基研研究会「場の理論と弦理論」
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E弦理論のSeiberg-Witten解とNekrasov型公式

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Plan

1. What is the E-string theory?

2. Compactification down to 4D: Nekrasov-type expression

3. Nekrasov-type expressions with Wilson line parameters

String Theory predicts the existence of nontrivial QFTs in 6D.

- The worldvolume theory of multiple M5 branes

  --- (2,0) SUSY (16 supercharges)

What about theories with (1,0) SUSY?

- Heterotic string theories on K3 (16 $\longrightarrow$ 8 supercharges)

What happens when instantons in K3 shrink to zero size?

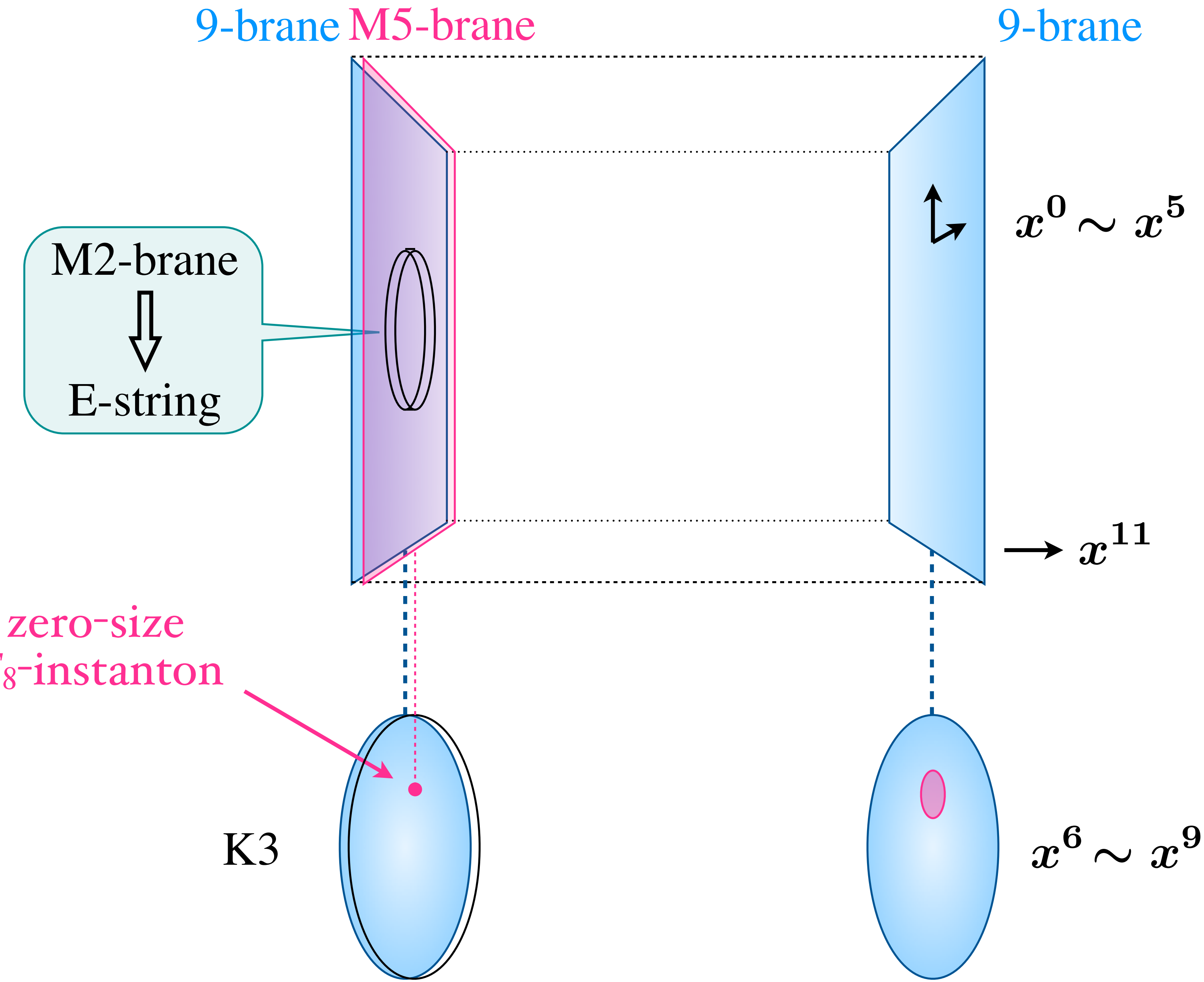
- Small SO(32) instantons

  $\Rightarrow$ extra Sp(n) gauge symmetry (Witten '95)

  $\Leftrightarrow$ worldvolume theory of n type I D5-branes

- What about small E_8 instantons?

M-theory description of the $E_8 \times E_8$ heterotic string theory on K3



E-string theory

(Ganor-Hanany '96) (Seiberg-Witten '96) (Klemm-Mayr-Vafa '96)
(Ganor-Morrison-Seiberg '96) (Minahan-Nemeschansky-Vafa-Warner '98)

- 6D (1,0)-supersymmetric local QFT
- decoupled from gravity
- no vector multiplets
- Coulomb branch --- a tensor multiplet
- Higgs branch $\cong$ the moduli space of an E_8 instanton
- fundamental excitations --- strings
- global E_8 symmetry

The simplest interacting QFT with (1,0) SUSY in 6D!?

Toroidal compactification down to lower dimensions

6D $\mathcal{N} = (1,0)$ tensor multiplet

$\Downarrow$

5D $\mathcal{N} = 1$ vector multiplet

$\Downarrow$

4D $\mathcal{N} = 2$ vector multiplet

One can study the low energy theory in the Coulomb branch by means of Seiberg-Witten theory.

Seiberg-Witten theory

(Seiberg-Witten '94)

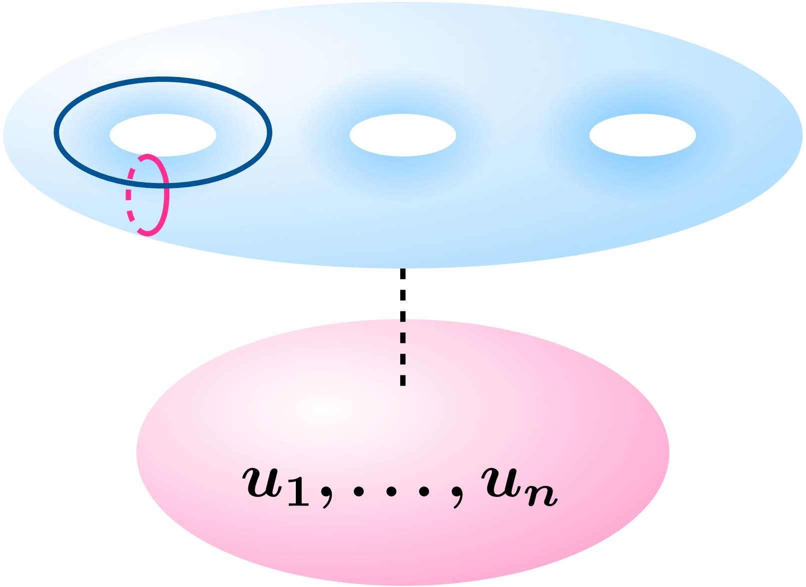
- Exact solution to the low energy theory of 4D $\mathcal{N}=2$ SYM

Low energy effective Lagrangian

$$\mathcal{L}_{\text{eff}} = \frac{1}{4\pi}\mathrm{Im}\left[\int d^4\theta \frac{\partial F_0(A)}{\partial A^i}\bar{A}^i + \int d^2\theta \frac{1}{2}\frac{\partial^2 F_0(A)}{\partial A^i \partial A^j} W^i_\alpha W^{\alpha j}\right]$$

prepotential $F_0(a_1, \ldots, a_n)$: holomorphic function

Seiberg-Witten curve



λ_{SW} : Seiberg-Witten differential

$$a_i = \oint_{\alpha_i} \lambda_{\text{SW}}, \qquad \frac{\partial F}{\partial a_i} = \oint_{\beta_i} \lambda_{\text{SW}}$$

Nekrasov partition function

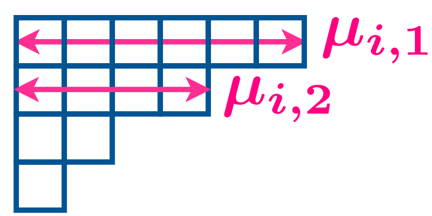
(Nekrasov '02)

(for pure SU(N) theory; instanton part; $\epsilon_1 = -\epsilon_2 = \hbar$)

for 5D theory (in $\mathbb{R}^4 \times S^1$)

$$Z = \sum_{R} \Lambda^{|R|} \prod_{i,j=1}^{N} \prod_{k,l=1}^{\infty} \frac{\sinh \beta \left(a_{ij} + \hbar(\mu_{i,k} - \mu_{j,l} + l - k)\right)}{\sinh \beta \left(a_{ij} + \hbar(l - k)\right)}$$

$R = (R_1, \ldots, R_N)$ $\quad$ R_i : partition $\quad$ $(a_{ij} := a_i - a_j)$



$$Z = \exp \sum_{g=0}^{\infty} F_g \hbar^{2g-2}$$

$\Rightarrow$ F_0 : prepotential

Gauge theories and the E-string theory

--- similarities and differences ---

SU(n) gauge theories	E-string theory
Seiberg-Witten curves are known	
realized by String Theory on certain CY_3	
$\exists$ Lagrangian description	no Lagrangian description
toric CY_3	non-toric CY_3
Nekrasov partition functions are known	Any analogue? $\Rightarrow$ Yes! (at least partly)

Heterotic -- F-theory duality

(Morrison-Vafa '96)

Heterotic on K3 ⇨ E-strings on $\mathbb{R}^6$

⇕

F-theory on CY$_3$ ⇨ E-strings on $\mathbb{R}^6$

⇕

M-theory on CY$_3$ ⇨ E-strings on $\mathbb{R}^5 \times S^1$

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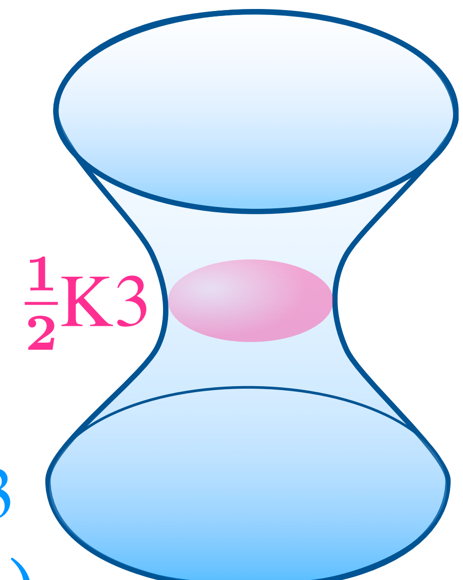
local $\frac{1}{2}$K3 surface

(the total space of
the canonical bundle
of a $\frac{1}{2}$K3 surface)



local $\frac{1}{2}$K3
(non-compact CY$_3$)

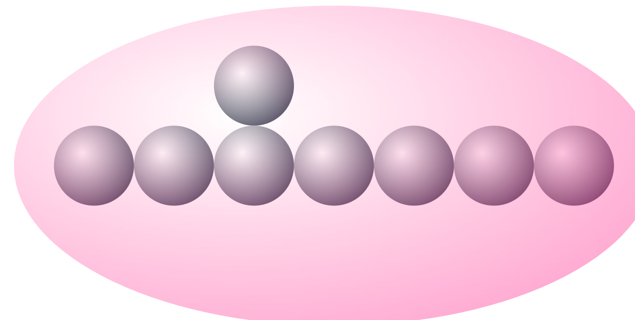
BPS states counting in M-theory on CY_3

(Gopakumar-Vafa '98)

$5D\ \mathcal{N}=1$ theory in $\mathbb{R}^4 \times S^1$

BPS states

= M2-branes wrapped over holomorphic 2-cycles in CY_3



Counting of BPS states = Counting of holomorphic 2-cycles

Seiberg-Witten prepotential of $5D\ \mathcal{N}=1$ theory in $\mathbb{R}^4 \times S^1$

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BPS partition function

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(genus-zero) topological string amplitude of the CY_3

Winding number expansion of the prepotential

$$F_0(\varphi,\tau) = \sum_{n=1}^{\infty} Q^n Z_n(\tau) \qquad Q := e^{2\pi i\varphi+\pi i\tau}$$

φ : scalar vev (tension) $\quad \tau$: complex modulus of T^2

$$Z_1 = \frac{E_4}{\eta^{12}}, \qquad Z_2 = \frac{E_2E_4^2 + 2E_4E_6}{24\eta^{24}}, \qquad \cdots$$

$$Z_n = \frac{P_{6n-2}(E_2,E_4,E_6)}{\eta^{12n}}$$

$E_{2n}(\tau)$: Eisenstein series $\quad \eta(\tau)$: Dedekind eta function

- Z_n at low orders can be determined by using the Seiberg-Witten curve or the modular anomaly equation

(Minahan-Nemeschansky-Warner '97)

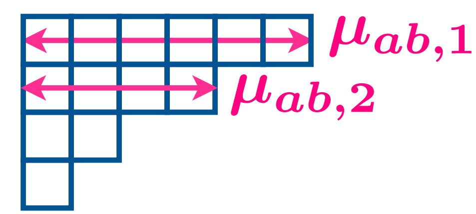
Nekrasov-type expression for the prepotential

(K.S. '12)

$$F_0 = \left(2\hbar^2 \ln \mathcal{Z}\right)\Big|_{\hbar=0}$$

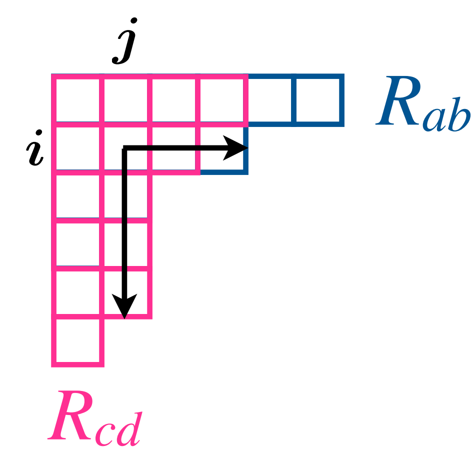
$$\mathcal{Z} = \sum_{\vec{R}} Q^{|\vec{R}|} \prod_{a,b,c,d} \prod_{(i,j)\in R_{ab}} \frac{\vartheta_{ab}\left(\frac{1}{2\pi}(j-i)\hbar, \tau\right)^2}{\vartheta_{1-|a-c|,1-|b-d|}\left(\frac{1}{2\pi}h_{ab,cd}(i,j)\hbar, \tau\right)^2}$$

$\vec{R} = (R_{11}, R_{10}, R_{00}, R_{01})$ $\quad$ R_{ab} : partition



$a, b, c, d = 0, 1$

$\vartheta_{ab}(z, \tau)$: Jacobi theta functions



$$h_{ab,cd}(i,j) := \mu_{ab,i} + \mu^{\vee}_{cd,j} - i - j + 1$$

(relative hook-length)

Prepotential = BPS partition function of wound E-strings

$$F_0(\varphi,\tau) = \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} N_{n,k} \sum_{m=1}^{\infty} \frac{1}{m^3} e^{2\pi i m(n\varphi + k\tau)}$$

n : winding number $\qquad$ k : momentum

$N_{n,k}$: BPS multiplicity (instanton number)

(Klemm-Mayr-Vafa '96)

n \ k	0	1	2	3	4	5	$\cdots$
1	1	252	5130	54760	419895	2587788	
2	0	0	−9252	−673760	−20534040	−389320128	
3	0	0	0	848628	115243155	6499779552	
4	0	0	0	0	−114265008	−23064530112	
5	0	0	0	0	0	18958064400	
$\vdots$							$\ddots$

The prepotential represents $g = 0$ topological string amplitude

$$F_0 = F_0^{\frac{1}{2}\mathrm{K3}}$$

However,

$$\mathcal{Z} \neq \mathcal{Z}^{\frac{1}{2}\mathrm{K3}}$$

$$:= \exp\left(\sum_{g=0}^{\infty} \hbar^{2g-2} F_g^{\frac{1}{2}\mathrm{K3}}\right)$$

Difference of modular anomalies

$$\partial_{E_2}\mathcal{Z} = \frac{1}{12}\hbar^2\partial_\phi^2\mathcal{Z}$$

$$\partial_{E_2}\mathcal{Z}^{\frac{1}{2}\mathrm{K3}} = \frac{1}{24}\hbar^2\partial_\phi(\partial_\phi + 1)\mathcal{Z}^{\frac{1}{2}\mathrm{K3}}$$

(Hosono-Saito-Takahashi '99)

⇓ ⇓

$$\partial_{E_2}F_0 = \frac{1}{24}\left(\partial_\phi F_0\right)^2$$

$$\partial_\phi = \frac{1}{2\pi i}\partial_\varphi = Q\partial_Q$$

Compactification with nontrivial Wilson line parameters

- One can introduce eight Wilson line parameters

$$\vec{\mu} = (m_1, \ldots, m_8) \in \mathbb{C}^8$$

and break the E_8 global symmetry

- Today we consider the cases with

$$\vec{\mu} = (m_1, m_2, m_3, m_4, m_1, m_2, m_3, m_4)$$

$$\Longleftrightarrow$$

$$\vec{\mu} = (0, 0, 0, 0, m_1 + m_2, m_1 - m_2, m_3 + m_4, m_3 - m_4)$$

- Notation

$$\vec{m} = (m_1, m_2, m_3, m_4)$$

Seiberg-Witten curve

- Explicit forms with general $\vec{\mu} = (m_1, \ldots, m_8)$ are known

(Ganor-Morrison-Seiberg '96) (Eguchi-K.S. '02)

--- expressed in terms of E_8-invariant Jacobi forms

- For the cases with $\vec{\mu} = (m_1, m_2, m_3, m_4, m_1, m_2, m_3, m_4)$,

we find a new expression based on
the SW curve for the 4D SU(2) $N_f = 4$ theory!

(K.S. '12)

Nekrasov-type expression

(K.S. '12)

$$\mathcal{Z} = \sum_{\vec{R}^{(4)}} Q^{|\vec{R}^{(4)}|} \prod_{a,b,c,d} \prod_{(i,j)\in R_{ab}} \frac{\vartheta_{ab}\left(\frac{1}{2\pi}(j-i)\hbar + m_{cd}, \tau\right) \vartheta_{ab}\left(\frac{1}{2\pi}(j-i)\hbar - m_{cd}, \tau\right)}{\vartheta_{1-|a-c|,1-|b-d|}\left(\frac{1}{2\pi} h_{ab,cd}(i,j)\hbar, \tau\right)^2}$$

$$\vec{m} = (m_1, m_2, m_3, m_4) = (m_{11}, m_{10}, m_{00}, m_{01})$$

$$F_0 = \left(2\hbar^2 \ln \mathcal{Z}\right)\Big|_{\hbar=0}$$

This is in perfect agreement with the prepotential computed from the Seiberg-Witten curve (checked up to order Q^{10}).

- Elliptic analogue of the Nekrasov partition function for the SU(N) gauge theory with $N_f = 2N$ flavors

(Nekrasov '02) (Hollowood-Iqbal-Vafa '03)

$$\mathcal{Z}^{\mathrm{SU}(N)}_{N_\mathrm{f}=2N}(\hbar;\varphi,\tau;a_1,\ldots,a_N;m_1,\ldots,m_{2N})$$

$$:= \sum_{\vec{R}^{(N)}} \left(-e^{2\pi i\varphi}\right)^{|\vec{R}^{(N)}|} \prod_{k=1}^{N} \prod_{(i,j)\in R_k} \frac{\prod_{n=1}^{2N} \vartheta_1\left(a_k + m_n + \frac{1}{2\pi}(j-i)\hbar,\tau\right)}{\prod_{l=1}^{N} \vartheta_1\left(a_k - a_l + \frac{1}{2\pi}h_{kl}(i,j)\hbar,\tau\right)^2}$$

- Our formula can be expressed as a special case of this function

$$\mathcal{Z} = \mathcal{Z}^{\mathrm{SU}(4)}_{N_\mathrm{f}=8}\left(\hbar;\varphi,\tau;0,\frac{1}{2},-\frac{1+\tau}{2},\frac{\tau}{2};m_1,m_2,m_3,m_4,-m_1,-m_2,-m_3,-m_4\right)$$

Global symmetries and counting of holomorphic curves in CY$_3$

Wilson line parameters	unbroken global symmetry	local geometry
$\vec{m} = (0,0,0,0)$	E_8	E_8 del Pezzo
$\vec{m} = \left(0,0,0,\frac{1}{2}\right)$	$E_7 \oplus A_1$	E_7 del Pezzo
$\vec{m} = \left(0,\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$	$E_6 \oplus A_2$	E_6 del Pezzo
$\vec{m} = \left(0,\frac{1}{4},\frac{1}{4},\frac{1}{2}\right)$	$E_5 \oplus A_3$	E_5 del Pezzo
$\vec{m} = \left(0,0,\frac{1}{2},\frac{1}{2}\right)$	D_8	$\mathbb{P}^1 \times \mathbb{P}^1$

Simplification

$$\vec{m} = (0, m_1, m_2, m_3)$$

$\Longrightarrow$

$$\mathcal{Z} = \mathcal{Z}^{\mathrm{SU}(3)}_{N_{\mathrm{f}}=6}\left(\hbar; \varphi, \tau; \frac{1}{2}, -\frac{1+\tau}{2}, \frac{\tau}{2}; m_1, m_2, m_3, -m_1, -m_2, -m_3\right)$$

- All the cases with $E_n \oplus A_{8-n}$ $(n = 8, 7, 6, 5)$ and D_8 are of this type

$$\vec{m} = \left(0, \frac{1}{2}, m_1, m_2\right)$$

$\Longrightarrow$

$$\mathcal{Z} = \mathcal{Z}^{\mathrm{SU}(2)}_{N_{\mathrm{f}}=4}\left(\hbar; \varphi, \tau; -\frac{1+\tau}{2}, \frac{\tau}{2}; m_1, m_2, -m_1, -m_2\right)$$

- The cases with $E_7 \oplus A_1$, $E_5 \oplus A_3$ and D_8 are of this type

- Example: the $E_7 \oplus A_1$ case

$$\mathcal{Z} = \sum_{\vec{R}^{(2)}} Q^{|\vec{R}^{(2)}|} \prod_{k,l=1}^{2} \prod_{(i,j)\in R_k} \frac{\vartheta_{k+2}\left(\frac{1}{2\pi}(j-i)\hbar, \tau\right)^2}{\vartheta_{|k-l|+1}\left(\frac{1}{2\pi}h_{kl}(i,j)\hbar, \tau\right)^2}$$

$$\left(Q = e^{2\pi i\varphi + \pi i\tau}\right)$$

$$= \mathcal{Z}^{\mathrm{SU}(2)}_{N_f=4}\left(\hbar; \varphi, \tau; -\frac{1+\tau}{2}, \frac{\tau}{2}; 0, 0, 0, 0\right)$$

$$F_0(\varphi, \tau) = \left(2\hbar^2 \ln \mathcal{Z}\right)\Big|_{\hbar=0}$$

$$= \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} N_{n,k} \sum_{m=1}^{\infty} \frac{1}{m^3} e^{2\pi i m(n\varphi + k\tau)}$$

2× BPS multiplicities

⇔ Instanton numbers of local E_7 del Pezzo surface

Summary

- We have found a Nekrasov-type expression for the Seiberg-Witten prepotential for the E-string theory.

- We have generalized the expression to the cases with four Wilson line parameters.

- Our formulas give very simple, closed expressions for the genus-zero topological string amplitudes of some basic non-toric Calabi-Yau threefolds.