# Exact Results in Supersymmetric Lattice Gauge Theories

Keio University　　Tatsuhiro Misumi

So Matsuura (Keio), TM & Kazutoshi Ohta (Meiji-Gakuin) in preparation.

Strings&Fields2014@YITP 7/22/2014

# 1. Introduction

# Localization in SUSY QFT

Witten (88)
Pestun (07)

- Path integral for Q-closed actions (operators) localizes to BPS locus
- Q-exact deformation helps know fixed points & obtain exact results
- For now, known to work for manifolds with isometry

$$\langle \mathcal{O}_{BPS} \rangle = \lim_{t\to\infty} \int [\mathcal{D}X]\, \mathcal{O}_{BPS}\; e^{-S[X]-tQ\Xi_F[X]}$$

$$= \sum_{X_0} \mathcal{O}[X_0] e^{-S[X_0]}\; \mathrm{Sdet}\left[\frac{\delta^2(Q\Xi_F[X_0])}{\delta X_0^2}\right]^{-1}$$

$X_0 \in$ BPS locus

BPS locus $: \Psi = \Psi^\dagger = 0, \quad Q\Psi = Q\Psi^\dagger = 0$

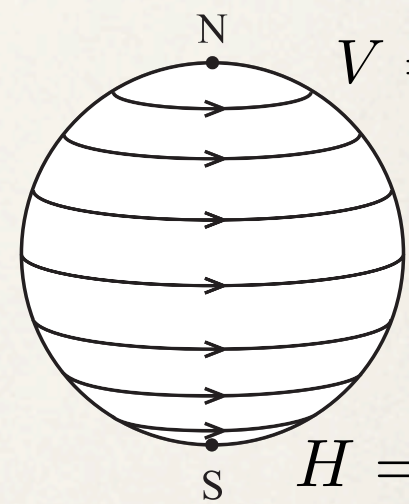
**Notion of ``Localization" is simpler in finite-dimensional integral**

# Equivariant localization

Duistermaat-Heckman(82)
Berline-Vergne(83)
Atiyah-Bott(84)

- Symplectic manifold $(M, \omega)$ with Hamiltonian $H$ for circle action
- Associated Hamiltonian vector field $V$ satisfies $dH = i_V \omega$
- Equivariant cohomology $d_V(H - \omega) = 0$ with $d_V = d + i_V$

| Equivariant localization | SUSY localization |
|---|---|
| $d_V$ | $Q$ |
| $d_V(H - \omega) = 0$ | $QS = 0$ |
| $\int e^{-(H-\omega)+\beta(K-\Omega)} = \int e^{-(H-\omega)}$ | $\int [\mathcal{D}X] e^{-S-tQ\Xi} = \int [\mathcal{D}X] e^{-S}$ |
| $dH = 0$ | $\Psi = 0,\ Q\Psi = 0$ |




**Harish-Chandra Itzykson-Zuber integral (unitary matrix model) is exactly evaluated by this. $\rightarrow$ How about lattice gauge theory?**

# Localization in lattice models?

- Consider 2D lattice models with BRST SUSY on simplicial complex
- Evaluate the path integral by the localization technique

- **Strategy**
  - Extension of 2D $\mathcal{N}$=(2,2) lattice model to simplicial complex
  - Application of localization to the system
- **Potential gains**
  - Reduction of numerical costs in SUSY simulations
  - Feedback to study in (quiver) matrix models

# 2. Localization in HCIZ integral

# HCIZ integral as SUSY

Harish-Chandra(57)
Itzykson-Zuber(80)

$$Z_{\mathrm{HCIZ}} = \int DU\, e^{-\mathrm{Tr}AUBU^\dagger} = \frac{\det e^{-a_i b_j}}{\Delta(a)\Delta(b)}$$

$A, B$ : Diagonal matrices $a_i$, $b_i$
$U$ : $U(N)$ unitary matrix
$\Delta(a), \Delta(b)$ : Vandermonde of $A, B$

$$H \equiv \mathrm{Tr}AUBU^\dagger \equiv \mathrm{Tr}AX_B$$

- Phase space ~ $U(N)/U(1)^N$ with symplectic 2-form $\omega = \mathrm{Tr}(X_B\,\theta\wedge\theta)$
- Localized to $dH$=0 due to the equivariant localization $d_V(H\text{-}\omega)$=0
- Identify MC 1-form $\theta = -idUU^\dagger$ as fermion $\psi$ → **SUSY localization**

- **SUSY algebra** $QX_B = \Psi_B, \quad Q\Psi_B = [A, X_B] \quad (\,\Psi_B = i[\psi, UBU^\dagger]\,)$
- **Q-exact term** $Q\Xi = Q\,\mathrm{Tr}[\Psi_B(Q\Psi_B)] = \mathrm{Tr}[A, X_B]^2 + \mathrm{Tr}\Psi_B[A, \Psi_B]$

# HCIZ integral as SUSY

Harish-Chandra(57)
Itzykson-Zuber(80)

- **t-indep. deform.** $Z_t = \dfrac{1}{\Delta(b)} \displaystyle\int \mathcal{D}U \mathcal{D}\psi \, e^{-(H-\omega)-tQ\Xi}$ $\quad \left( \omega = -\frac{1}{2}\mathrm{Tr}\psi[X_B, \psi] \right)$

- **Fixed points** $Q\Psi_B = [A, X_B] = [A, UBU^\dagger] = 0, \qquad \Psi_B = 0$

  ➡ $U = \Gamma_\sigma$ (permutation group)

- **One-loop det.** $Q\Xi = \mathrm{Tr}[A, X_B]^2 \;+\; \mathrm{Tr}\Psi_B[A, \Psi_B]$

  $\mathrm{Tr}[A, X_B]^2 \downarrow |\Delta(a)|^{-2}|\Delta(b)|^{-2}$, $\quad \mathrm{Tr}\Psi_B[A, \Psi_B] \downarrow \Delta(a)|\Delta(b)|^2$

  ➡ $(-1)^{|\sigma|}\Delta(a)^{-1}$

➡ $$Z_{\mathrm{HCIZ}} = \sum_\sigma \frac{(-1)^{|\sigma|}}{\Delta(a)\Delta(b)} e^{-\sum_i a_i b_{\sigma(i)}} = \frac{\det e^{-a_i b_j}}{\Delta(a)\Delta(b)}$$

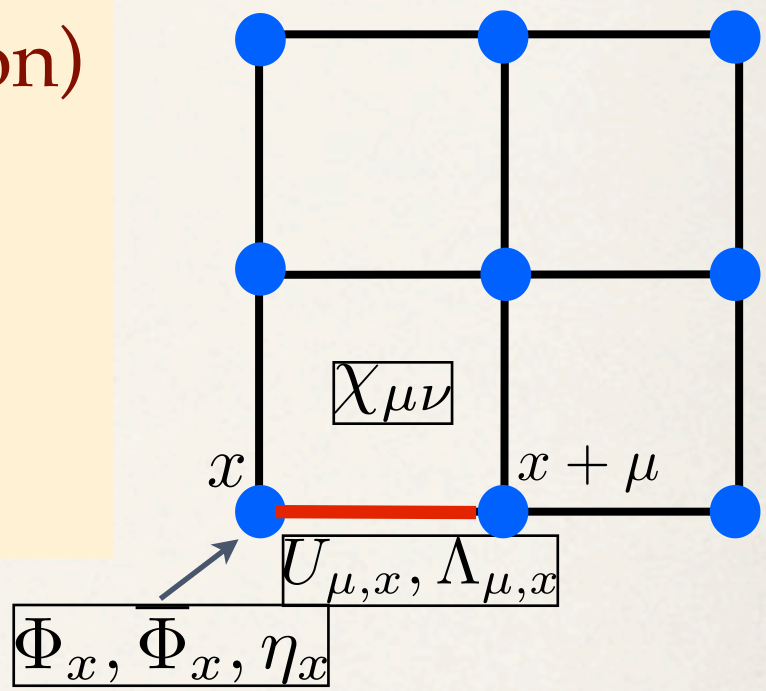
**reproduces HCIZ integral**

# 3. Localization on the lattice

# Lattice 2D $\mathcal{N}$=(2,2) SYM model

Sugino(03)

- Lattice model with scalar SUSY (Q-exact action)
- Variables in topologically-twisted form
- Site, link & face variables
- Rest of SUSY will restore in the cont. limit




- **BRST SUSY algebra**

$$
\begin{aligned}
&QU_{\mu,x} = \Lambda_{\mu,x}, \quad Q\Lambda_{\mu,x} = -i(\Phi_x U_{\mu,x} - U_{\mu,x}\Phi_{x+\mu}), \\
&Q\Phi_x = 0, \\
&Q\overline{\Phi}_x = \eta_x, \qquad Q\eta_x = -i[\overline{\Phi}_x, \Phi_x], \\
&QY_{\mu\nu,x} = -i[\chi_{\mu\nu,x}, \Phi_x], \quad Q\chi_{\mu\nu,x} = Y_{\mu\nu,x}
\end{aligned}
$$

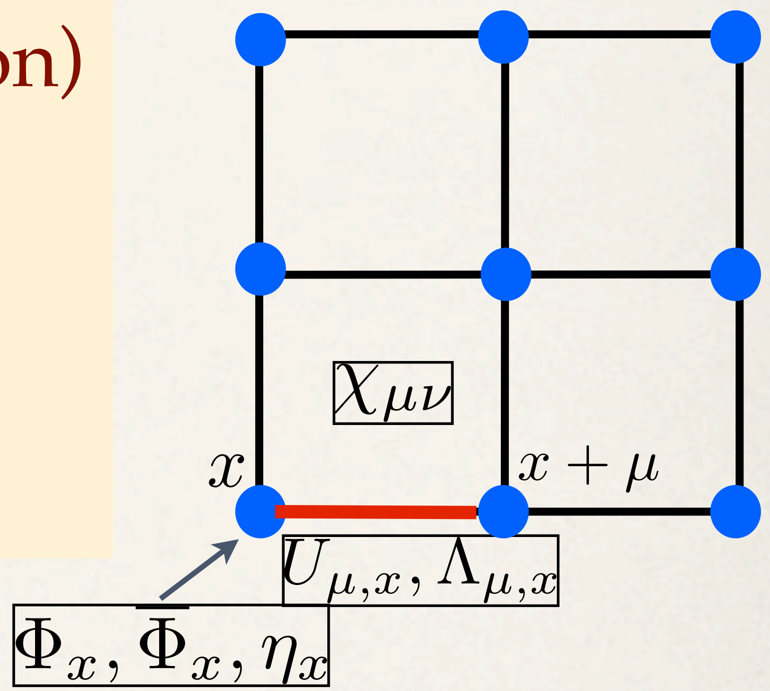
➡ $Q^2 = \delta_{gauge}(\Phi)$

nilpotent on
gauge-invariant operator

# Lattice 2D $\mathcal{N}$=(2,2) SYM model

Sugino(03)

- Lattice model with scalar SUSY (Q-exact action)
- Variables in topologically-twisted form
- Site, link & face variables
- Rest of SUSY will restore in the cont. limit




- **Q-exact action**

$\mu_{\mu\nu} \sim U_P - U_P^\dagger \;\rightarrow\; F_{\mu\nu}$

$$
\begin{aligned}
S_{\text{sugino}} &= \frac{1}{2g^2}\sum_x Q\,\mathrm{Tr}\left[\mathcal{F}\cdot\overline{Q\mathcal{F}} + 2\chi_{\mu\nu}\mu_{\mu\nu}\right] \\
&= \frac{1}{2g^2}\sum_x Q\,\mathrm{Tr}\left[i\Lambda_\mu(\overline{\Phi}_{x+\mu}U_\mu^\dagger - U_\mu^\dagger\overline{\Phi}_x) + i\eta[\Phi,\overline{\Phi}] - \chi_{\mu\nu}(Y_{\mu\nu} - 2\mu_{\mu\nu})\right] \\
&= \frac{1}{2g^2}\sum_x \mathrm{Tr}\left[|U_\mu\Phi_{x+\mu} - \Phi_x U_\mu|^2 + [\Phi,\overline{\Phi}]^2 + \mu_{\mu\nu}^2 + \cdots\right]
\end{aligned}
$$

# Extension to generic simplicial complex

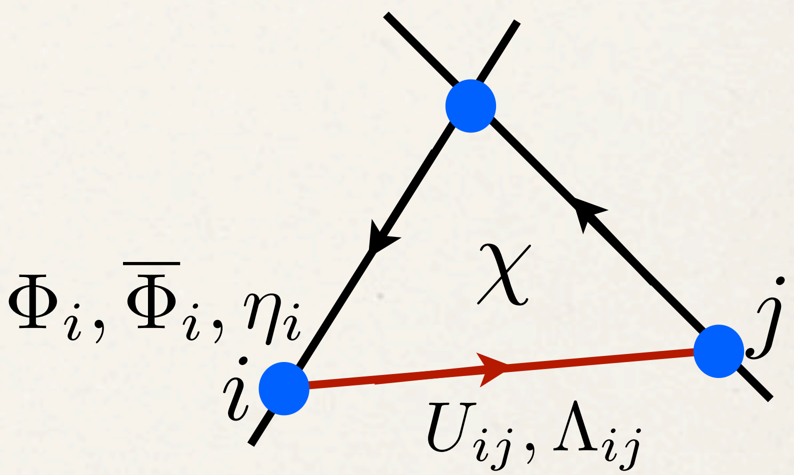
- Extension to simplicial complex by *labeling sites* and *orienting links*
- Metric & connection are defined from the vielbein
- Topological field theory on generic Riemann surface in $a \to 0$

- **Labeling sites for variables**

Site variables : $\Phi_x, \overline{\Phi}_x, \eta_x \;\to\; \Phi_i, \overline{\Phi}_i, \eta_i$

Link variables : $U_{\mu,x}, \Lambda_{\mu,x} \;\to\; U_{ij}, \Lambda_{ij}$

Face variables : $\chi_{\mu\nu,x} \;\to\; \chi_i$




- **From Vielbein to Metric** $\quad U_{ij} = \exp\left[ia\, e^{\mu}_{ij} A_\mu(i)\right] \qquad \Lambda_{ij} = e^{\mu}_{ij}\Lambda_\mu$

$$a^2 \sum_i \;\to\; \int d^2x \sqrt{g}$$

$$\sum_{j \in \langle i, \cdot \rangle \text{ outgoing links}} e^{\mu}_{ij} e^{\nu}_{ij} \equiv g^{\mu\nu}(i) \;\to\; g^{\mu\nu}(x)$$

# Extension to generic simplicial complex

- **Supersymmetric BRST algebra**

$$
\begin{aligned}
&QU_{ij} = \Lambda_{ij}, && Q\Lambda_{ij} = -i(\Phi_i U_{ij} - U_{ij}\Phi_j), \\
&Q\Phi_i = 0, && \\
&Q\bar{\Phi}_i = \eta_i, && Q\eta_i = -i[\bar{\Phi}_i, \Phi_i] \\
&QY_i = -i[\chi_i, \Phi_i], && Q\chi_i = Y_i.
\end{aligned}
$$

➡ $Q^2 = \delta_{gauge}(\Phi)$

nilpotent on gauge-invariant operator
⇆ equivariant cohomology

- **Q-exact action on simplicial complex**

$$
\begin{aligned}
S &= \frac{1}{2g^2}\sum_i Q\,\mathrm{Tr}\left[i\Lambda_{ij}(U^\dagger_{ij}\bar{\Phi}_i - \bar{\Phi}_j U^\dagger_{ij}) + i\eta_i[\bar{\Phi}_i, \Phi_i] - \chi_i(Y_i - 2\mu_i)\right] \\
&= \frac{1}{2g^2}\sum_i \mathrm{Tr}\left[|\Phi_i U_{ij} - U_{ij}\Phi_j|^2 + |[\Phi_i, \bar{\Phi}_i]|^2 - Y_i(Y_i - 2\mu_i) + \cdots\right]
\end{aligned}
$$

$$
\sum_{j\in\langle i,\cdot\rangle} e^\mu_{ij} e^\nu_{ij} \lambda_\mu(x)\mathcal{D}_\nu\bar{\Phi}(x) \rightarrow g^{\mu\nu}(x)\lambda_\mu(x)\mathcal{D}_\nu\bar{\Phi}(x)
$$

Metric & connection emerge

# Localization on the lattice

Calculate path integral with the Q-exact action by localization ($g \to 0$)

- **Fixed points (BPS)** Gauge fixing : $\Phi_i = \mathrm{diag}(\phi_{i,1}, \phi_{i,2}, \ldots, \phi_{i,N})$

$$\Lambda_{ij} = 0$$
$$Q\Lambda_{ij} = -i(\Phi_i U_{ij} - U_{ij}\Phi_j) = 0$$

→

$U_{ij} = \Gamma_{ij}$ Permutation group

$\Phi_j = \Gamma_{ij}^{\dagger}\Phi_i\Gamma_{ij}$ indep. of $i$ up to permutation

- **One-loop determinant**

$$\text{1-loop det.} = \prod_{i,j}\prod_{a<b} \frac{(\phi_{i,a}-\phi_{i,b})^2_{c,\bar{c}} \times (\phi_{i,a}-\phi_{i,b})_{\chi}}{(\phi_{i,a}-\phi_{j,b})_{U_{ij}} \times (\phi_{i,a}-\phi_{i,b})_{\bar{\Phi}}}$$

$$= \frac{\prod_{i\in V}\prod_{a<b}(\phi_{i,a}-\phi_{i,b})\prod_{i\in F}\prod_{a<b}(\phi_{i,a}-\phi_{i,b})}{\prod_{<ij>\in L}\prod_{a\leq b}(\phi_{i,a}-\phi_{j,b})}$$

Contributions from site, link & face variables

# Localization on the lattice

- **Partition function**

$$Z = \sum_{\sigma_{ij}} \int \prod_i \prod_{a=1}^{N} d\phi_{i,a} \prod_{a<b} (\phi_{i,a} - \phi_{i,b})^{\chi}$$

permutation elements

Euler characteristic

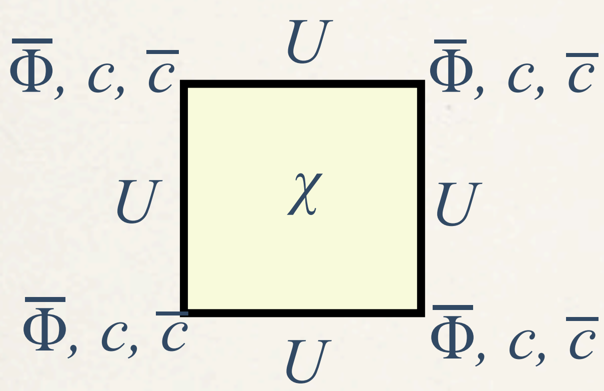
$\chi \equiv \dim V - \dim L + \dim F$

#sites #links #faces

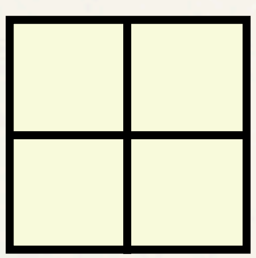
- The result depends only on the topology of the 2D surface
- Independent of simplicial decomposition (2D YM is topological)
- Multiple integrals remain due to flat direction of SUSY $\rightarrow \; e^{-S-\Phi^2}$

# Examples of Riemann surfaces

- **Disks**



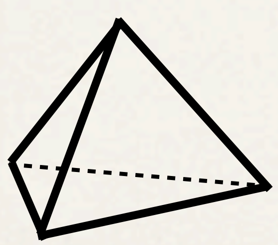

\# of $\overline{\Phi}, c, \overline{c} = 4$
\# of $U = 4$
\# of $\chi = 1$



\# of $\overline{\Phi}, c, \overline{c} = 9$
\# of $U = 12$
\# of $\chi = 4$

$\rightarrow$ $$\prod_{a<b} (\phi_a - \phi_b)^1$$

- **Spheres**



\# of $\overline{\Phi}, c, \overline{c} = 4$
\# of $U = 6$
\# of $\chi = 4$

$\rightarrow$ $$\prod_{a<b} (\phi_a - \phi_b)^2$$

The path integral depends only on the topology of the 2D surface.

# Examples of Q-closed operators

- **Kazakov-Migdal Q-closed operator**

$$\mathcal{O} = \sum_{i,j} \mathrm{Tr}\left[\Phi_i U_{ij} \Phi_j U_{ij}^\dagger\right] + \frac{1}{2}\sum_i \mathrm{Tr}\Lambda_{ij}[\Phi_j, \Lambda_{ij}^\dagger]$$ Multi-matrix HCIZ operator

Fixed points $\Phi_i U_{ij} \Phi_j U_{ij}^\dagger = \Phi_i^2$ → $\langle \mathcal{O} \rangle = \sum_{\sigma_{ij}} \int \prod_i \prod_{a=1}^N d\phi_{i,a} \prod_{a<b} (\phi_{i,a} - \phi_{i,b})^\chi \sum_{a=1}^N \phi_{i,a}^2$

- **Ward-Takahashi identity**

$$Q\,\mathrm{Tr}[i\Lambda_{ij}\Phi_j U_{ij}^\dagger] = \sum_{i,j} \mathrm{Tr}\left[\Phi_i U_{ij} \Phi_j U_{ij}^\dagger\right] + \frac{1}{2}\sum_i \mathrm{Tr}\Lambda_{ij}[\Phi_j, \Lambda_{ij}^\dagger] - \sum_i \Phi_i^2$$

Q-exact operator | Q-closed | Q-closed

$\langle Q\,\mathrm{Tr}[\Lambda_{ij}\Phi_j U_{ij}^\dagger]\rangle = 0$ → $\langle \quad \rangle = \langle \quad \rangle$ consistent to the above result

# What is appropriate Q-exact deformation ?

$$Q\Xi = Q(\mathcal{F} \cdot \overline{Q\mathcal{F}})$$

## Inappropriate Q-exact terms

- **Has contribution from *boundaries***

$$\int [\mathcal{D}X]\, Q\left(\Xi e^{-S-tQ\Xi}\right) \neq 0 \quad \rightarrow \quad \text{t-dependent integral !}$$

- **Restrict *configuration space*** (broken sym? structure changed?)

theory structure changed → fixed points can be mutilated !

**cf.) Kazakov-Migdal** $S = \sum_{i,j} \mathrm{Tr}\left[\Phi_i U_{ij} \Phi_j U_{ij}^\dagger\right] + \sum_i \mathrm{Tr} V(\Phi_i) + \frac{1}{2}\sum_i \mathrm{Tr}\Lambda_{ij}[\Phi_j, \Lambda_{ij}^\dagger]$ Q-closed under two different Q

1. $QU_{ij} = \Lambda_{ij}, \quad Q\Lambda_{ij} = -i\Phi_i U_{ij}$ ➡ $Q\Xi = \mathrm{Tr}[\Phi_i, U_{ij}\Phi_j U_{ij}^\dagger]^2 + \cdots$ ➡ $U_{ij} = \Gamma_{ij}$
2. $QU_{ij} = \Lambda_{ij}, \quad Q\Lambda_{ij} = -i(\Phi_i U_{ij} - U_{ij}\Phi_j)$ ➡ $Q\Xi = \mathrm{Tr}|\Phi_i - U_{ij}\Phi_j U_{ij}^\dagger|^2 + \cdots$ ➡ $U_{ij} = \Gamma_{ij} \quad \phi_{j,a} = \phi_{i,\sigma_{ij}(a)}$

⋮ $\mathcal{N}$=(2,2) BRST algebra

**Different results**

# Summary

- We reduce the SUSY lattice gauge theory to the simpler integral via the localization technique.
- We extend the lattice SUSY model to generic lattice surfaces.
- We evaluate KM operator and find useful Ward-Takahashi identities.
- We discuss that inappropriate Q-exact deformations do not give correct answer of the original integral.

# Back-up slides

# localization in HCIZ integral

$$\begin{aligned}\int \mathcal{D}\psi_R\, e^{\beta\omega} &= \int \mathcal{D}\psi_R\, e^{-\frac{\beta}{2}\mathrm{Tr}\,\psi_R[X_B,\psi_R]} \\ &= \int \mathcal{D}\psi_L\, e^{-\frac{\beta}{2}\mathrm{Tr}\,\psi_L[B,\psi_L]} \\ &= \beta^{N(N-1)/2}\Delta(b),\end{aligned}$$

← $\psi_L = U^\dagger \psi_R U$

Left-invariant MC form

- **One-loop determinant**

$$\begin{aligned} K &= \mathrm{Tr}[A,[Z,B]]^2 + \cdots\,, \\ \Omega &= \mathrm{Tr}[\psi_R, B_\sigma][A,[\psi_R,B_\sigma]] + \cdots\,, \end{aligned}$$

$$\begin{aligned} K &= 2\sum_{\alpha>0}\alpha(a)^2\alpha(b_\sigma)^2 z^\alpha z^{-\alpha} + \cdots\,, \\ \Omega &= -2\sum_{\alpha>0}\alpha(a)\alpha(b_\sigma)^2\psi_R^\alpha\psi_R^{-\alpha} + \cdots\,, \end{aligned}$$

Cartan-Weyl basis

$$Z = z^i H_i + z^\alpha E_\alpha, \qquad \psi_R = \psi_R^i H_i + \psi_R^\alpha E_\alpha,$$

$$[H_i, H_j] = 0,$$

$$[H_i, E_\alpha] = \alpha_i E_\alpha,$$

$$\mathrm{Tr}(E_\alpha E_\beta) = \delta_{\alpha+\beta,0}.$$

$$|\Delta(a)|^{-2}|\Delta(b)|^{-2} \qquad \Delta(a)|\Delta(b)|^2 \quad \Rightarrow \quad (-1)^{|\sigma|}\Delta(a)^{-1}$$

# Q-exact action on simplicial complex

- **Q-exact action on simplicial complex**

$$
\begin{aligned}
S &= \frac{1}{2g^2}\sum_i Q\,\mathrm{Tr}\left[i\Lambda_{ij}(U^\dagger_{ij}\bar\Phi_i - \bar\Phi_j U^\dagger_{ij}) + i\eta_i[\bar\Phi_i,\Phi_i] - \chi_i(Y_i - 2\mu_i)\right] \\
&= \frac{1}{2g^2}\sum_i \mathrm{Tr}\Big[|\Phi_i U_{ij} - U_{ij}\Phi_j|^2 + |[\Phi_i,\bar\Phi_i]|^2 - Y_i(Y_i - 2\mu_i) \\
&\qquad -i\Lambda_{ij}(U^\dagger_{ij}\eta_i - \eta_j U^\dagger_{ij}) + i\Lambda_{ij}(U^\dagger_{ij}\Lambda_{ij}U^\dagger_{ij}\bar\Phi_i - \bar\Phi_j U^\dagger_{ij}\Lambda_{ij}U^\dagger_{ij}) \\
&\qquad +i\eta_i[\Phi_i,\eta_i] + i\chi_i[\Phi_i,\chi_i] - 2\chi_i\frac{\delta\mu_i}{\delta U_{ij}}\Lambda_{ij}\Big]
\end{aligned}
$$

# One-loop determinant SUSY

- **Action** $S = tQ\,\mathrm{Tr}\left[g_{IJ}\mathcal{F}^I\overline{Q\mathcal{F}^J}\right]$

$$= t\,\mathrm{Tr}\left[||Q\vec{\mathcal{F}}||^2 - \mathcal{F}^I Q(g_{IJ}\overline{Q\mathcal{F}^J})\right]$$

$$\mathcal{B}^I = \mathcal{B}^I_0 + \frac{1}{\sqrt{t}}\tilde{\mathcal{B}}^I,$$

$$\mathcal{F}^I = \mathcal{F}^I_0 + \frac{1}{\sqrt{t}}\tilde{\mathcal{F}}^I$$

- **Action at quadratic order**

$$S = \mathrm{Tr}\left[G_{IJ}\tilde{\mathcal{B}}^I\tilde{\mathcal{B}}^J - \Omega_{IJ}\tilde{\mathcal{F}}^I\tilde{\mathcal{F}}^J\right] + \mathcal{O}(1/\sqrt{t})$$

$$G_{IJ} = \frac{\delta^2}{\delta\mathcal{B}^I\delta\mathcal{B}^J}||Q\vec{\mathcal{F}}||^2\Big|_{\vec{\mathcal{B}}=\vec{\mathcal{B}}_0},$$

$$\Omega_{IJ} = \frac{1}{2}\left(\frac{\delta}{\delta\mathcal{F}^I}Q(g_{JK}\overline{Q\mathcal{F}^K}) - \frac{\delta}{\delta\mathcal{F}^J}Q(g_{IK}\overline{Q\mathcal{F}^K})\right)\Big|_{\vec{\mathcal{F}}=\vec{\mathcal{F}}_0}$$

$$G_{IJ}(Q\tilde{\mathcal{B}}^I)\tilde{\mathcal{B}}^J = \Omega_{IJ}(Q\tilde{\mathcal{F}}^I)\tilde{\mathcal{F}}^J$$

Q-closed

# One-loop determinant SUSY

- **To look into the Hessian**

$$Q\mathcal{F}^I = Q\mathcal{F}^I\big|_{\vec{\mathcal{B}}=\vec{\mathcal{B}}_0} + \frac{1}{\sqrt{t}} \frac{\delta Q\mathcal{F}^I}{\delta\mathcal{B}^J}\bigg|_{\vec{\mathcal{B}}=\vec{\mathcal{B}}_0} \tilde{\mathcal{B}}^J,$$
$$Q\mathcal{B}^I = Q\mathcal{B}^I\big|_{\vec{\mathcal{F}}=\vec{\mathcal{F}}_0} + \frac{1}{\sqrt{t}} \frac{\delta Q\mathcal{B}^I}{\delta\mathcal{F}^J}\bigg|_{\vec{\mathcal{F}}=\vec{\mathcal{F}}_0} \tilde{\mathcal{F}}^J$$

⇔

$$Q\mathcal{F}^I = Q\mathcal{F}^I_0 + \frac{1}{\sqrt{t}} Q\tilde{\mathcal{F}}^I,$$
$$Q\mathcal{B}^I = Q\mathcal{B}^I_0 + \frac{1}{\sqrt{t}} Q\tilde{\mathcal{B}}^I.$$

⇒

$$Q\tilde{\mathcal{F}}^I = \frac{\delta Q\mathcal{F}^I}{\delta\mathcal{B}^J}\bigg|_{\vec{\mathcal{B}}=\vec{\mathcal{B}}_0} \tilde{\mathcal{B}}^J,$$
$$Q\tilde{\mathcal{B}}^I = \frac{\delta Q\mathcal{B}^I}{\delta\mathcal{F}^J}\bigg|_{\vec{\mathcal{F}}=\vec{\mathcal{F}}_0} \tilde{\mathcal{F}}^J$$

- **Substituting them**

$$G_{IJ} \frac{\delta Q\mathcal{F}^I}{\delta\mathcal{B}^K}\bigg|_{\vec{\mathcal{B}}=\vec{\mathcal{B}}_0} = \Omega_{IK} \frac{\delta Q\mathcal{B}^I}{\delta\mathcal{F}^J}\bigg|_{\vec{\mathcal{F}}=\vec{\mathcal{F}}_0}$$

⇒

$$\frac{\mathrm{Det}\, G_{IJ}}{\mathrm{Det}\, \Omega_{IJ}} = \frac{\mathrm{Det}\, \frac{\delta Q\mathcal{B}^I}{\delta\mathcal{F}^J}}{\mathrm{Det}\, \frac{\delta Q\mathcal{F}^I}{\delta\mathcal{B}^J}},$$