# On Noether Charge for Theories with Chern-Simons Terms and Fluid/Gravity Correspondence

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# Introduction

# Black Hole Entropy

Bekenstein-Hawking formula for Einstein gravity

$$S_{BH} = \frac{A}{4G_N}$$

Higher derivative corrections

under diffeo/gauge trans.

- For Lagrangian satisfying $\delta_\chi \bar{L}_{cov} = \mathcal{L}_\xi \bar{L}_{cov}$ $\chi = \{\xi, \Lambda\}$
  → Wald formula [Wald, Iyer-Wald, Jacobson-Kang-Myers]

$$S_W = 2\pi \int_{Bif} \varepsilon_b{}^a \varepsilon^{cd} \frac{\delta \bar{L}_{cov}}{\delta R^a{}_{bcd}}$$

- For Chern-Simons terms → Tachikawa formula

[Tachikawa, Bonora et al.]

→ Main subject of this presentation

# Chern-Simons Term

Consider a system with gravity + U(1) gauge field

Examples of CS terms in (2n+1)-dim $\quad I_{CS}[\mathbf{A}, \mathbf{F}, \mathbf{\Gamma}, \mathbf{R}]$

(ex) $\quad \mathrm{tr}\left(\mathbf{\Gamma} \wedge d\mathbf{\Gamma} + \frac{2}{3}\mathbf{\Gamma}^3\right) \qquad \mathbf{A} \wedge \mathrm{tr}[\mathbf{R} \wedge \mathbf{R}]$

Non-covariance of Chern-Simons terms under diffeo/gauge trans.

$$\delta_\chi I_{CS} = \mathcal{L}_\xi I_{CS} + d(\ldots) \qquad \chi = \{\xi, \Lambda\}$$

Why Chern-Simons terms in gravity ?

- String theory
- Holographic duals of 2n-dim CFTs with anomalies

CS terms in the bulk ⇔ Anomalies in CFT

(ex) 3d topologically massive gravity (in $AdS_3$)

(ex) anomaly-induced transports

Before talking about CS-terms …

Let's start with
a sketch of the Wald formalism

MAIN IDEA

*\`Black hole entropy is the Noether charge'*

[Robert M Wald Phys. Rev. D48 (1993)3427-3431]

# Noether Procedure

How to construct charges for diffeomorphis/gauge trans.?

## Step 1. Pre-symplectic current

defining eq. $\nabla_a(\not\delta^2\Omega_{\rm PSympl})^a = -\frac{1}{\sqrt{-G}}\,\delta_1\left(\sqrt{-G}\not\delta_2\bar{\mathcal{E}}\right) + \frac{1}{\sqrt{-G}}\,\delta_2\left(\sqrt{-G}\not\delta_1\bar{\mathcal{E}}\right)$

with $\quad \not\delta\mathcal{E} = \frac{1}{2}\delta g_{ab}T^{ab} + J_a\delta A^a$

## Step 2. Noether's theorem

Existence of on-shell vanishing current for diffeomorphism/gauge trans.

$$\nabla_a \mathrm{N}^a = \not\delta_\chi\bar{\mathcal{E}} \qquad \mathrm{N}^a = \xi_b T^{ab} + J^a(\Lambda + \xi^c A_c)$$

$$\longrightarrow \nabla_a\left[(\not\delta\not\delta_\chi\bar{\Omega}_{\rm PSympl})^a + \xi^a\not\delta\bar{\mathcal{E}} - \frac{1}{\sqrt{-G}}\delta\left(\sqrt{-G}\;\mathrm{N}^a\right)\right] = 0$$

## Step 3. Differential Noether charge

$$-\nabla_b(\not\delta\bar{Q}_{\rm Noether})^{ab} = (\not\delta\not\delta_\chi\bar{\Omega}_{\rm PSympl})^a + \xi^a\not\delta\bar{\mathcal{E}} - \frac{1}{\sqrt{-G}}\delta\left(\sqrt{-G}\;\mathrm{N}^a\right)$$

How to integrate by parts ?

# Wald formalism

[Wald, Lee-Wald, Iyer-Wald]

## (Key 1) Pre-symplectic potential → Pre-symplectic current

Defining eq : variation of Lagrangian $\not\delta \bar{\mathcal{E}} + \nabla_a \not\delta \Theta^a_{\rm PSympl} = \frac{1}{\sqrt{-G}} \delta \left( \sqrt{-G}\ \bar{L} \right)$

$$\longrightarrow \quad \not\delta^2 \Omega^a_{\rm PSympl} = -\frac{1}{\sqrt{-G}} \delta_1 \left( \sqrt{-G} \not\delta_2 \bar{\Theta}^a_{\rm PSympl} \right) + \frac{1}{\sqrt{-G}} \delta_2 \left( \sqrt{-G} \not\delta_1 \bar{\Theta}^a_{\rm PSympl} \right)$$

## (Key 2) Komar decomposition → Differential Noether charge

$$\mathrm{N}^a = \xi^a \bar{L} - \not\delta_\chi \bar{\Theta}^a_{\rm PSympl} + \nabla_b \bar{\mathcal{K}}^{ab}$$

Substituting into

$$-\nabla_b (\not\delta \bar{Q}_{\rm Noether})^{ab} = (\not\delta \not\delta_\chi \bar{\Omega}_{\rm PSympl})^a + \xi^a \not\delta \bar{\mathcal{E}} - \frac{1}{\sqrt{-G}} \delta \left( \sqrt{-G}\ \mathrm{N}^a \right)$$

$$\longrightarrow \quad \not\delta \bar{Q}^{ab}_{\rm Noether} = \frac{1}{\sqrt{-G}} \delta \left( \sqrt{-G}\ \bar{\mathcal{K}}^{ab}_\chi \right) + \xi^a (\not\delta \bar{\Theta}_{\rm PSympl})^b - (\not\delta \bar{\Theta}_{\rm PSympl})^a \xi^b$$

# BH entropy = Noether Charge

For stationary solutions & on-shell

For the time-like Killing vector (vanishing at the bifurcation surface)

$$\xi = \partial_t + \Omega_H \partial_\phi$$

Conservation of differential Noether charge

$$\nabla_b (\not\delta \bar{Q}^{ab}_{\mathrm{Noether}}) \simeq 0$$

$$\longrightarrow \quad \int_{Bif} \not\delta \mathbf{Q}_{\mathrm{Nooether}} = \int_{\infty} \not\delta \mathbf{Q}_{\mathrm{Nooether}}$$

Wald entropy $\qquad\qquad \delta M + \Omega_H \delta J + \cdots$

1st law of thermodynamics

# Extension to CS Term

For theories with Chern-Simons terms [Tachikawa, Bonora et al.]

## Pre-symplectic current

Straightforward generalization of Wald formalism

$$\delta\!\!\!/^2 \Omega_{\mathrm{PSympl}} = \ldots + d\left(\delta_1 \mathbf{A} \cdot \frac{\partial^2 \mathbf{I}_{CS}}{\partial \mathbf{F} \partial \mathbf{F}} \cdot \delta_2 \mathbf{A} + \delta_1 \mathbf{\Gamma}^c{}_b \frac{\partial^2 \mathbf{I}_{CS}}{\partial \mathbf{R}^c{}_b \partial \mathbf{R}^g{}_h} \delta_2 \mathbf{\Gamma}^g{}_h\right) + d\left(\delta_1 \mathbf{A} \cdot \frac{\partial^2 \mathbf{I}_{CS}}{\partial \mathbf{F} \partial \mathbf{R}^g{}_h} \delta_2 \mathbf{\Gamma}^g{}_h - \delta_2 \mathbf{A} \cdot \frac{\partial^2 \mathbf{I}_{CS}}{\partial \mathbf{F} \partial \mathbf{R}^g{}_h} \delta_1 \mathbf{\Gamma}^g{}_h\right)$$

Non-covariant boundary terms in 5d and higher

## Differential Noether charge

Non-covariance of pre-symplectic current is inherited

Non-covariant bulk & boundary terms in 5d and higher

# Short Summary

- Wald formalism

For theories satisfying $\delta_\chi \bar{L}_{cov} = \mathcal{L}_\xi \bar{L}_{cov}$

Pre-symplectic potential → Pre-symplectic current
→ Differential Noether charge → Wald entropy formula

- Tachikawa's extension

Simple generalization of Wald formalism to CS terms

→ Not covariant in 5d and higher !

Our Goal : Manifestly covariant formulation of Noether charge

# Our Formulation

# Noether Procedure

How to construct charges for the local/global symmetry ?

Step 1. Pre-symplectic current

defining eq.

$$\nabla_a(\not\delta^2\Omega_{\rm PSympl})^a = -\frac{1}{\sqrt{-G}}\,\delta_1\left(\sqrt{-G}\not\delta_2\bar{\mathcal{E}}\right) + \frac{1}{\sqrt{-G}}\,\delta_2\left(\sqrt{-G}\not\delta_1\bar{\mathcal{E}}\right)$$

Step 2. Noether's theorem

Existence of on-shell conserved current for diffeomorphism/gauge trans.

$$\nabla_a \mathrm{N}^a = \not\delta_\chi\bar{\mathcal{E}} \qquad \mathrm{N}^a = \xi_b T^{ab} + J^a(\Lambda + \xi^c A_c)$$

$$\longrightarrow \nabla_a\left[(\not\delta\not\delta_\chi\bar{\Omega}_{\rm PSympl})^a + \xi^a\not\delta\bar{\mathcal{E}} - \frac{1}{\sqrt{-G}}\delta\left(\sqrt{-G}\ \mathrm{N}^a\right)\right] = 0$$

Step 3. Differential Noether charge

$$-\nabla_b(\not\delta\bar{Q}_{\rm Noether})^{ab} = (\not\delta\not\delta_\chi\bar{\Omega}_{\rm PSympl})^a + \xi^a\not\delta\bar{\mathcal{E}} - \frac{1}{\sqrt{-G}}\delta\left(\sqrt{-G}\ \mathrm{N}^a\right)$$

How to integrate by parts ?

# Manifestly Covariant Formalism

Our strategy

Integrate by parts RHS of the defining eqs. directly

Remark Anomaly polynomial $\mathbf{P}_{CFT}[\mathbf{F},\mathbf{R}] = d\mathbf{I}_{CS}$

(ex) $\mathrm{tr}\left(\mathbf{\Gamma}\wedge d\mathbf{\Gamma} + \frac{2}{3}\mathbf{\Gamma}^3\right) \longrightarrow \mathrm{tr}[\mathbf{R}^2]$

$$\mathbf{A}\wedge \mathrm{tr}[\mathbf{R}\wedge\mathbf{R}] \longrightarrow \mathbf{F}\wedge\mathrm{tr}[\mathbf{R}^2]$$

(Derivatives of) anomaly polynomial is covariant

# Manifestly Covariant Formalism

<u>Pre-symplectic current</u>

[TA-Loganayagam-Ng-Rodriguez]

$$(\not{\delta}^2\bar{\Omega}_{\rm PSympl})^a_H = \frac{1}{2}\frac{1}{\sqrt{-G}}\delta_1\left[\sqrt{-G}\,(\Sigma_H)^{(bc)a}\right]\delta_2 G_{bc} - \frac{1}{2}\frac{1}{\sqrt{-G}}\delta_2\left[\sqrt{-G}\,(\Sigma_H)^{(bc)a}\right]\delta_1 G_{bc}$$
$$+\delta_1 A_e\cdot(\bar{\sigma}^{FF}_H)^{efa}\cdot\delta_2 A_f + \delta_1\Gamma^c{}_{be}\cdot(\bar{\sigma}^{RR}_H)_{cg}^{bhefa}\cdot\delta_2\Gamma^g{}_{hf}$$
$$+\delta_1 A_e\cdot(\bar{\sigma}^{FR}_H)_g^{hefa}\,\delta_2\Gamma^g{}_{hf} - \delta_2 A_e\cdot(\bar{\sigma}^{FR}_H)_g^{hefa}\,\delta_1\Gamma^g{}_{hf}\,.$$

<u>Differential Noether charge</u>

$$(\not{\delta}\bar{Q}^{ab}_{\rm Noether})_H = \left[\nabla_h\xi^g(\bar{\sigma}^{RR}_H)_{gd}^{hcabf} + (\Lambda+\xi^e A_e)\cdot(\bar{\sigma}^{FR}_H)_d^{cabf}\right]\delta\Gamma^d{}_{cf}$$
$$+\left[\nabla_h\xi^g(\bar{\sigma}^{RF}_H)_g^{habf} + (\Lambda+\xi^e A_e)\cdot(\bar{\sigma}^{FF}_H)^{abf}\right]\cdot\delta A_f$$
$$+\frac{1}{2}\left[(\Sigma_H)^{(cd)a}\xi^b - (\Sigma_H)^{(cd)b}\xi^a\right]\delta G_{cd}$$
$$+\frac{1}{2}\frac{\xi^d}{\sqrt{-G}}\delta\left[\sqrt{-G}\,G_{cd}\left(\Sigma_H^{acb}+\Sigma_H^{bac}+\Sigma_H^{cab}\right)\right]\,.$$

where

$\Sigma_H \sim (\partial \mathbf{P}_{CFT}/\partial\mathbf{R})$ $\quad\bar{\sigma}^{RR}_H \sim (\partial^2\mathbf{P}_{CFT}/\partial\mathbf{R}\partial\mathbf{R})$ etc.

Covariant!

# ‘Tachikawa Formula’

[Tachikawa, Bonora et al.]

Entropy formula for theories with CS term

$$S_{\text{W-T}} = \int_{Bif} 2\pi \varepsilon_b{}^a \frac{\delta \bar{L}_{cov}}{\delta R^a{}_{bcd}} \varepsilon^{cd} + \int_{Bif} \sum_{k=1}^{\infty} 8\pi k \mathbf{\Gamma}_N (d\mathbf{\Gamma}_N)^{2k-2} \frac{\partial \mathbf{P}_{CFT}}{\partial \mathrm{tr}\mathbf{R}^{2k}}$$

with $$\mathbf{\Gamma}_N \equiv \left[\frac{1}{2}\varepsilon_a{}^b \mathbf{\Gamma}^a{}_b\right]_{Bif}$$

Subtlety in the derivation

To derive Tachikawa formula, one needs extra assumptions like choices of specific coordinates/gauges

Derivation in our formalism :
Just compute our charge at the bifurcation surface!
No extra assumptions!

[TA-Loganayagam-Ng-Rodriguez]

# Comparison

- Manifestly covariant

  covariant pre-symplectic current
  → covariant differential Noether charge

- Covariant proof of Tachikawa formula

  No extra assumptions for the derivation

  Straightforward generalization to multi-trace cases

# Application

Rotating charged AdS BHs in fluid/gravity expansion
(Einstein-Maxwell-Chern-Simons system)

[TA-Loganayagam-Ng-Rodriguez]

↕

conformal fluid with anomalies

<u>CFT predictions:</u> Replacement rule

Various anomaly-induced quantities are determined
from the anomaly polynomial through a simple rule

[Loganayagam-Surowka, Jensen-Loganayagam-Yarom, …]

Anomaly-induced currents computed through our formalism
indeed reproduce the CFT replacement rules !

# Summary

We propose a manifestly covariant formulation of differential Noether charge in the presence of Chern-Simons terms in higher dimensions

Our formulation covariantly proves the Tachikawa formula for Chern-Simons contribution to black hole entropy

Application to rotating charged AdS BH reproduces CFT replacement rules for anomaly-induced transports