobi theta

$$\tau_j(u) = \exp\left(\frac{1}{2}\tilde{A}u^2 + bu + \tilde{b}j\right)\vartheta_\nu(\pm u + \kappa j + a, \tau) \ \rightarrow \ \text{Nahm eqns.}$$

Defining $u_j := \pm u + \kappa j + a$, we find $(\nu = 0, 1, 2, 3)$

$$A + C^{-2}\left(\log \vartheta_\nu(u_j)\right)'' = \frac{\vartheta_\nu(u_j - \kappa)\vartheta_\nu(u_j + \kappa)}{\vartheta_\nu^2(u_j)}$$

A differential eqn. for Jacobi theta (M.Toda (1967)), if we choose

$$C^{-2} = \left(\frac{\vartheta_1(\kappa)}{\vartheta_1'(0)}\right)^2, \ A = \tilde{A}C^{-2} = \left(\frac{\vartheta_0(\kappa)}{\vartheta_0(0)}\right)^2 - \frac{\vartheta_0''(0)}{\vartheta_0(0)}\left(\frac{\vartheta_1(\kappa)}{\vartheta_1'(0)}\right)^2$$

♠ The solutions are:

$$f_j^2 = -C^2\frac{\vartheta_\nu(u_{j-1})\vartheta_\nu(u_{j+1})}{\vartheta_\nu(u_j)^2}, \ p_j = \frac{d}{du}\left(\log\frac{\vartheta_\nu(u_j)}{\vartheta_\nu(u_{j-1})}\right)$$

Hermiticity

Elements of Bulk Nahm data

$$f_j = iC\frac{\sqrt{\vartheta_\nu(u_{j-1})\vartheta_\nu(u_{j+1})}}{\vartheta_\nu(u_j)},\ p_j = \frac{d}{du}\left(\log\frac{\vartheta_\nu(u_j)}{\vartheta_\nu(u_{j-1})}\right)$$

For hermiticity, we need $f_j, p_j \in \mathbb{R}$

- $\vartheta_\nu > 0$ (or < 0) for all j with appropriate range of u
- $C = \frac{\vartheta_1{}'(0)}{\vartheta_1(\kappa)} \in i\,\mathbb{R}$

It's sufficient to choose:

- The modulus $\tau \in i\mathbb{R}$ ($q := e^{i\pi\tau}$),
  and $\nu = 0$ or 3 (ϑ_2 has negative region on $\mathbb{R}$)
- Since $\vartheta_1'(0) = 2\pi q^{1/4}\prod_{m=1}^{\infty}(1-q^{2m})^3$, taking the branch $q^{1/4} \in i\,\mathbb{R}$

Reality

For the reality to be enjoyed, we need a change of basis

$$\tilde{T}_j = U_N T_j U_N^{-1} \Rightarrow \tilde{T}_j^t(u) = \tilde{T}_j(-u)$$

We refer to

- T_j: C_N-symmetric basis
- $\tilde{T}_j$: Reality basis

◊

♠ To find U_N is a non-trivial problem

Boundary Data

The boundary data is an N-dim. quaternion vector

$$W = (\lambda_1, \lambda_2, \cdots, \lambda_N), \ \lambda_j \in \mathbb{H}$$

which enjoys

$$T_j(-\mu) - T_j(\mu) = \frac{1}{2}\mathrm{tr}_2\, \sigma_j W^\dagger W$$

The boundary data is C_N-symmetric, if it satisfies

$$R_N \otimes R_2\, W^\dagger = W^\dagger \hat{q}$$

for a unit quaternion ($\hat{q}\hat{q}^\dagger = 1$)

$\diamondsuit$

♠ Quite non-trivial to find a C_N-symmetric boundary data

N=3 Bulk Data

$$T_1 = \frac{1}{2}\begin{bmatrix} 0 & f_1 & f_0 \\ f_1 & 0 & f_2 \\ f_0 & f_2 & 0 \end{bmatrix}, T_2 = \frac{i}{2}\begin{bmatrix} 0 & -f_1 & f_0 \\ f_1 & 0 & -f_2 \\ -f_0 & f_2 & 0 \end{bmatrix}$$

$$T_3 = \frac{1}{2}\ \text{diag.}\ [p_1, p_2, p_0],$$

The solutions ($j = 0, 1, 2,\ \nu = 0$ or 3):

$$f_j(u) = iC\frac{\sqrt{\vartheta_\nu(u_{j+1})\vartheta_\nu(u_{j-1})}}{\vartheta_\nu(u_j)},\ p_j(u) = \frac{d}{du}\log\frac{\vartheta_\nu(u_j)}{\vartheta_\nu(u_{j-1})},$$

$$u_j = \pm u + \frac{j}{3} \quad (\kappa = \frac{1}{3})$$

Into the reality basis

By a unitary transformation, the bulk data in reality basis are

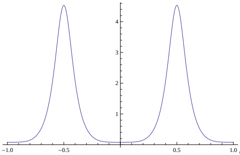
$$\tilde{T}_1 = \frac{1}{2}\begin{bmatrix} 0 & f_+ - if_- & f_0 \\ f_+ + if_- & 0 & f_+ - if_- \\ f_0 & f_+ + if_- & 0 \end{bmatrix}$$

$$\tilde{T}_2 = \frac{1}{2}\begin{bmatrix} f_0 & -f_+ - if_- & 0 \\ -f_+ + if_- & 0 & f_+ + if_- \\ 0 & f_+ - if_- & -f_0 \end{bmatrix}$$

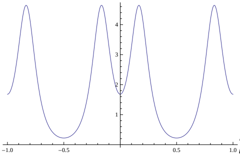
$$\tilde{T}_3 = \frac{1}{4}\begin{bmatrix} -p_2 & 0 & i(p_0 - p_1) \\ 0 & 2p_2 & 0 \\ -i(p_0 - p_1) & 0 & -p_2 \end{bmatrix}$$

where $f_\pm = (f_1 \pm f_2)/2$

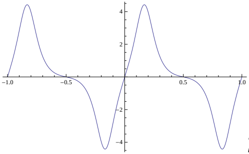
Elements of the bulk data: ϑ_3, $q = 0.5$



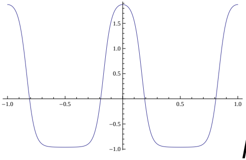
$f_0(u)$



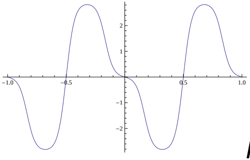
$f_+(u)$



$f_-(u)$

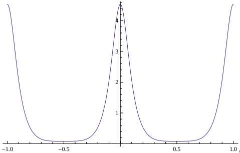


$p_2(u)$

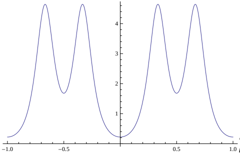


$p_0(u) - p_1(u)$

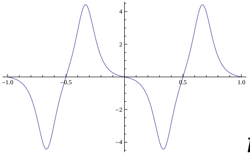
Elements of the bulk data: ϑ_0, $q = 0.5$



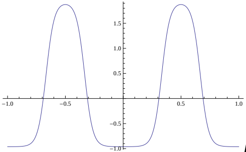
$f_0(u)$



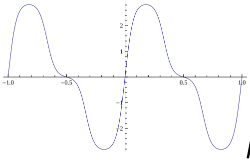
$f_+(u)$



$f_-(u)$



$p_2(u)$



$p_0(u) - p_1(u)$

Boundary data: Matching Conditions

$$\tilde{T}_j(-\mu) - \tilde{T}_j(\mu) = \frac{1}{2}\mathrm{tr}_2\,\sigma_j \tilde{W}^\dagger \tilde{W}, \quad \text{where} \quad \tilde{W} = (\lambda, \rho, \chi)$$

Only the odd-function elements contribute to LHS:

$$\tilde{T}_1(-\mu) - \tilde{T}_1(\mu) = \begin{bmatrix} 0 & -ig(\mu) & 0 \\ ig(\mu) & 0 & -ig(\mu) \\ 0 & ig(\mu) & 0 \end{bmatrix}$$

$$\tilde{T}_2(-\mu) - \tilde{T}_2(\mu) = \begin{bmatrix} 0 & -ig(\mu) & 0 \\ ig(\mu) & 0 & ig(\mu) \\ 0 & -ig(\mu) & 0 \end{bmatrix}$$

$$\tilde{T}_3(-\mu) - \tilde{T}_3(\mu) = \begin{bmatrix} 0 & 0 & ih(\mu) \\ 0 & 0 & 0 \\ -ih(\mu) & 0 & 0 \end{bmatrix}$$

Boundary data: Matching Conditions

, where

$$g(\mu) := \frac{1}{2}(f_-(-\mu) - f_-(\mu)) = -f_-(\mu)$$

$$h(\mu) := \frac{1}{4}\{(p_0(-\mu) - p_1(-\mu)) - (p_0(\mu) - p_1(\mu))\}$$
$$= -\frac{1}{2}(p_0(\mu) - p_1(\mu))$$

A solution to Boundary data $\tilde{W} = (\lambda, \rho, \chi)$:

$$\lambda = i\lambda_1(\sigma_1 + \sigma_2),\ \rho = \rho_0 = -\frac{2}{\lambda_1}g(\mu),$$
$$\chi = -i\lambda_1(\sigma_1 - \sigma_2)$$

with a constraint: $h(\mu) = 2\lambda_1^2 > 0 \ \Rightarrow\ {}^{\exists}\lambda_1 \in \mathbb{R}$

Boundary data: C_3-symmetry

In the C_3-symmetric basis, we can show

$$R_3 \otimes R_2 \, W^\dagger = W^\dagger \hat{q}$$

where

$$R_2 = \hat{q} = \cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right) \sigma_3$$

◊

Therefore, we have found the caloron Nahm data $\{T_j, W\}$ with C_3-symmetry!

Large Scale Limit

The bulk data have no poles on the real axis

$\Downarrow$

There is no large scale, or monopole, limit

$^\exists$ Calorons which do NOT have monopole limit

Large Period Limit: $\mu \to 0$

Bulk data: Only the values at $\mu = 0$ remain $\Rightarrow$ ADHM data

$$\tilde{T}_1 \to \frac{1}{2}\begin{bmatrix} 0 & f_+^0 & f_0^0 \\ f_+^0 & 0 & f_+^0 \\ f_0^0 & f_+^0 & 0 \end{bmatrix}, \quad \tilde{T}_2 \to \frac{1}{2}\begin{bmatrix} f_0^0 & -f_+^0 & 0 \\ -f_+^0 & 0 & f_+^0 \\ 0 & f_+^0 & -f_0^0 \end{bmatrix},$$

$$T_3 \to \frac{1}{4}\begin{bmatrix} -p_2^0 & 0 & 0 \\ 0 & 2p_2^0 & 0 \\ 0 & 0 & -p_2^0 \end{bmatrix}$$

where ($iC \in \mathbb{R}$, $\kappa = 1/3$, taking the ϑ_0 solution)

$$f_+^0 := f_+(0) = iC\sqrt{\frac{\vartheta_0(0)}{\vartheta_0(\kappa)}}, \quad f_0^0 := f_0(0) = iC\frac{\vartheta_0(\kappa)}{\vartheta_0(0)}$$

$$p_2^0 := p_2(0) = 2\frac{\vartheta_0'(\kappa)}{\vartheta_0(\kappa)}$$

Large Period Limit: ADHM data

$$\Delta = \begin{bmatrix} \tilde{\lambda} & \tilde{\rho} & \tilde{\chi} \\ & & \\ & i\tilde{T}_j \otimes \sigma_j & \\ & & \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{bmatrix}$$

$$\Downarrow$$

$$\frac{1}{2}\begin{bmatrix} 2i\tilde{\lambda}_1(\sigma_1+\sigma_2) & 2\tilde{\rho}_0 & -2i\tilde{\lambda}_1(\sigma_1-\sigma_2) \\ i(f_0^0\sigma_2 - \frac{1}{2}p_2^0\sigma_3) & if_+^0(\sigma_1-\sigma_2) & if_0^0\sigma_1 \\ if_+^0(\sigma_1-\sigma_2) & ip_2^0\sigma_3 & if_+^0(-\sigma_1+\sigma_2) \\ if_0^0\sigma_1 & if_+^0(\sigma_1+\sigma_2) & -i(f_0^0\sigma_2+\frac{1}{2}p_2^0\sigma_3) \end{bmatrix}$$

Large Period Limit: ADHM data

For the Δ to be ASD ($\Leftrightarrow \mathrm{tr}\sigma_j \Delta^\dagger \Delta = 0$)

$$4\tilde{\lambda}_1^2 = -C^2 \left\{ \left(\frac{\vartheta_0(\kappa)}{\vartheta_0(0)} \right)^2 - \frac{\vartheta_0(0)}{\vartheta_0(\kappa)} \right\} > 0$$

$$\tilde{\rho}_0 = \frac{iC}{4\tilde{\lambda}_1} \left(-3\vartheta_0'(\kappa) \right) \sqrt{\frac{\vartheta_0(0)}{\vartheta_0^3(\kappa)}}$$

$$\Rightarrow \ {}^{\exists}\tilde{\lambda}_1,\ \tilde{\rho}_0 \in \mathbb{R} \text{ for } \vartheta_0$$

We have confirmed the existence of ADHM (instanton) limit for ϑ_0, but this is not for $\vartheta_3 \Rightarrow$ $^{\exists}$ or $^{\nexists}$ the instanton limit

♠ Is the “cyclic instanton” new type?

In summary

We have found:

- The Nahm data of calorons with C_3-symmetry in terms of Jacobi theta.
- Existence of calorons without monopole limit.
- For ϑ_0, $^\exists$ ADHM limit, but ϑ_3 is not.

Future direction:

- Generalization to C_N-symmetric calorons, Riemann theta, Other class of solutions rather than TL ansatz, $\cdots$
- To find the gauge field profile by Nahm transform – numerical analysis