Trace Anomaly Matching and Exact Results For Entanglement Entropy

Shamik Banerjee
Kavli IPMU

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Introduction

- Entanglement entropy is an important and useful quantity which finds applications in many branches of physics, starting from black holes to quantum critical phenomena.
- In general it is a difficult thing to compute even for free field theories.
- Many exact results are known for conformal field theories but non-conformal field theories are even more difficult to deal with.
- Some exact results are known for two dimensional non-conformal field theories and for strongly coupled theories via gauge-gravity duality (Ryu-Takayanagi formula).

Goal

- Our goal is to propose a general algorithm for computing entanglement entropy in non-conformal field theories.
- It turns out that the techniques developed by Komargodski and Schwimmer to prove the a-theorem in four dimensions is useful for this purpose.

Replica Trick

- Entanglement entropy is usually computed by replica trick.
- In replica trick the entanglement entropy is defined as,

$$S_E = n\frac{\partial}{\partial n}(F(n) - nF(1)) \,|_{n=1} \tag{1}$$

where $F(n)$ is the free energy of the Euclidean field theory on a space with conical singularities. The angular excess at each conical singularity is given by $2\pi(n-1)$. The detailed geometry of the space is determined by the geometry of the background space and the geometry of the entangling surface.

Two Dimwnsions

- Let us consider a massive scalar filed of mass m in two dimensions described by the Euclidean action,

$$S = \frac{1}{2}\int ((\partial\phi)^2 + m^2\phi^2) \tag{2}$$

- We want to compute the entanglement entropy of a subsystem which want to keep arbitrary.
- It could be an infinite half-line or it could be an interval of finite length. In order to do this one has to compute the free energy of this theory on a space with conical singularities.
- One way to do this is to use the identity (Calabrese-Cardy, Casini),

$$\frac{\partial}{\partial m^2} lnZ_n = -\frac{1}{2}\int G_n(\vec{r},\vec{r})d^2\vec{r} \tag{3}$$

- $G_n(\vec{r},\vec{r}\,')$ is the Green's function of the operator $(-\nabla^2 + m^2)$, on the singular space.

- Now instead of doing this one could also use the following identity,

$$m^2 \frac{\partial}{\partial m^2} lnZ_n = -\frac{1}{2} \frac{\partial}{\partial \tau}|_{\tau=0}\ lnZ_n(\tau) \tag{4}$$

- $-lnZ_n(\tau)$, is the free energy computed on the cone for the theory defined by the euclidean action,

$$S(\tau) = \frac{1}{2} \int ((\partial\phi)^2 + m^2 e^{-2\tau} \phi^2) \tag{5}$$

- Now this is precisely the coupling of the dilaton to the massive theory.
- So we can interpret the number τ as a constant background dilaton field.
- This shows that we can calculate the entanglement entropy once we know the dilaton effective action on the cone.

More general case in two dimensions

- Consider a UV-CFT deformed by a relevant operator.
- When the subsystem is an infinite half-line, Calabrese and Cardy proved a general result.
- They proved that,

$$\int_{cone} (< T_\mu^\mu >_n - < T_\mu^\mu >_1) = -\pi n \frac{c_{UV} - c_{IR}}{6} (1 - \frac{1}{n^2}) \quad (6)$$

- $< T_\mu^\mu >_n$ denotes the expectation value of the trace on the cone and $< T_\mu^\mu >_1$ denotes the expectation value of the trace on the plane.
- The above formula computes the contribution of the conical singularity to the trace of the energy-momentum of the non-conformal theory.
- Let us first show that this result can also be obtained by coupling the theory to a constant background dilation field on the cone.

Brief review of the Komargodski-Schwimmer method

- Our deformed field theory is not conformal but it can be made conformally invariant by coupling to a background dilaton field.
- The dilaton, τ, couples to the deformed theory as,

$$S = S_{CFT}^{UV} + \int d^2x\sqrt{h}\, g(e^{\tau(x)}\Lambda)\Lambda^{2-\Delta} O \tag{7}$$

- This is conformally invariant if the metric and the background field are transformed as,

$$h_{ab} \to e^{2\sigma} h_{ab},\;\; \tau(x) \to \tau(x) + \sigma \tag{8}$$

- To first order dilaton couples to the trace of the energy momentum tensor, $\sim \int \tau(x) T^{\mu}_{\mu}(x)$.
- So to compute the integrated trace we can couple to a constant dilaton field.

- We need to compute the dilaton effective action for a constant dilaton background field.
- KS have shown that this action consists of two parts. One is the Weyl non-invariant universal term which is completely determined by the conformal anomaly matching between the UV and the IR.
- The other part is the Weyl invariant part of the effective action which can be written as a functional of the Weyl invariant combination $e^{-2\tau} h_{ab}$.

Universal Part In Two dimensions

- The trace of the energy-momentum of a conformal field theory of central charge c on the cone is given by (Cardy-Peschel, Holzhey et.al),

$$\int_{cone} \sqrt{h} < T^{\mu}_{\mu} >= \frac{c}{24\pi} \, \frac{1}{2}(1+\frac{1}{n}) \int_{cone} \sqrt{h} R(h) \qquad (9)$$

- This is the response of the 2-D CFT on the cone to a scale transformation.
- Using this and the anomaly matching condition gives us the universal (Weyl non-invariant) part of the dilaton effective action for a constant dilaton field to be,

$$F(n,\tau) = -\frac{c_{UV} - c_{IR}}{24\pi} \, \frac{1}{2}(1+\frac{1}{n}) \, \tau \int_{cone} \sqrt{h} R(h) \qquad (10)$$

- So we get,

$$\int_{cone} < T^{\mu}_{\mu} >_{n,universal} = -\frac{c_{UV} - c_{IR}}{24\pi} \frac{1}{2}(1 + \frac{1}{n}) \int_{cone} \sqrt{h}R(h) \tag{11}$$

- The non-universal contribution is purely bulk contribution in this case because there is no other length scale in the problem and hence cancelled in the combination $\int_{cone}(< T^{\mu}_{\mu} >_n - < T^{\mu}_{\mu} >_1)$.
- Hence we arrive at the Calabrese-Cardy result once we note that,

$$\int_{cone} \sqrt{h}R(h) = 4\pi(1 - n) \tag{12}$$

- Now let μ denote the mass scale associated with the relevant operator.
- Since μ is the only dimensionful parameter associated with the theory a scale transformation is equivalent to a change in the parameter. (Calabrese-Cardy)
- So,

$$\mu\frac{d}{d\mu}S_{EE} = n\frac{\partial}{\partial n}|_{n=1}\ (\mu\frac{d}{d\mu}F(n) - n\mu\frac{d}{d\mu}F(1)) \tag{13}$$

- And,

$$\mu\frac{d}{d\mu}F = -\int\sqrt{h} < T^{\mu}_{\mu} > \tag{14}$$

- This gives us,

$$\mu\frac{d}{d\mu}S_{EE} = -\frac{c_{UV} - c_{IR}}{6} \tag{15}$$

- This is precisely the Calabrese-Cardy answer,

$$S_{EE} = -\frac{c_{UV}}{6}ln(\mu a) + \frac{c_{IR}}{6}ln(\mu L_{IR}) \tag{16}$$

Higher Dimensions

- Same Principle !
- Non-trivial non-universal terms in dilaton effective action / entanglement entropy. (See arXiv: 1405.4876, arXiv: 1406.3038, SB ; for more details on the type of terms it gives rise to)
- No symmetry principle fixes the non-universal terms of the dilaton effective action except that they are Weyl-invariant under a simultaneous transformation of the metric and the field τ.
- But now we have a precise thing to compute in higher dimensions which is valid for any field theory !

Four Dimensions

- In Four dimensions dimensions the universal (Weyl non-invariant) part of the dilaton effective action for a constant dilaton filed is given by,

$$F(n,\tau) = -\tau \int_{cone} d^4x\sqrt{h}\,(\frac{c_{UV}-c_{IR}}{16\pi^2}W^2 - 2(a_{UV}-a_{IR})E_4) \tag{17}$$

- This gives rise to a term which is universal,

$$S_{EE} \supset -n\frac{\partial}{\partial n}|_{n=1} \int_{cone} d^4x\sqrt{h}\,(\frac{c_{UV}}{16\pi^2}W^2 - 2a_{UV}E_4)\; ln(\mu a) \tag{18}$$

- In fact, this term always appears if you compute holographic entanglement entropy in RG-flow geometries.
- Our method extends this to any field theory and explains this as the consequence of trace-anomaly matching.