Tracy-Widom distribution as instanton sum of 2D IIA superstrings

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§I. 2D IIA superstrings and Kuroki-Sugino SUSY matrix model

2D IIA superstrings

Kutasov-Seiberg '90

target space

$$X^\mu = \begin{pmatrix} x \\ \varphi \end{pmatrix} \begin{matrix} \cdots \text{compactified to selfdual } R=1 \\ \cdots \text{noncompact, linear dilaton b.g.} \end{matrix} , \quad \Psi^\mu = \begin{pmatrix} \psi_x \\ \psi_\varphi \end{pmatrix} \underset{\text{bosonize}}{\rightarrow} \frac{\psi_x \pm i\psi_\varphi}{\sqrt{2}} = \exp(\mp iH)$$

action

$$S_{\mathrm{CFT}} = \int \frac{d^2 z}{2\pi}\left[\partial x \bar{\partial} x + \partial\varphi\bar{\partial}\varphi + \frac{1}{2}\hat{R}\varphi + \partial H \bar{\partial} H \quad + \text{ghosts}\right]$$

stress tensor

$$T(z) = -\frac{1}{2}(\partial x)^2 - \frac{1}{2}(\partial\varphi)^2 + \partial^2\varphi - \frac{1}{2}\psi_x\partial\psi_x - \frac{1}{2}\psi_\varphi\partial\psi_\varphi \underbrace{-2b\partial c - \partial b c}_{\text{conformal ghost}} \underbrace{-\frac{3}{2}\beta\partial\gamma - \frac{1}{2}\partial\beta\gamma}_{\text{superconformal ghost}} \underset{\text{bosonize}}{\rightarrow} \begin{matrix} \gamma = \mathrm{e}^{\phi}\eta \\ \beta = \partial\xi\, \mathrm{e}^{-\phi} \end{matrix}$$

vertex op.

$$NS: \quad T_k(z) = \exp\left(-\phi + ikx + p\varphi\right)$$

$$R\ : \quad V_k^{\pm}(z) = \exp\left(-\frac{1}{2}\phi \pm \frac{i}{2}H + ikx + p\varphi\right)$$

scale inv.,
Seiberg bnd. $\rightarrow$

physical vertex op.

$$T_k(z) = \exp\left(-\phi + ikx + (1-|k|)\varphi\right)$$

$$V_k^{\pm}(z) = \exp\left(-\frac{1}{2}\phi \pm \frac{i}{2}H + ikx + (1-|k|)\varphi\right) \qquad (\pm k > 0)$$

supercharge

$$Q_+ = \oint \frac{dz}{2\pi i} V_{-1}^{-}(z) \qquad \bar{Q}_- = \oint \frac{d\bar{z}}{2\pi i} \bar{V}_1^{+}(\bar{z}) \qquad Q_+^2 = \bar{Q}_-^2 = \left\{Q_+, \bar{Q}_-\right\} = 0$$

2D IIA superstrings

Ita-Nieder-Oz '05

spectrum of winding b.g.

$(NS,NS):\quad T_k(z)\bar{T}_{-k}(\bar{z})\quad k\in\mathbf{Z}+\frac{1}{2}$: tachyon [wind.]

$(R+,R-):\quad V_k^+(z)\bar{V}_{-k}^-(\bar{z})\quad k\in\mathbf{N}+\frac{1}{2}$

$(R-,R+):\quad V_{-k}^-(z)\bar{V}_k^+(\bar{z})\quad k\in\mathbf{N}$

: RR 2-form strength [wind.]

$(NS,R-):\quad T_k(z)\bar{V}_k^-(\bar{z})\quad k\in\mathbf{N}+\frac{1}{2}$: fermion(−) [momentum]

$(R+,NS):\quad V_k^+(z)\bar{T}_k(\bar{z})\quad k\in\mathbf{N}+\frac{1}{2}$: fermion(+) [momentum]

- locality w/supercurrent =GSO
- mutual locality
- superconformal inv.
- level matching

'Louville' interaction

$$S_{\text{Liu}}=\omega\int d^2z\ \hat{T}^{(0)}_{-1/2}(z)\hat{\bar{T}}^{(0)}_{-1/2}(\bar{z})$$

maximal set of $(R-,R+)$ vertex op.
preserving TS SUSY, incl. non-local (Seiberg)

correlation function in (R-,R+) b.g.

$$\langle\langle\cdots\rangle\rangle_{(R-,R+)\text{b.g.}}:=\int D(x,\varphi,H,\text{ghosts})\ \exp\left(-S_{\text{CFT}}-S_{\text{Liu}}-W_{RR}\right)\cdots$$

$$W_{RR}=q_{RR}\sum_{k\in\mathbf{Z}}a_k\omega^{k+1}\int d^2z\ V_{-k}^-(z)\bar{V}_k^+(\bar{z})$$

Double-well SUSY Matrix Model

Kuroki-Sugino '10~13

quest for a nonperturbative definition of 2D IIA superstring

⇒ directly simulate TS SUSY on Matrix Model, identical spectrum

(ϕ,ψ,B): 0D superfield, $N\times N$ hermitian

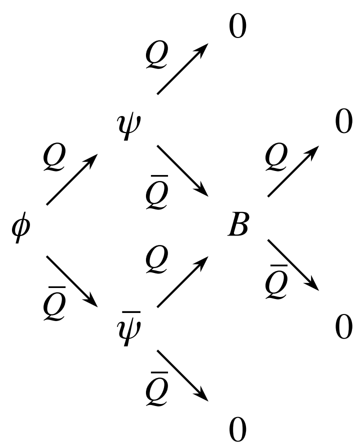
and try the simplest, non-random-triang.-motivated

$$W(\phi)=\frac{1}{3}\phi^3-\mu^2\phi: \text{ superpotential } \rightarrow V(\phi)=\left(\phi^2-\mu^2\right)^2: \text{ double-well potential}$$

$$S_{\rm MM}=N\ \mathrm{tr}\left[\frac{1}{2}B^2+iB(\phi^2-\mu^2)+\bar\psi\phi\psi+\bar\psi\psi\phi\right]$$

$$=N\ \mathrm{tr}\ \frac{1}{2}\left(\phi^2-\mu^2\right)^2-\mathrm{logdet}\left(\phi\otimes\mathbf{1}_N+\mathbf{1}_N\otimes\phi\right)$$

'SUSY'



$$Q^2=\bar Q^2=\{Q,\bar Q\}=0$$

SUSY order parameter

$$\mathrm{tr}\ B^k=Q\ \mathrm{tr}\left(iB^{k-1}\bar\psi\right)=\bar Q\ \mathrm{tr}\left(iB^{k-1}\psi\right) \quad\Rightarrow\quad \left\langle\frac{1}{N}\mathrm{tr}\ B^k\right\rangle\begin{cases}=0 & \text{if SUSY preserved}\\ \neq 0 & \text{if SUSY spont.broken}\end{cases}$$

Identifying the spectra

Kuroki-Sugino '10~13

◇ Let us assume the correspondence of supercharges between the matrix model and the type IIA theory:

matrix IIA string

$$(Q,\bar{Q}) \Leftrightarrow (Q_+,\bar{Q}_-).$$

⇒ SUSY transformation properties etc lead to

$$\Phi_1 = \frac{1}{N}\mathrm{tr}\,\phi \Leftrightarrow c_0\, g_s^2 \int d^2z\, V_{\frac{1}{2},+1}(z)\,\bar{V}_{-\frac{1}{2},-1}(\bar{z}) \qquad (\mathrm{R}+,\ \mathrm{R}-),$$

$$\Psi_1 = \frac{1}{N}\mathrm{tr}\,\psi \Leftrightarrow d_0\, g_s^2 \int d^2z\, T_{-\frac{1}{2}}(z)\,\bar{V}_{-\frac{1}{2},-1}(\bar{z}) \qquad (\mathrm{NS},\ \mathrm{R}-),$$

$$\bar{\Psi}_1 = \frac{1}{N}\mathrm{tr}\,\bar{\psi} \Leftrightarrow \bar{d}_0\, g_s^2 \int d^2z\, V_{\frac{1}{2},+1}(z)\,\bar{T}_{\frac{1}{2}}(\bar{z}) \qquad (\mathrm{R}+,\ \mathrm{NS}),$$

$$\frac{1}{N}\mathrm{tr}(-iB) \Leftrightarrow g_s^2 \int d^2z\, T_{-\frac{1}{2}}(z)\,\bar{T}_{\frac{1}{2}}(\bar{z}) \qquad (\mathrm{NS},\ \mathrm{NS}).$$

Quartet w.r.t. $(Q,\bar{Q})$ ⇔ Quartet w.r.t. $(Q_+,\bar{Q}_-)$

c_0, d_0, $\bar{d}_0$: numerical consts. , $\frac{1}{N} \Leftrightarrow g_s$

(Single trace operators in the matrix model) ⇔ (Integrated vertex operators in IIA)

(Powers of matrices) ⇔ (Windings or Momenta)

Double-well SUSY Matrix Model

Kuroki-Sugino '10~13

$$Z_{\rm MM}(\mu) = \int d^{N^2}\phi \; \exp\left(-\frac{N}{2}{\rm tr}\,\left(\phi^2-\mu^2\right)^2 - \log\det\left(\phi\otimes\mathbf{1}_N + \mathbf{1}_N\otimes\phi\right)\right)$$

$$\propto \int_{-\infty}^{\infty} d^N\lambda \; \underbrace{\prod_{i=1}^{N}\exp\left(-\frac{N}{2}\left(\lambda_i^2-\mu^2\right)^2\right)}_{\text{from potential}} \underbrace{\prod_{j>k}^{N}\left(\lambda_j-\lambda_k\right)^2}_{\text{from measure (Jacobian)}} \underbrace{\prod_{j>k}^{N}\left(\lambda_j+\lambda_k\right)^2\prod_{l=1}^{N}\lambda_l}_{\text{from fermion det }\psi,\bar{\psi}}$$

$$= \int_{-\infty}^{\infty} d^N\lambda \; \prod_{i=1}^{N}\lambda_i \; \exp\left(-\frac{N}{2}\left(\lambda_i^2-\mu^2\right)^2\right) \prod_{j>k}^{N}\left(\lambda_j^2-\lambda_k^2\right)^2 \quad = \sum_{\nu_+ + \nu_- = 1}\frac{N!}{\nu_+ N!\,\nu_- N!} Z_{(\nu_+,\nu_-)}(\mu)$$

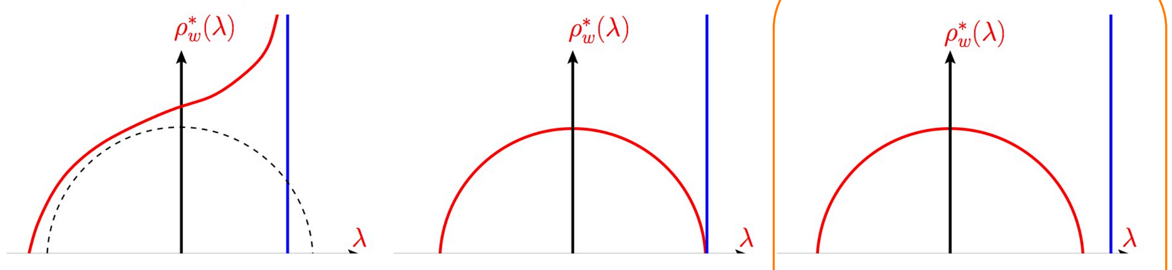
$\left[x_i := \mu^2 - \lambda_i^2\right]$

(ν_+, ν_-) filling fraction of R, L well

⇨ consider as fixed free parmeter

$$Z_{(1,0)}(\mu) = \int_{-\infty}^{\mu^2} d^N x \; \prod_{i=1}^{N}\exp\left(-\frac{N}{2}x_i^2\right) \prod_{j>k}^{N}\left(x_j-x_k\right)^2 \quad = {\rm Prob}\left[\text{all GUE-EV } x_i \le \mu^2\right]$$

⇒ 2 phases in the large-N limit .... MM computation in ⇩ phase



$\mu^2 < 2$ $\mu^2 = 2$ $\mu^2 > 2$

Identifying the correlators

Kuroki-Sugino '10~13

- $\langle N\mathrm{tr}(-iB)\,\Phi_{2k+1}\rangle_{\mathrm{cylinder}} = -\frac{1}{4}\partial_\omega \langle\Phi_{2k+1}\rangle_{\mathrm{disk}} \sim (\nu_+ - \nu_-)\,\omega^{k+1}\ln\omega$

$$\Updownarrow$$

$$\mathcal{N} g_s^{-2} \left\langle\!\!\left\langle \frac{1}{4}\left(\int T_{-\frac{1}{2}}\bar{T}_{\frac{1}{2}}\right)\left(c_k g_s^2 \int V_{k+\frac{1}{2},+1}\bar{V}_{-k-\frac{1}{2},-1}\right)\right\rangle\!\!\right\rangle$$

$$= \mathcal{N} c_k \frac{1}{4}(\nu_+ - \nu_-)\sum_{\ell\in Z} a_\ell\,\omega^{\ell+1}\left\langle\left(\int T_{-\frac{1}{2}}\bar{T}_{\frac{1}{2}}\right)\left(\int V_{k+\frac{1}{2},+1}\bar{V}_{-k-\frac{1}{2},-1}\right)\mathcal{V}_\ell^{\mathrm{RR}}\right\rangle$$

$$= -\mathcal{N} c_k \frac{1}{2}(\nu_+ - \nu_-)\,a_k\,(\omega^{k+1}\ln\omega)\,e^{i2\pi\beta(-k^2-\frac{1}{2}k+\frac{1}{4})}$$

↑ cocycle factor

RR b.g. is treated perturbatively

- various other correlators agree

- $\left\langle N^{-1}\mathrm{tr}\,(-iB)\right\rangle_{\mathrm{disk}} = 0$

$$\Updownarrow$$

$$g_s^2\left\langle\!\left\langle\int T_{-\frac{1}{2}}\bar{T}_{\frac{1}{2}}\right\rangle\!\right\rangle_{\mathrm{disk}} = 0$$

... SUSY unbroken perturbatively in g_s

Liouville coupling
↓
$4\omega = \mu^2 - 2$
↑
DWMM coupling

string coupling
↓
$g_s = N^{-1}$
↑
Matrix rank

Computation in the type IIA side reproduces the $(\nu_+ - \nu_-)$-dependence and the ω-dependence in the matrix model result.

Moreover, relations among numerical coefficients seem consistent.

Double-scaling limit

[from now on] SMN-Sugino '14

$$Z^{(1,0)}(\mu) = C \int_{-\infty}^{\mu^2} d^N x \prod_{i=1}^{N} e^{-\frac{N}{2}\operatorname{tr} x_i^2} \prod_{j>k}^{N} \left(x_j - x_k\right)^2$$

$$= \mathrm{Prob}\left[\text{all } x_i \le \mu^2\right]$$

Liouville coupling → $4\omega = \mu^2 - 2$ ← DWMM coupling

string coupling → $g_s = N^{-1}$ ← Matrix rank

⟶ $g_s^{-2/3}\, 4\omega = N^{2/3}(\mu^2 - 2) = s$:fixed

should capture all perturbative & nonperturbative contributions

but Tracy & Widom already gave the answer back in 1993!

$$Z_{\mathrm{MM}} \xrightarrow{N\to\infty,\ N^{2/3}(\mu^2-2)=s} \det\left(1 - \hat{K}^{\mathrm{Airy}}_{[s,\infty)}\right) = \exp\left(-\int_s^{\infty} dx\,(x-s)\,q_{\mathrm{HM}}(x)^2\right)$$

…just need to 'reinterpret' TW result in terms of String Theory

§II. Tracy-Widom distribution

Spectral kernel for GUE

$$d\mu(H) = d^{N^2}H\,\mathrm{e}^{-\mathrm{tr}\,H^2} \propto \prod_i d\lambda_i\,\underbrace{\mathrm{e}^{-\lambda_i^2}\left|\det\left[\lambda_i^{j-1}\right]_{i,j=1}^N\right|^2}$$

harmonic osc. WF (Hermite polyn.)

$$\det\left[\sum_{k=0}^{N-1}\psi_k(\lambda_i)\psi_k(\lambda_j)\right]_{i,j=1}^N := \det\left[K(\lambda_i,\lambda_j)\right]_{i,j=1}^N$$

- Recursion for p-point correlators

$$\int d\lambda_k \det\left[K(\lambda_i,\lambda_j)\right]_{i,j=1}^k = (N-k+1)\det\left[K(\lambda_i,\lambda_j)\right]_{i,j=1}^{k-1}$$

"det process"

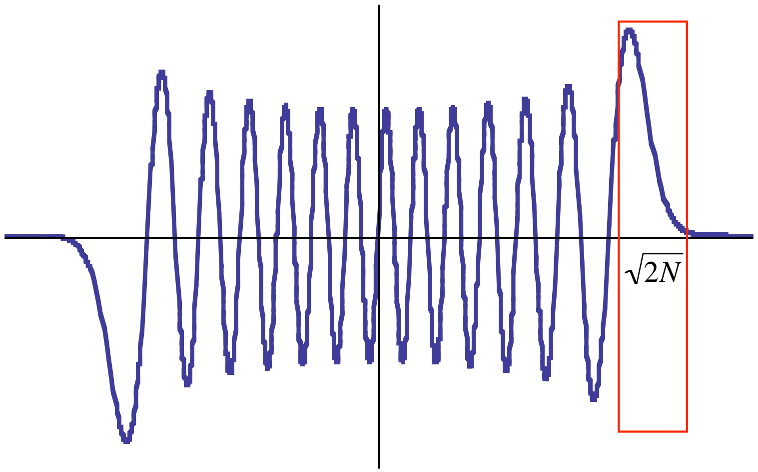
$$\Rightarrow R_p(\lambda_1,\ldots,\lambda_p) := \frac{N!}{(N-k)!}\int d\lambda_{p+1}\cdots d\lambda_N \det\left[K(\lambda_i,\lambda_j)\right]_{i,j=1}^N = \det\left[K(\lambda_i,\lambda_j)\right]_{i,j=1}^p$$

- Christoffel-Dalboux formula

$$K(\lambda,\lambda') = \sum_{k=0}^{N-1}\psi_k(\lambda)\psi_k(\lambda') = \frac{\|\psi_N\|}{\|\psi_{N-1}\|}\frac{\psi_N(\lambda)\psi_{N-1}(\lambda') - \psi_{N-1}(\lambda)\psi_N(\lambda')}{\lambda-\lambda'}$$

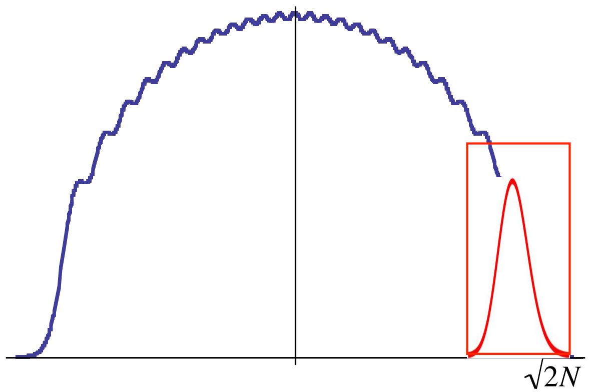
Local EV distribution

$\psi_n(\lambda)$: harmonic osc. WF



⇨

$\rho(\lambda)$: EV density



soft edge unfolding $$\lambda = \sqrt{2N} + \frac{1}{N^{1/6}} x$$

$$\Rightarrow \quad \psi_n(x) \sim \mathrm{Ai}(x)$$

$$K^{\mathrm{Airy}}(x,y) = \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(x)}{x-y}$$

TW: largest EV distribution
described by Airy kernel

⇓

YITP workshop on KPZ growth

Gap probability

$$\tau(I) := \left(\int_{\mathbf{R}} - \int_I d\lambda_1\right)\cdots\left(\int_{\mathbf{R}} - \int_I d\lambda_N\right) P(\lambda_1,\ldots,\lambda_N)$$

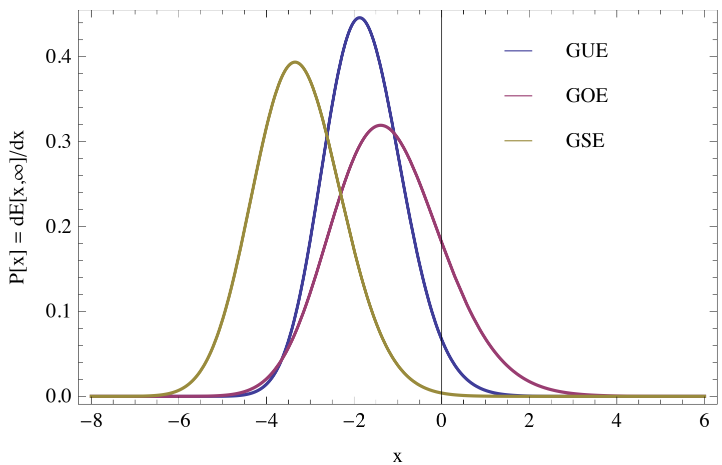
$$= \int_{\mathbf{R}} d\lambda_1 \cdots d\lambda_N P(\{\lambda\}) - \int_I d\lambda_1 \underbrace{{}_N C_1 \int_{\mathbf{R}} d\lambda_2 \cdots d\lambda_N P(\{\lambda\})}_{\text{1-EV correlator } R_1(\lambda_1)} + \int_I d\lambda_1 d\lambda_2 \underbrace{{}_N C_2 \int_{\mathbf{R}} d\lambda_3 \cdots d\lambda_N P(\{\lambda\})}_{\text{2-EV correlator } R_2(\lambda_1,\lambda_2)} - \cdots$$

$$= 1 - \int_I d\lambda_1 K(\lambda_1,\lambda_1) + \frac{1}{2!}\int_I d\lambda_1 d\lambda_2 \begin{vmatrix} K(\lambda_1,\lambda_1) & K(\lambda_1,\lambda_2) \\ K(\lambda_2,\lambda_1) & K(\lambda_2,\lambda_2) \end{vmatrix} - \frac{1}{3!}\int_I d\lambda_1 d\lambda_2 d\lambda_3 \begin{vmatrix} K(\lambda_1,\lambda_1) & K(\lambda_1,\lambda_2) & K(\lambda_1,\lambda_3) \\ K(\lambda_2,\lambda_1) & K(\lambda_2,\lambda_2) & K(\lambda_2,\lambda_3) \\ K(\lambda_3,\lambda_1) & K(\lambda_3,\lambda_2) & K(\lambda_3,\lambda_3) \end{vmatrix} + \cdots$$

$$= \det\left(1 - \hat{K}_I\right)$$

$$(\hat{K}_I \circ f)(\lambda) := \int_I d\lambda' \, K(\lambda,\lambda') f(\lambda')$$

$$p_{\text{largest}}(x) = \frac{d}{dx}\text{Prob}\left[\text{no EV in } [x,\infty)\right] = \frac{d}{dx}\det\left(1 - \hat{K}_{[x,\infty)}\right)$$



Tracy-Widom method for Fredhom det

T-W '93

$$K(x,y)=\frac{A(x)B(y)-B(x)A(y)}{x-y} \quad \text{s.t.} \quad \frac{d}{dx}\begin{pmatrix} A(x) \\ B(x) \end{pmatrix} = \underbrace{\begin{pmatrix} * & * \\ * & * \end{pmatrix}}_{\text{rational func. in } x} \begin{pmatrix} A(x) \\ B(x) \end{pmatrix}$$

(#) integrable integral op.

$$\left[K^{\text{Airy}}(x,y)=\frac{\text{Ai}(x)\text{Ai}'(y)-\text{Ai}'(x)\text{Ai}(y)}{x-y} \quad : \quad \frac{d}{dx}\begin{pmatrix} \text{Ai}(x) \\ \text{Ai}'(x) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix} \begin{pmatrix} \text{Ai}(x) \\ \text{Ai}'(x) \end{pmatrix} \right]$$

$$q(x)=\frac{1}{1-\hat{K}_I}\circ A(x), \quad p(x)=\frac{1}{1-\hat{K}_I}\circ B(x) \quad \Rightarrow \quad R(x,y):=\left\langle x \left| \frac{\hat{K}_I}{1-\hat{K}_I} \right| y \right\rangle = \frac{q(x)p(y)-p(x)q(y)}{x-y}$$

$$\text{for } I=[s,\infty), \quad \left\{ q(s),\ p(s),\ R(s,s)=\frac{d}{ds}\log\det\left(1-\hat{K}_{[s,\infty)}\right) \right\}$$

satisfy a closed set of eqs inherited from (#)

$$\log\det\left(1-\hat{K}^{\text{Airy}}_{[s,\infty)}\right) = -\int_s^\infty dx (x-s) q(x)^2 ,$$

$$q''(x)=x\,q(x)+2q(x)^3$$: Painleve $\text{II}_{\alpha=0}$

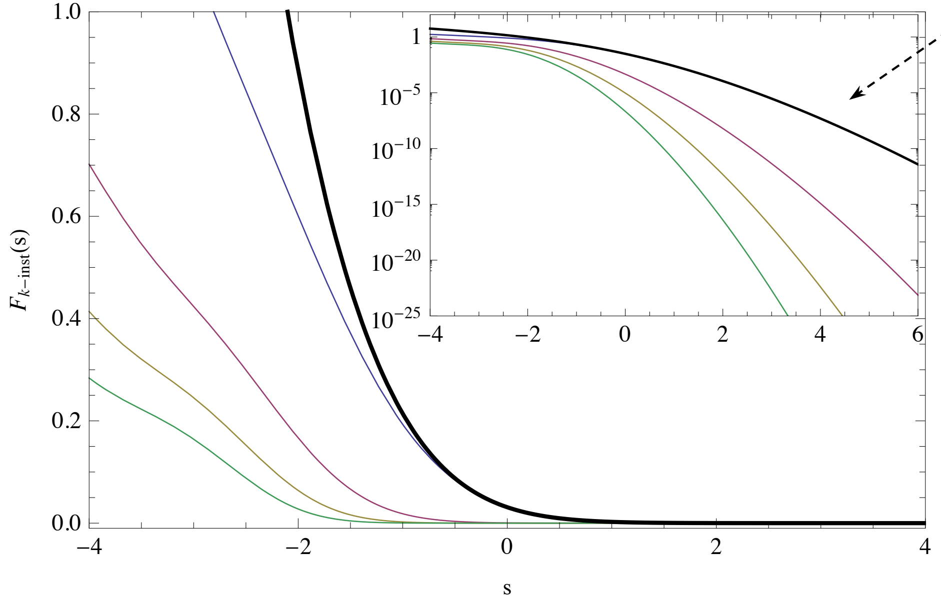
$$q(x)\xrightarrow{x\to\infty}\text{Ai}(x)$$: Hastings-McLeod sol.

§III. SUSY breaking & weak/strong coupling expansions

Double-scaling limit

$$Z^{(1,0)}(\mu^2) \longrightarrow \tau(s) = \det\left(1 - \hat{K}^{\text{Airy}}_{[s,\infty)}\right) = \exp\Big(-\underbrace{\int_s^\infty dx(x-s)q_{\text{HM}}(x)^2}_{\text{Free Energy } F(s)}\Big)$$

$g_s^{-2/3}\, 4\omega \;=\; N^{2/3}(\mu^2 - 2) = s$:fixed



$$g_s^2 \left\langle\left\langle \int T_{-\frac{1}{2}} \bar{T}_{\frac{1}{2}} \right\rangle\right\rangle = \left\langle N^{-1}\text{tr}\,(-iB)\right\rangle \;=\; \frac{d}{ds}F(s) \neq 0 \quad \Rightarrow$$ TS-SUSY always spont. broken

smooth for all s $\Rightarrow$ 3rd order transition: planar artifact, CrossOver in the DSL

Strong coupling expansion

$g_s^{-2/3}\, 4\omega \;=\; N^{2/3}(\mu^2-2) = s$: fixed $\quad\Rightarrow\quad$ strong string-coupling: $s \ll 1$

$$F(s) = \int_s^\infty dx (x-s) q_{\mathrm{HM}}(x)^2$$: smooth at $s=0$

numerical sol. to PII

$$F(s) = 0.0311059853 - 0.0690913807s + 0.0673670913s^2 - 0.0361399144s^3 + 0.0102959400s^4 - 0.000675999388s^5 - 0.000468453645s^6 + 0.0000815342772s^7 \cdots$$

strongest string coupling $s=0$ is nothing special

Weak coupling expansion

$g_s^{-2/3}\,4\omega = N^{2/3}(\mu^2-2) = s$: fixed $\Rightarrow$ weak string-coupling: $s >> 1$

$$F(s) = \int_s^\infty dx(x-s)q_{\mathrm{HM}}(x)^2$$

substitute trans-series to PII & equate like terms

$$q_{\mathrm{HM}}(x) = \sum_{k\geq 0} Q_k(x)\,, \quad \begin{cases} Q_k(x) = \dfrac{\mathrm{e}^{-\frac{4k+2}{3}s^{3/2}}}{s^{\frac{6k+1}{4}}} \displaystyle\sum_{n\geq 0} \frac{a_n^{(k)}}{s^{\frac{3n}{2}}} \\ Q_0(x) = \mathrm{Ai}(x) \end{cases}$$

⇩

trans-series for F:

$$F(s) = \sum_{k\geq 1} F_{1-\mathrm{inst}}(s) \sim \frac{e^{-\frac{4}{3}s^{3/2}}}{16\pi s^{3/2}}\left(1 - \frac{35}{24s^{3/2}} + \frac{3745}{1152s^3} - \frac{805805}{82944s^{9/2}} + \cdots\right),$$

$$F_{2-\mathrm{inst}}(s) \sim \frac{1}{2}\left(\frac{e^{-\frac{4}{3}s^{3/2}}}{16\pi s^{3/2}}\right)^2\left(1 - \frac{35}{12s^{3/2}} + \frac{619}{72s^3} - \frac{592117}{20736s^{9/2}} + \cdots\right),$$

$$F_{3-\mathrm{inst}}(s) \sim \frac{1}{3}\left(\frac{e^{-\frac{4}{3}s^{3/2}}}{16\pi s^{3/2}}\right)^3\left(1 - \frac{35}{8s^{3/2}} + \frac{2059}{128s^3} - \frac{184591}{3072s^{9/2}} + \cdots\right),$$

$$F_{4-\mathrm{inst}}(s) \sim \frac{1}{4}\left(\frac{e^{-\frac{4}{3}s^{3/2}}}{16\pi s^{3/2}}\right)^4\left(1 - \frac{35}{6s^{3/2}} + \frac{3701}{144s^3} - \frac{1112077}{10368s^{9/2}} + \cdots\right),$$

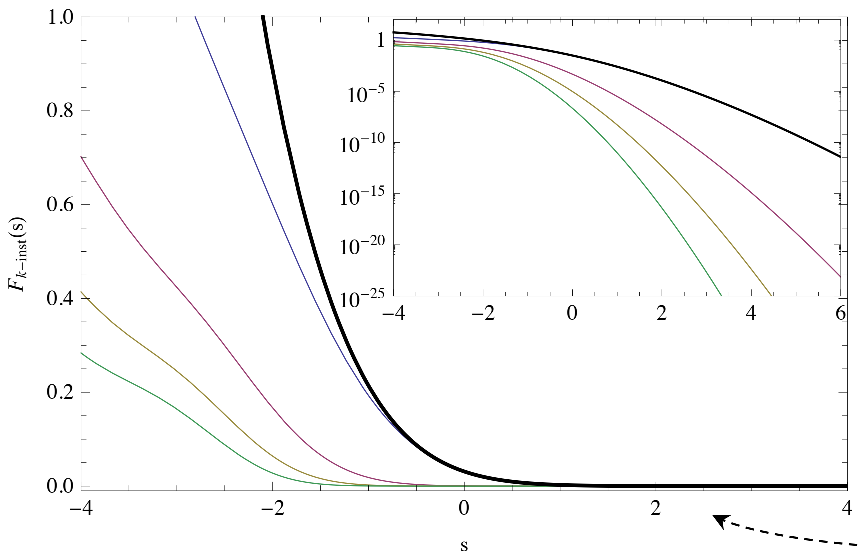
perturbative part absent

expansion in N^{-1} = open strings

Weak coupling expansion

$g_s^{-2/3}\,4\omega = N^{2/3}(\mu^2-2) = s$: fixed $\Rightarrow$ weak string-coupling: $s >> 1$

$$F(s) = \int_s^\infty dx(x-s)q_{\rm HM}(x)^2$$



$$F_{1-\rm inst}(s) \sim \frac{e^{-\frac{4}{3}s^{3/2}}}{16\pi s^{3/2}}\left(1-\frac{35}{24s^{3/2}}+\frac{3745}{1152s^3}-\frac{805805}{82944s^{9/2}}+\cdots\right),$$

$$F_{2-\rm inst}(s) \sim \frac{1}{2}\left(\frac{e^{-\frac{4}{3}s^{3/2}}}{16\pi s^{3/2}}\right)^2\left(1-\frac{35}{12s^{3/2}}+\frac{619}{72s^3}-\frac{592117}{20736s^{9/2}}+\cdots\right),$$

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$$F_{4-\rm inst}(s) \sim \frac{1}{4}\left(\frac{e^{-\frac{4}{3}s^{3/2}}}{16\pi s^{3/2}}\right)^4\left(1-\frac{35}{6s^{3/2}}+\frac{3701}{144s^3}-\frac{1112077}{10368s^{9/2}}+\cdots\right),$$

nonperturbative (=inst.) effects always break TS-SUSY, even in the weak coupling

"Negative weak" coupling expansion

$g_s^{-2/3}\, 4\omega = N^{2/3}(\mu^2 - 2) = s$: fixed ⇒ $s < 0$: no interpretation as 2D IIA strings

$$F(s) = \int_s^\infty dx (x-s) q_{\rm HM}(x)^2 = \int^s dy \int^y dx\, q_{\rm HM}(x)^2$$

substitute trans-series to PII & equate like terms

$$q(-z;C) = \sum_{k\geq 0} C^k q_k(z)\,, \quad \begin{cases} q_k(z) = \dfrac{e^{-k\frac{2\sqrt{2}}{3} z^{3/2}}}{s^{\frac{3k-2}{4}}} \displaystyle\sum_{n\geq 0} \frac{b_n^{(k)}}{z^{\frac{3n}{2}}} \\ q_0(z) \sim \sqrt{z/2} \end{cases}$$

⇩

H-M solution given as "median" Borel resum.

$$q_{\rm HM}(-z) = q_\pm(-z : \mp S/2) = \Re\mathrm{e}\left[q_{0,\pm}(z) - \frac{S^2}{4} q_{2,\pm}(z) + \frac{5S^4}{16} q_{4,\pm}(z) + \cdots \right]$$

⇩

$S = -i/\sqrt{2\pi}$: Stokes const for PII Kawai-Kuroki-Matsuo '05

trans-series for F:

$$F(-z) \sim \frac{z^3}{12} + \frac{1}{8}\log z - \frac{1}{24}\log 2 - \zeta'(-1) - \frac{3}{2^6 z^3} - \frac{63}{2^8 z^6} + \cdots$$

← perturbative, N^{-2} : closed

$$+ \frac{e^{-\frac{4\sqrt{2}}{3} z^{3/2}}}{2\pi z^{3/2}} \left(\frac{1}{2^{11/2}} - \frac{71}{2^9 3 z^{3/2}} + \frac{13465}{2^{27/2} 3^2 z^3} - \frac{5083145}{2^{17} 3^4 z^{9/2}} + \cdots \right)$$

← "2"-instanton, N^{-1} : open

$$+ \frac{e^{-\frac{8\sqrt{2}}{3} z^{3/2}}}{(2\pi)^2 z^3} \left(\frac{3}{2^{10}} - \frac{65}{2^{25/2} z^{3/2}} + \frac{3905}{2^{15} 3 z^3} - \frac{3132385}{2^{39/2} 3^3 z^{9/2}} + \cdots \right) + \cdots$$

← "4"-instanton, N^{-1} : open

§IV. Instanton condensation

Instanton fugacity

$$F(s) = \mathrm{tr}\,\log\left(1 - \hat{K}^{\mathrm{Airy}}_{[s,\infty)}\right) = \sum_{k\geq 1} \frac{1}{k} \int_s^\infty dx_1 \cdots dx_k K^{\mathrm{Airy}}(x_1, x_2) K^{\mathrm{Airy}}(x_2, x_3) \cdots K^{\mathrm{Airy}}(x_k, x_1)$$

$$= \int_s^\infty (x-s) q_{\mathrm{HM}}(x)^2 \qquad q_{\mathrm{HM}}(x) \sim \mathrm{Ai}(x)$$

$$= \sum_{k\geq 1} F_{k\text{-inst}}(s)$$

what if we include the spectral parameter ξ ?

Instanton fugacity

$$F(\xi,s)=\mathrm{tr}\ \log\left(1-\xi\hat{K}^{\mathrm{Airy}}_{[s,\infty)}\right)=\sum_{k\geq 1}\frac{\xi^k}{k}\int_s^\infty dx_1\cdots dx_k K^{\mathrm{Airy}}\left(x_1,x_2\right)K^{\mathrm{Airy}}\left(x_2,x_3\right)\cdots K^{\mathrm{Airy}}\left(x_k,x_1\right)$$

$$=\int_s^\infty (x-s)q_\xi(x)^2 \qquad q_\xi(x)\sim\sqrt{\xi}\,\mathrm{Ai}(x)$$

$$=\sum_{k\geq 1}\xi^k F_{k\text{-inst}}(s)$$

trans-series parameter ξ = 'Instanton Fugacity',

at least for the weak-cpl. regime $s>>1$

where individual instanton contributions are discernible

for $I\subsetneq\mathbf{R}$: $\quad\mathrm{Spec}\left(\hat{K}_I\right)=\left\{(1>)\Lambda_1\geq\Lambda_2\geq\cdots(>0)\right\}$

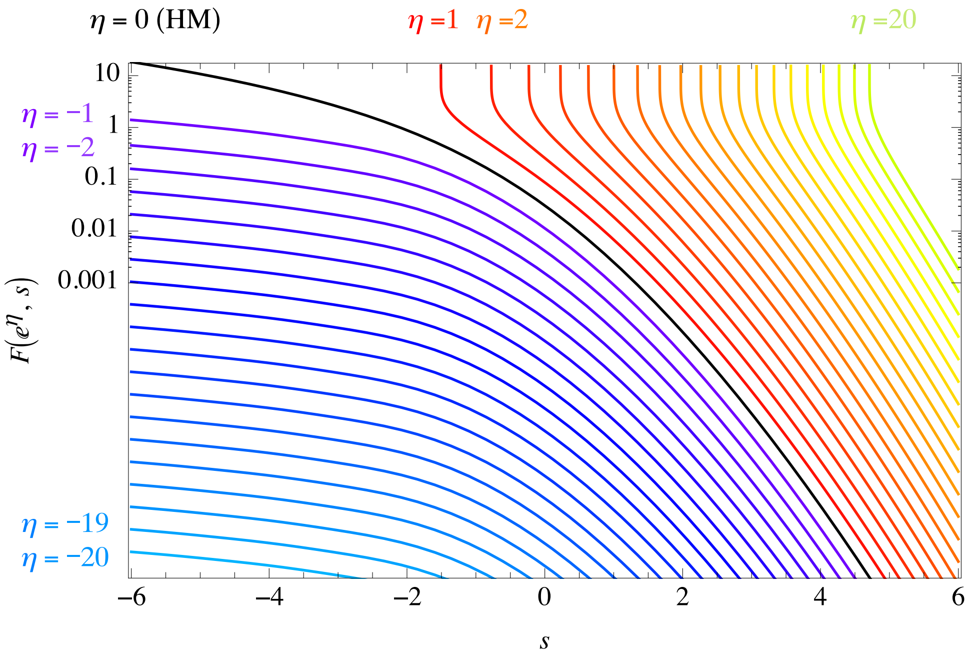
$$\Rightarrow\quad \tau(\xi;s)=\det\left(1-\xi\hat{K}^{\mathrm{Airy}}_{[s,\infty)}\right)=\prod_n\left(1-\xi\,\Lambda_n(s)\right)\quad\text{vanishes when}\quad \xi\to\frac{1}{\Lambda_1(s)}>1$$

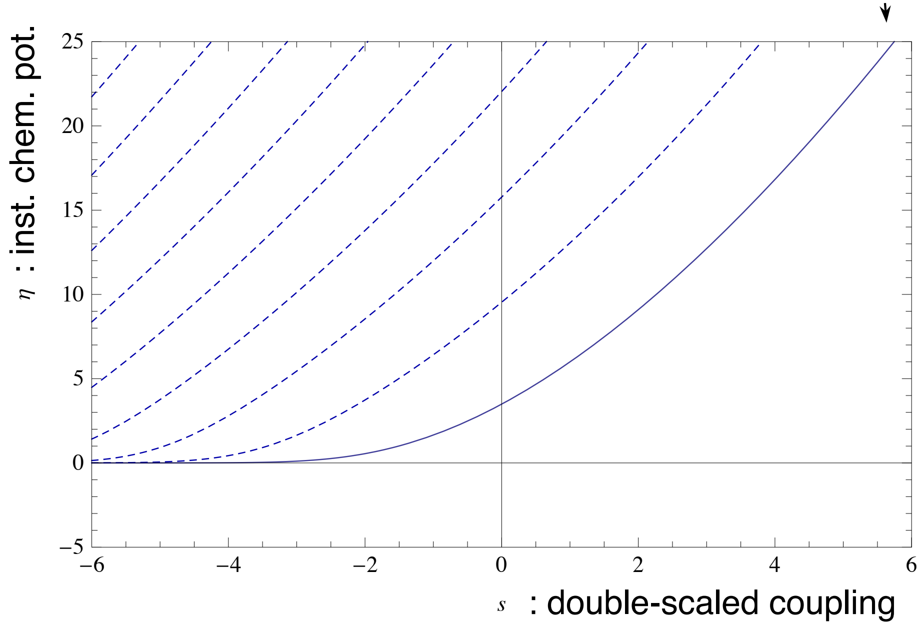
$$[\log]\ \Rightarrow\quad F\left(e^\eta;s\right)=-\sum_n\log\left(1-e^\eta\,\Lambda_n(s)\right)\qquad\text{diverges when}\quad \eta\to-\log\Lambda_1(s)>0$$

Instanton condensation?

$$F\left(e^{\eta};s\right) = -\sum_n \log\left(1 - e^{\eta}\,\Lambda_n(s)\right) \quad \text{diverges when} \quad \eta \to -\log\Lambda_1(s) > 0$$

$$\Lambda_1(s) = \max \mathrm{Spec}\left(\hat{K}^{\mathrm{Airy}}_{[s,\infty)}\right)$$





even at the weakest coupling $s \gg 1$,

turning on instanton chem. pot $\eta > 0$ leads to singularity

⇒ condensation transition

§V. Conclusion

V. Conclusion

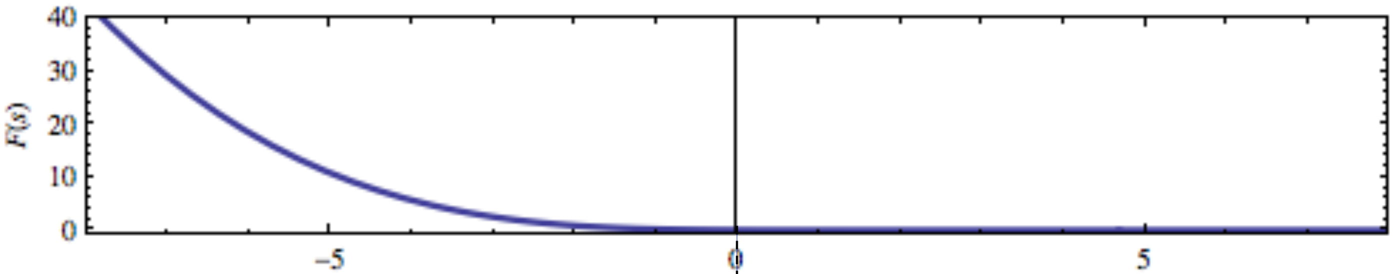
<u>SUMMARY</u>

- DSL of 2D IIA superstrings at R_{selfdual}, in RR b.g. ⇦ KS SUSY MM

  $Z \to$ TW distribution, $F'' \to$ HM solution to Painleve II ⇨ Trans-series = Instanton sum

- TS SUSY is always broken $F'(s) = \left\langle N^{-1}\mathrm{tr}(-iB)\right\rangle = \langle 0|\{Q,*\}|0\rangle = \langle 0|\{\bar{Q},*\}|0\rangle = O(\mathrm{e}^{-\frac{4}{3}s^{3/2}}) \neq 0$

  by the Airy tail of EV's WF, planar : 3rd order GW transition → DSL : crossover



- 2D U(N) YM: weak [2-cut] ←→ strong [1-cut]

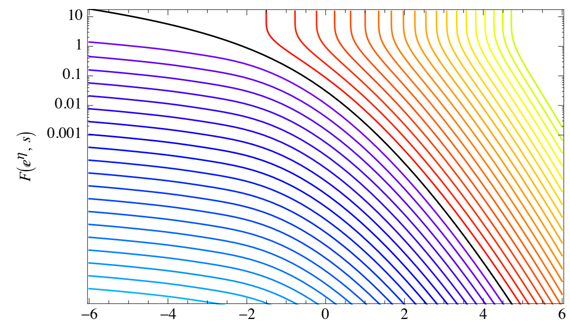
  KS SUSY MM: planar SUSY × ←→ planar SUSY √

  String Theory: weak ← 1D 0B strings | 1D 0A strings → weak

  strong ← 2D IIA superstrings → weak

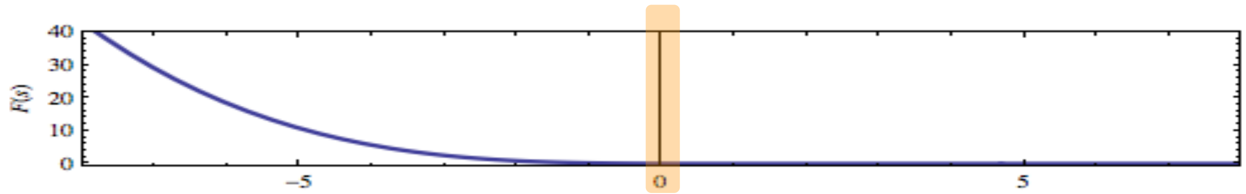
- Spectral parameter ξ in $\det\left(1-\xi\hat{K}^{\mathrm{Airy}}_{[s,\infty)}\right)$ = instanton fugacity

  phase boundary of instanton condensation identified



To-Do List

- construct a dual theory "offcritical M" which reproduces the strong couping exp. as perturbative series



- correlators of other operators of IIA string theory, at higher genera

- GOE & GSE analogues vs non-oriented strings

- counterparts for $\mathrm{PII}_{\alpha\neq 0}$, higher critical TW,...

- 2D Liouville conformal block ⟷ [AGT '10] 4D N=2 SQCD instanton sum ⟷ [Gamayun '13] τ of PIII, PV, PVI

  ? ⟷ 2D IIA strings instanton sum ⟷ [NS '14] τ of PII