BFV and AKSZ Formalism of Current Algebras

Noriaki Ikeda
Maskawa Institute
Kyoto Sangyo University
and
Ritsumeikan University

YITP 2014

NI and Xiaomeng Xu, arXiv:1301.4805, arXiv:1308.0100.

§1. Introduction

Purpose

Unify current algebras formulation a la Batalin-Fradkin-Vilkovisky formalism

Unified and simple formulation including currents of algebroids, which recently appear in the string theory with flux or nongeometric backgrouds, etc.

General theory of possible anomaly terms and anomaly cancellation conditions

Construct new current algebras and new physical theories

Ingredients of BRST-BV-BFV formalism

1, Φ, Φ^*: super combinations of physical fields and unphysical antifields
graded (super) manifold

2, $\{-,-\}$: odd Poisson bracket (antibracket)
graded symplectic structure

3, S: Generator of the BRST symmetry $\delta = \{S, -\}$ (BV action) such that $\{S, S\} = 0$ (master equation), which is equivalent to $\delta^2 = 0$.

(A homological vector field $Q = \delta$ and its Hamiltonian function $\Theta = S$.)

It is called a **QP manifold**, or a **differential graded symplectic manifold**, or recently a **symplectic NQ manifold**.

Plan of Talk

BFV formalism (supergeometry) of Poisson brackets

Supergeometric formalism of current algebras

(Examples)

§2. BFV Formalism of Poisson Brackets

A current algebra is a Lie algebra under a Poisson bracket. Therefore, we start with the Poisson bracket.

Poisson brackets

$\boldsymbol{x}^I = (x^i, p_i)$: canonical conjugates

The Poisson bracket is $\{f(\boldsymbol{x}), g(\boldsymbol{x})\}_{PB} = -\pi^{IJ}(\boldsymbol{x})\frac{\partial f(\boldsymbol{x})}{\partial \boldsymbol{x}^I}\frac{\partial g(\boldsymbol{x})}{\partial \boldsymbol{x}^J}$, which satisfies the Jacobi identitity

$$\{\{f(\boldsymbol{x}), g(\boldsymbol{x})\}_{PB}, h(\boldsymbol{x})\}_{PB} + (f, g, h \text{ cyclic}) = 0.$$

BFV Formalism

1, The graded cotangent bundle $T^*[1]M$.

$\boldsymbol{x}^I = (x^i, p_i)$ of degree 0, physical canonical quantities

$\boldsymbol{\xi}_I = (\xi_i, \eta^i)$ of degree 1 (Grassman odd), antifields

2, Set an **odd Poisson bracket**

$$\{\boldsymbol{x}^I, \boldsymbol{x}^J\} = 0, \quad \{\boldsymbol{\xi}_I, \boldsymbol{\xi}_J\} = 0, \quad \{\boldsymbol{x}^I, \boldsymbol{\xi}_J\} = \delta^I{}_J.$$

3, Introduce a degree 2 function as a generator:

$$S = \Theta \equiv \frac{1}{2}\pi^{IJ}(\boldsymbol{x})\boldsymbol{\xi}_I\boldsymbol{\xi}_J.$$

Note that $\pi^{IJ}(\boldsymbol{x})$ is antisymmetric because $\boldsymbol{\xi}_I$ is odd.

An original Poisson bracket is reconstructed by $\{-,-\}_{PB} = \{\{-,\Theta\},-\}$, which is called a **derived bracket**. In fact,

$$\{\{f(\boldsymbol{x}),\Theta\},g(\boldsymbol{x})\} = \{f(\boldsymbol{x}),g(\boldsymbol{x})\}_{PB}.$$

Theorem 1.

$$\{\Theta,\Theta\} = 0 \Longleftrightarrow \{\{f(\boldsymbol{x}),g(\boldsymbol{x})\}_{PB},h(\boldsymbol{x})\}_{PB} + (f,g,h \textit{ cyclic}) = 0.$$

'Current algebra' in our talk

Definition 1. *A a current algebra is a Lie algebra of a Poisson bracket (Poisson algebra) of functions on a mapping space Σ to M, where Σ is a space of a worldvolume and M is a target space.*

Functions of the original canonical quantities $\boldsymbol{x} = (x, p)$ are commutative by the odd Poisson bracket $\{-,-\}$. And classical currents must be closed in the derived bracket: $\{-,-\}_{PB} \equiv \{\{-,\Theta\},-\}$.

Definition 2. ***Physical classical currents*** *are functions on a Lagrangian submanifold in a grarded symplectic manifold and are closed by the derived bracket $\{\{-,\Theta\},-\}$.*

§3. Target Space in n Dimensions

QP Manifold (Symplectic NQ Manifold)

is a graded version of a BFV structure.

Definition 3. *A following triple $(\mathcal{M}, \omega, Q)$ is called a QP-manifold (symplectic NQ manifold) of degree n $(n \in \boldsymbol{Z}_{\geq 0})$.*

*1, $\mathcal{M}$ is a **graded manifold** of nonnegative integer degree, which is called a **N-manifold**.*

2, ω is a graded symplectic form of degree n on $\mathcal{M}$.

3, Q is a vector field of degree $+1$ such that $Q^2 = 0$, which satisfies $\mathcal{L}_Q \omega = 0$. Take $\Theta \in C^\infty(\mathcal{M})$ such that $Q(-) = \{\Theta, -\}$. $Q^2 = 0$ is equivalent to $\{\Theta, \Theta\} = 0$.

Theorem 2. *A QP manifold of degree 1 is a Poisson structure on M.*

§4. BFV Structure on Mapping Space

$X_n = \boldsymbol{R} \times \Sigma_{n-1}$ is an n dimensional manifold, which is a spacetime.

Then we can construct the BFV formalism of the Poisson bracket on the mapping space $\mathrm{Map}(T[1]\Sigma_{n-1}, \mathcal{M})$, which is the field theory setting.

AKSZ Construction Alexandrov, Kontsevich, Schwartz, Zaboronsky '97

induces an BFV structure on a mapping space, $\mathrm{Map}(T[1]\Sigma_{n-1}, \mathcal{M})$.

$\mathcal{X} = T[1]\Sigma_{n-1}$ is a worldvolume supermanifold with a Berezin measure μ.

$(\mathcal{M}, \omega, Q)$: A target space QP-manifold of degree n

Theorem 3. [AKSZ] $\mathrm{Map}(\mathcal{X}, \mathcal{M})$ *is a QP manifold of degree* 1.

$$\int_{T[1]\Sigma_{n-1}} d^{n-1}\sigma d^{n-1}\theta \{F(\boldsymbol{x}(\sigma,\theta)), G(\boldsymbol{x}(\sigma,\theta))\}_{-1}^{\mathrm{Map}} = \ \{F(\boldsymbol{x}), G(\boldsymbol{x})\}_{-n}^{target}.$$

$$S_{b,2}^{\mathrm{Map}} = \int_{T[1]\Sigma_{n-1}} d^{n-1}\sigma d^{n-1}\theta \ \Theta_{n+1}^{target}(\boldsymbol{x}, \boldsymbol{\xi})(\sigma, \theta).$$

§5. Functions on Mapping Space

Our strategy: First we prepare functions on a target space and next pullback them to the mapping space by the AKSZ construction.

Functions on a target space (Seed of currents)

$C_{n-1}(\mathcal{M}) = \{f \in C^\infty(\mathcal{M}) | |f| \leq n-1\}$: A space of functions of degree equals to or less than $n-1$ on a target space.

$C_{n-1}(\mathcal{M})$ is closed not only under the **graded Poisson bracket** $\{-,-\}$, but also under the **derived bracket** $\{\{-,\Theta\},-\}$.

AKSZ construction of 'currents'

For a function $J \in C_{n-1}(\mathcal{M})$, the AKSZ construction induces a function on $\mathrm{Map}(T[1]\Sigma_{n-1}, \mathcal{M})$, $\mathcal{J}(\epsilon) = \mu_* \epsilon \, \mathrm{ev}^* J$, where ϵ is a test function on $T[1]\Sigma_{n-1}$ of degree $n-1-|J|$. Note that $|\mathcal{J}| = 0$.

$$\mathcal{CA}_{n-1}(\Sigma_{n-1}, \mathcal{M}) = \{\mathcal{J} = \mu_* \epsilon \, \mathrm{ev}^* J \in C^\infty(\mathrm{Map}(T[1]\Sigma_{n-1}, \mathcal{M})) | J \in C_{n-1}(\mathcal{M})\},$$

is a Poisson algebra.

Problem

This Poisson algebras do not have **anomaly terms**, because this is closed by the Poisson bracket. Simple geometrical procedure introduces possible anomaly terms in this formalism.

§6. Canonical Transformation and Current Algebras

Canonical Transformation (Twisting)

Definition 4. *Let $(\mathcal{M}, \omega, \Theta)$ be a QP manifold of degree n. Let $\alpha \in C^\infty(\mathcal{M})$ be a function of degree n. A* ***canonical transformation*** *e^{δ_α} is defined by $f' = e^{\delta_\alpha} f = f + \{f, \alpha\} + \frac{1}{2}\{\{f, \alpha\}, \alpha\} + \cdots$.*

e^{δ_α} is also called twisting.

A canonical transformation preserves the master equation. If Θ is homological $\{\Theta, \Theta\} = 0$, so is Θ'. $\{\Theta', \Theta'\} = e^{\delta_\alpha}\{\Theta, \Theta\} = 0$ for any twisting.

Twisting by Small Canonical 1-Form

Take a symplectic structure ω_s for the derived Poisson bracket $\{-,-\}_s$ and consider the canonical 1-form ϑ_s for ω_s such that $\omega_s = -\delta\vartheta_s$. In a local coordinate, it is $\vartheta_s = p_i \delta x^i$.

Define a function S_s of degree 1 on the mapping space by the AKSZ construction:

$$\alpha = S_s = \iota_{\hat{D}} \mu_* \mathrm{ev}^* \vartheta_s.$$

Definition 5. *A* $\boldsymbol{BFV}$ ***current*** $\boldsymbol{J}(\epsilon)$ *is defined by twisting by* S_s*:*

$\boldsymbol{J}(\epsilon) := e^{\delta_{S_s}} \mathcal{J}|_{\mathrm{Map}(T[1]\Sigma_{n-1}, \mathcal{L})}$.

Theorem 4. *[NI, Xu] For currents* $\boldsymbol{J}_{J_1}$ *and* $\boldsymbol{J}_{J_2}$ *associated to current functions*

J_1, $J_2 \in C_{n-1}(\mathcal{M})$ *respectively, the commutation relation is given by*

$$\begin{aligned}
\{\boldsymbol{J}_{J_1}(\epsilon_1), \boldsymbol{J}_{J_2}(\epsilon_2)\}_{PB} &= \big(-e^{\delta_{S_s}}\mu_*\epsilon_1\epsilon_2\mathrm{ev}^*\{\{J_1, \Theta\}, J_2\} \\
&\quad -e^{\delta_{S_s}}\iota_{\hat{D}}\mu_*(d\epsilon_1)\epsilon_2\mathrm{ev}^*\{J_1, J_2\}\big)\,|_{\mathrm{Map}(T[1]\Sigma_{n-1}, \mathcal{L})} \\
&= -\boldsymbol{J}_{[J_1, J_2]_D}(\epsilon_1\epsilon_2) \\
&\quad -e^{\delta_{S_s}}\iota_{\hat{D}}\mu_*(d\epsilon_1)\epsilon_2\mathrm{ev}^*\{J_1, J_2\}|_{\mathrm{Map}(T[1]\Sigma_{n-1}, \mathcal{L})},
\end{aligned}$$

where ϵ_i *are test functions for* J_i *on* $\mathrm{Map}(T[1]\Sigma_{n-1}, \mathcal{M})$ *and* $[J_1, J_2]_D$ *is the bracket defined from the drived bracket on a target space* $\mathcal{M}$*.*

Corollary 1. *Let* $Comm$ *be a commutative subspace of* $C_{n-1}(\mathcal{M})$*, that is,* $\{J_1, J_2\} = 0$ *under the graded Poisson bracket for* $J_1, J_2 \in Comm$*. If target space functions are in* $(Comm, \{\{-, \Theta\}, -\})$*, then anomalies vanish,*

'Holographic formulation'

$$\begin{array}{c}\textbf{Graded Poisson algebra on BFV Theory } \mathcal{M} \\ \Big\downarrow \text{Twisting and reduction to Lagrangian submfd} \\ \textbf{Current algebra with anomaly terms on physical space } \mathcal{L}\end{array}$$

A generalization of

the Wess-Zumino consistency condition, which requires an extended closedness condition for $\delta_{BRST} + d$,

the Wess-Zumino terms in n dimensional quantum theories are realized by $n + 1$ dimensional gauge invariant terms.

§7. Example $n = 2$: Current Algebras of Courant Algebroid and Dirac Structure

Alekseev, Strobl '05, NI, Koizumi '11

$X_2 = S^1 \times \boldsymbol{R}$ with a local coordinate $(\sigma, \tau) \longrightarrow M$

$x^I(\sigma), p_I(\sigma)$: canonical conjugates. The canonical commutation relation twisted by a closed 3-form H:

$$\{x^I, x^J\}_{PB} = 0, \quad \{x^I, p_J\}_{PB} = \delta^I{}_J \delta(\sigma - \sigma'),$$
$$\{p_I, p_J\}_{PB} = -H_{IJK}(x)\partial_\sigma x^K \delta(\sigma - \sigma').$$

A generalization of a current algebra on a target space $TM \oplus T^*M$:

$$J_{0(f)}(\sigma) = f(x(\sigma)), \quad J_{1(u,\alpha)}(\sigma) = a_I(x(\sigma))\partial_\sigma x^I(\sigma) + u^I(x(\sigma))p_I(\sigma),$$

where $f(x(\sigma))$ is a function, $a(x) = a_I(x)dx^I$ is a 1-form and $u(x) = u^I(x)\partial_I$ is a vector field.

$$
\begin{aligned}
\{J_{0(f)}(\sigma), J_{0(g)}(\sigma')\}_{PB} &= 0, \\
\{J_{1(u,a)}(\sigma), J_{0(g)}(\sigma')\}_{PB} &= -u^I \frac{\partial g}{\partial x^I}(x(\sigma))\delta(\sigma - \sigma'), \\
\{J_{1(u,a)}(\sigma), J_{1(v,b)}(\sigma')\}_{PB} &= -J_{1([(u,a),(v,b)]_D)}(\sigma)\delta(\sigma - \sigma') \\
&\quad +\langle (u,a),(v,b)\rangle(\sigma')\partial_\sigma\delta(\sigma - \sigma'),
\end{aligned}
$$

where $[(u,a),(v,b)]_D = ([u,v], L_u b - L_v a + d(i_v a) + H(u,v,\cdot\ ))$: **Dorfman bracket** on $TM \oplus T^*M$.
$\langle (u,\alpha),(v,b)\rangle = i_v\alpha + i_u b$: symmetric scalar product on $TM \oplus T^*M$.

•Anomaly cancellation condition

$\langle (u,a),(v,b)\rangle = 0.$

This condition is satisfied on the **Dirac structure** on M. The Dirac structure is a maximally isotropic subbundle of $TM \oplus T^*M$, whose sections are closed under the Dorfman bracket.

§8. BFV formalism of current algebras

Local coordinate on target space QP manifold $\mathcal{M} = T^*[2]T^*[1]M$

1, (x^I, p_I, q^I, ξ_I) of degree $(0, 1, 1, 2)$, where (x^I, p_I) is the $\mathcal{L}$ component.

2, $\omega_b = \delta x^I \wedge \delta \xi_I + \delta p_I \wedge \delta q^I$: graded symplectic structure of degree 2.

3, Homological function of degree 3:

$$\Theta = \xi_I q^I + \frac{1}{3!} H_{IJK}(x) q^I q^J q^K.$$

$\{\Theta, \Theta\} = 0$ if H is a closed 3-form on M, where $H = \frac{1}{3!} H_{IJK}(x) dx^I \wedge dx^J \wedge dx^K$.

The derived bracket $\{-, -\}_s = \{\{-, \Theta\}, -\}|_{\mathcal{L}}$ satisfies $\{x^I, p_J\}_s = \delta^I{}_J$.

Graded Poisson algebra on target space

Let us consider a space of functions $C_1(T^*[2]T^*[1]M) = \{f \in C^\infty(T^*[2]T^*[1]M) | |f| \leq 1\}$.

Elements are a function of degree 0, and a functions of degree 1,

$$J_{(0)f} = f(x), \qquad J_{(1)(u,a)} = a_I(x)q^I + u^I(x)p_I.$$

The graded Poisson bracket is a seed of the anomaly term:

$$\begin{aligned} \{J_{(0)(f)}, J'_{(0)(g)}\} &= 0, \\ \{J_{(1)(u,a)}, J'_{(0)(g)}\} &= 0. \\ \{J_{(1)(u,a)}, J'_{(1)(v,b)}\} &= a_I v^I + u^I b_I = \langle (u,a)\,, (v,b)\rangle. \end{aligned}$$

where $J'_{(0)(v)} = g(x)$ and $J'_{(1)(v,b)} = b_I(x)q^I + v^I(x)p_I$.

The derived bracket is a seed of the commutator term:

$$
\begin{aligned}
&\{\{J_{(0)(f)}, \Theta\}, J'_{(0)(g)}\} = 0, \\
&\{\{J_{(1)(u,a)}, \Theta\}, J'_{(0)(g)}\} = -u^I \frac{\partial J'_{(0)(g)}}{\partial x^I}, \\
&\{\{J_{(1)(u,a)}, \Theta\}, J'_{(1)(v,b)}\} \\
= -\Bigg[&\left(u^J \frac{\partial v^I}{\partial x^J} - v^J \frac{\partial u^I}{\partial x^J}\right) p_I + \left(u^J \frac{\partial b_I}{\partial x^J} - v^J \frac{\partial a_I}{\partial x^J} + v^J \frac{\partial a_J}{\partial x^I} + b_J \frac{\partial u^J}{\partial x^I} + H_{JKI} u^J v^K\right) \\
&= -J_{(1)([(u,a),(v,b)]_D)}.
\end{aligned}
$$

We have the same solution in the previous section.

Let us consider $\mathcal{X} = T[1]S^1$ with a local coordinate (σ, θ). The Berezin measure is $\mu = \mu_{T[1]S^1} = d\sigma d\theta$.

Twisting and Current algebras

Remember $\{x^I, p_J\}_s = \{\{x^I, \Theta\}, p_J\}|_{\mathcal{L}} = \delta^I{}_J$. Therefore the small canonical 1-from is

$$S_s = \iota_{\hat{D}} \mu_* \mathrm{ev}^* \vartheta_s = \int_{T[1]S^1} \mu\, \boldsymbol{p}_I \boldsymbol{d}\boldsymbol{x}^I.$$

This twisting changes $\boldsymbol{q}^I$ to $\boldsymbol{d}\boldsymbol{x}^I$.

The corresponding currents are

$$\boldsymbol{J}_{(0)(f)} = \int_{T[1]S^1} \mu \epsilon_{(1)} f(\boldsymbol{x}),\ \boldsymbol{J}_{(1)(u,a)} = \int_{T[1]S^1} \mu \epsilon_{(0)} (a_I(\boldsymbol{x}) \boldsymbol{d}\boldsymbol{x}^I + u^I(\boldsymbol{x}) \boldsymbol{p}_I),$$

where $\epsilon_{(i)}$ is a test function of degree i. BFV current formulae give the same result:

$$
\begin{aligned}
&\{\boldsymbol{J}_{(0)(f)}(\epsilon), \boldsymbol{J}'_{(0)(g)}(\epsilon')\}_{PB} = 0, \\
&\{\boldsymbol{J}_{(1)(u,a)}(\epsilon), \boldsymbol{J}'_{(0)(g)}(\epsilon')\}_{PB} = -u^I \frac{\partial \boldsymbol{J}'_{(0)(g)}}{\partial \boldsymbol{x}^I}(\epsilon\epsilon'), \\
&\{\boldsymbol{J}_{(1)(u,a)}(\epsilon), \boldsymbol{J}_{(1)(v,b)}(\epsilon')\}_{PB} \\
&= -\boldsymbol{J}_{(1)([(u,a),(v,b)]_D)}(\epsilon\epsilon') \\
&\quad - \int_{T[1]S^1} \mu(\boldsymbol{d}\epsilon_{(0)}\epsilon'_{(0)}\langle (a_I(\boldsymbol{x}), u^I(\boldsymbol{x}))\,, (b_I(\boldsymbol{x}), v^I(\boldsymbol{x}))\rangle,
\end{aligned}
$$

where $\boldsymbol{J}'_{(0)(g)} = \int_{T[1]S^1} \mu\epsilon_{(1)} g(\boldsymbol{x})$, $\boldsymbol{J}'_{(1)(v,b)} = \int_{T[1]S^1} \mu\epsilon_{(0)}(b_I(\boldsymbol{x})\boldsymbol{d}\boldsymbol{x}^I + v^I(\boldsymbol{x})\boldsymbol{p}_I)$.

§9. Summary and Outlook

We have proposed a **new formulation** of current algebra a la BFV formalism. A **derived bracket** and a **canonical transformation** on a **graded Poisson structure** derive a current algebra with anomaly terms on a Lagrangian submanifold.

Our Results

NI Xu 13-2

In our formulation, all known current algebras are included, such as Lie algebras (gauge currents), Kac-Moody algebras, Alekseev-Strobl types (algebroids), topological membranes, L_∞-algebra, etc.,excecpt for the energy-moment tensor.

We have constructed new current algebras of Lie n-algebroids.

Anomaly cancellation conditions are characterized in terms of supergeometry.

Physical examples of new current algebras appear in AKSZ sigma models.

Origin of anomaly terms is the **derived bracket**.

The derived bracket is not a graded Poisson bracket if $n > 1$.
not skew:

$$\{\{f,\Theta\},g\} = -(-1)^{(|f|-n+1)(|g|-n+1)}\{\{g,\Theta\},f\} - (-1)^{(|f|-n+1)}\{\Theta,\{f,g\}\}.$$

not Leibniz rule:

$$\begin{aligned}\{\{fg,\Theta\},h\} &= \{f\{g,\Theta\} + (-1)^{|g|}\{f,\Theta\}g, h\} \\ &= f\{\{g,\Theta\},h\} + (-1)^{|g|(|h|+1-n)}\{\{f,\Theta\},h\}g \\ &\quad + (-1)^{|g|}\{f,\Theta\}\{g,h\} + (-1)^{(|g|+1)(|h|-n)}\{f,h\}\{g,\Theta\}.\end{aligned}$$

Outlook

- String theory with flux and nongeometric backgrouds, with exiotic structures.

- Poisson Vertex Algebra — Li '02, Sole, Kac, Wakimoto '10

- String Field Theory — Hata, Zwiebach '93

- AKSZ sigma models, TQFT

- Higher groupoids and higher category

- Quantization?

Deformation, Geometric, Path integral,,,, anomaly terms in current algebras.

Thank you!