Calorons and the monopole limit

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0: INTRODUCTION

- Calorons are solutions of the anti-self dual Yang-Mills theory in $\mathbb{R}^3 \times S^1$. It is supposed that the calorons give connection between the instantons and the monopoles.
- It is well known that calorons can be produced generally through the Nahm construction, in which the dual space description of the gauge fields, called the Nahm data, plays central role. The transformation from the dual space into the configuration space is called the Nahm transform.
- The Nahm data of calorons have usually the monopole limit as well as the instanton limit. However, the Nahm data of 3-caloron with C_3-symmetry (i.e. cyclic 3-caloron) does not have the monopole limit. In this poster, we perform numerical Nahm transform and visualize the action densities of the cyclic 3-calorons. We discuss the behavior of the cyclic 3-caloron on the configuration space.

I: WHAT IS A CALORON

Calorons are finite solutions to the anti-self dual Yang-Mills equations on $\mathbb{R}^3 \times S^1$.

Action : $S = \int_{\mathbb{R}^3 \times S^1} d^4x \ \mathrm{tr}\ F_{\mu\nu}F^{\mu\nu}$. $\quad \mu, \nu = 1, 2, 3, 4.$

$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$: field strength.
$A_\mu \in \mathcal{G}$:gauge field, $\mathcal{G}$: Lie algebra

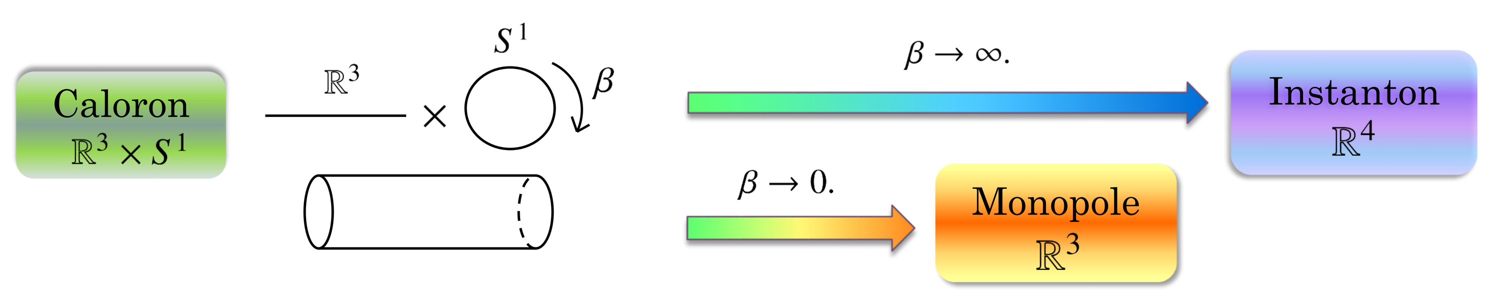
Calorons satisfy the Anti-self dual equations.

ASD(Anti-self dual) equations : $F_{\mu\nu} = \pm {}^*F_{\mu\nu}$, $\Longrightarrow \begin{cases} S = \mp\frac{1}{2}\int d^4x\ \mathrm{tr}[{}^*F_{\mu\nu}F^{\mu\nu}], & \text{minimum,} \\ D_\mu F^{\mu\nu} = 0, & \text{satisfy the Yang-Mills eq.} \end{cases}$

Hodge dual : ${}^*F^{\mu\nu} := \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$.

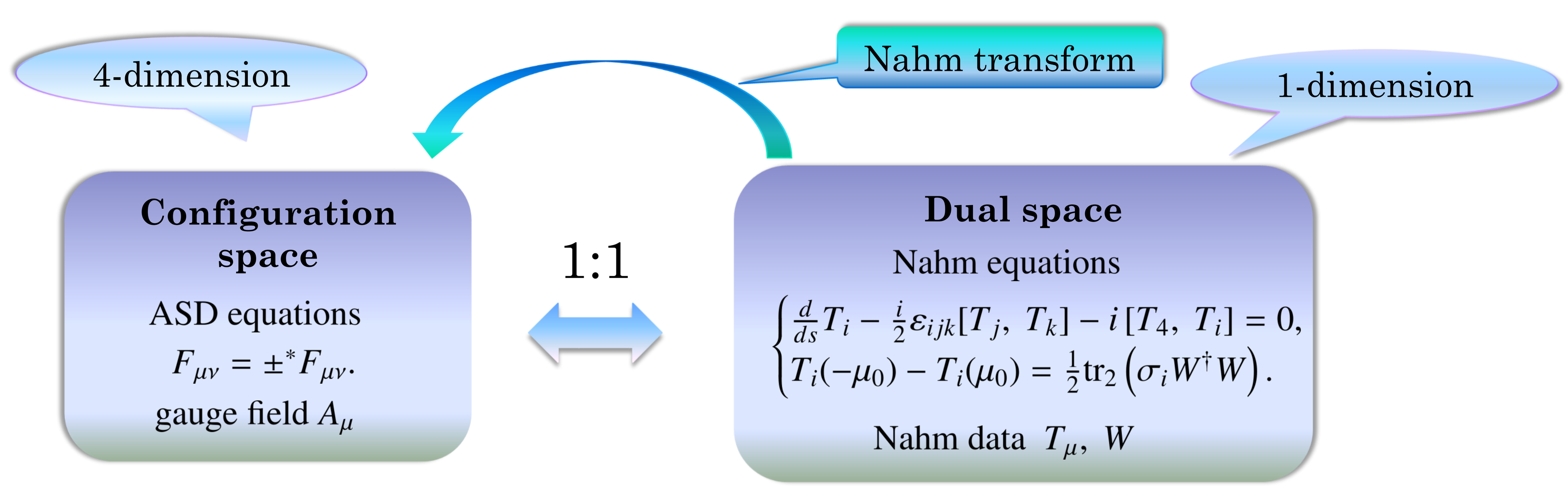
B.J. Harrington and H. K. Shepard, Phys. Rev. D **17** (1978) 2122-2125; *ibid.* **18** (1978) 2990-2994.

The calorons are closely related to monopoles and instantons.
Now, let β be period of S^1. When β goes to infinity, S^1 is identified as $\mathbb{R}$, then in this limit caloron corresponds with instanton. This limit is called "instanton limit" of the calorons.
On the other hand, when β goes to zero, S^1 shrinks to a point, then in this limit caloron corresponds with monopole. This limit is called "monopole limit" of the calorons.



II: NAHM CONSTRUCTION

Nahm construction is a complete construction of the caloron from "Nahm data".
In the Nahm construction, we solve the Nahm equations which is quite easy to treat compared with ASD equations. The Nahm equations and the ASD equations constitute a dual structure.
We are able to get the Nahm data in terms of solving the Nahm equations, which corresponds the gauge field on dual space.
After getting the Nahm data, we pull the data from dual space back to configuration space.
This transform is called "Nahm transform".



W. Nahm, *The construction of all self-dual monopoles by the ADHM method, in "Monopoles in Quantum Field Theory"*, Proceedings of the monopole meeting in Trieste 1981, World Scientific, Singapore, 1982.

II-1 :Nahm construction of the SU(2) massless k-caloron

For simplicity, we consider the case of SU(2) gauge group and also calorons of the massless limit.

$k \in \mathbb{Z}_+$:caloron's instanton charge, $l = 1, 2$:index,
$s \in (-1, 1)$: dual space coordinate, $\mu_0 \in (0, 1)$: scale parameter

We introduce the Nahm data for the caloron, which is equivalent of the gauge field on dual space. Caloron Nahm data usually consists of two elements; the bulk data and the boundary data, which each satisfy Nahm equations.

Nahm data $T_\mu(s, x^\mu)$:bulk, $W(\mu_0)$:boundary

$^\exists T_\mu$: $k \times k$ matrix, W : $2k \times 2$ matrix $\quad s.t. \quad \begin{cases}\text{bulk Nahm-eq. : } \frac{d}{ds}T_i - \frac{i}{2}\varepsilon_{ijm}[T_j,\ T_m] - i[T_4,\ T_i] = 0. \\ \text{boundary Nahm-eq. : } T_i(-\mu_0) - T_i(\mu_0) = \frac{1}{2}\mathrm{tr}_2\left(\sigma_i W^\dagger W\right).\end{cases}$

σ_i : Pauli matrices
$e_\mu = (-i\sigma_i, \mathbf{1}_2)$: quaternion basis

Once we obtain the Nahm data, we perform the Nahm transform by using such data to get the caloron gauge field.

Nahm transform

Weyl-eq. : $\left(\mathbf{1}_{2k}\frac{d}{ds} - i(T_\mu(s) + x_\mu \mathbf{1}_k) \otimes e_\mu\right)\mathbf{u}_l = iW^\dagger \mathbf{v}_l \delta(s - \mu_0).$

zero modes $\mathbf{u}_l$: $2k$-vector, $\mathbf{v}_l$: 2-vector

normalization : $\int_I \mathbf{u}_a^\dagger \mathbf{u}_b ds + \mathbf{v}_a^\dagger \mathbf{v}_b = \delta_{ab}.$

gauge field : $\left(A_\mu(x)\right)_{ab} = \int_I \mathbf{u}_a^\dagger \partial_\mu \mathbf{u}_b ds + \mathbf{v}_a^\dagger \partial_\mu \mathbf{v}_b. \qquad a, b = 1, 2.$

Usually it is impossible to perform the Nahm transform analytically, because we need to treat the Weyl equation which is coupled system of the ordinary differential equations.
So we consider performing the Nahm transform numerically.

II-2: How to numerically solve the Weyl-eq.

The essential point of the numerical Nahm transform is the method to solve the Weyl equations.
A key point of the strategy is that we regard the boundary Weyl equations as boundary conditions for the bulk Weyl equations.

Weyl-eq. : $\left(\mathbf{1}_{2k}\frac{d}{ds} - i(T_\mu(s) + x_\mu \mathbf{1}_k) \otimes e_\mu\right)\mathbf{u}_l = iW^\dagger \mathbf{v}_l \delta(s - \mu_0).$

decompose $\Rightarrow \begin{cases}\text{bulk Weyl-eq. : } \left(\mathbf{1}_{2k}\frac{d}{ds} - i(T_\mu(s) + x_\mu \mathbf{1}_k) \otimes e_\mu\right)\mathbf{u}_l = 0, \\ \text{boundary Weyl-eq. : } \mathbf{u}_l(-\mu_0) - \mathbf{u}_l(\mu_0) =: \Delta\mathbf{u}_l = iW^\dagger \mathbf{v}_l.\end{cases}$

$\mathbf{u}_l$: bulk zero modes, $\mathbf{v}_l$: boundary zero modes

Strategy of the numerical solution of the Weyl-eq.

- First, we find solutions to the bulk Weyl equations.
- Next, we solve the boundary Weyl equations with use of a degree of freedom of the linear combinations of the bulk solutions.

III: NAHM DATA

III-1: Conditions of the Nahm data

As is well-known that the caloron Nahm data and the monopole Nahm data has close relation, so it is useful to summarize now.
In the massless calorons and monopoles Nahm data we are able to take $T_4 = 0$ by using the gauge transformation.

Monopole Nahm data

Monopole Nahm data $T_i(s)$ satisfy:

- $\frac{d}{ds}T_i(s) = \frac{i}{2}\varepsilon_{ijk}\left[T_j(s),\ T_k(s)\right],$
- $T_i(s) = T_i^\dagger(s),$
- $T_i(-s) = {}^tT_i(s).$
- $T_i(s)$ has simple poles at $s = \pm 1$.
- The matrix residues of (T_1, T_2, T_3) at each pole form the irreducible k-dimensional representation of su(2).

Caloron Nahm data

Caloron Nahm data consists of two elements:bulk data $T_i(s)$ and boundary data W.

Bulk Nahm data satisfy:

- $\frac{d}{ds}T_i(s) = \frac{i}{2}\varepsilon_{ijk}\left[T_j(s),\ T_k(s)\right],$
- $T_i(s) = T_i^\dagger(s),$
- $T_i(-s) = {}^tT_i(s).$

Boundary Nahm data satisfy:

$T_i(-\mu_0) - T_i(\mu_0) = \frac{1}{2}\mathrm{tr}_2\left(\sigma_i W^\dagger W\right).$

- Caloron bulk Nahm data does not have to have simple pole at $s = -1, 1$.

We are easily to expect that the condition of the caloron bulk data is equivalent to a part of the condition of the monopole data because the conditions for the bulk data are almost equivalent.
So if we know a monopole Nahm data $T_i(s)$, then it can directly be applied for caloron bulk Nahm data.
Next, we introduce two types of the Nahm data. The first type has the monopole limit, which is called "symmetric 3-caloron". And second type doesn't have the monopole limit, which is called "cyclic 3-caloron".

III-2: Symmetric 3-caloron

R.S. Ward, Phys.Lett.B **582** (2004) 203.

Bulk Nahm data

$$T_1(s) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & b(s) - 2ia(s) \\ 0 & b(s) + 2ia(s) & 0 \end{pmatrix},\quad T_2(s) = \begin{pmatrix} 0 & 0 & b(s) + 2ia(s) \\ 0 & 0 & 0 \\ b(s) - 2ia(s) & 0 & 0 \end{pmatrix},$$

$$T_3(s) = \begin{pmatrix} 0 & b(s) - 2ia(s) & 0 \\ b(s) + 2ia(s) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$a(s) = -\frac{\omega\,\wp'\left(\frac{\omega}{3}(s+3)\right)}{12\wp\left(\frac{\omega}{3}(s+3)\right)},\quad b(s) = -\frac{\omega}{\sqrt{3}\wp\left(\frac{\omega}{3}(s+3)\right)},$$

$$\omega = \frac{\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{1}{3}\right)}{4\sqrt{\pi}},$$

Here $\wp$ is the Weierstrass p-function satisfying,

$$4\wp'(x)^2 = 4\wp(x)^3 + 4.$$

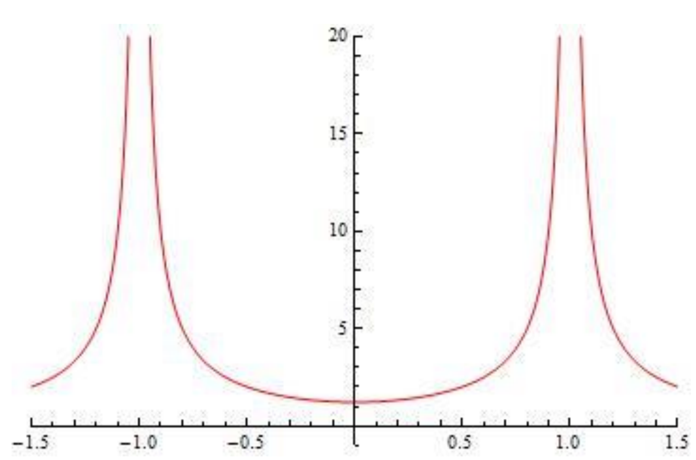


Fig: absolute value of $b(s) + 2ia(s)$

Boundary Nahm data: $W = \lambda\left(\mathbf{1}_2 \quad i\sigma_3 \quad -i\sigma_2\right)$. $\quad$ where $\lambda := 2\sqrt{a(\mu_0)}$.

- The Nahm data of symmetric 3-caloron has the simple pole at $s = -1$ and $s = 1$.

III-3: Cyclic 3-caloron

A. Nakamula and N.Sawado, N.Phys.B **868** (2013) 476-491.

Bulk Nahm data

$$T_1 = \frac{1}{2}\begin{pmatrix} 0 & f_+ - if_- & f_0 \\ f_+ + if_- & 0 & f_+ - if_- \\ f_0 & f_+ + if_- & 0 \end{pmatrix},\quad T_2 = \frac{1}{2}\begin{pmatrix} f_0 & -f_+ - if_- & 0 \\ -f_+ + if_- & 0 & f_+ + if_- \\ 0 & f_+ - if_- & -f_0 \end{pmatrix},$$

$$T_3 = \frac{1}{4}\begin{pmatrix} -p_2 & 0 & i(p_0 - p_1) \\ 0 & 2p_2 & 0 \\ -i(p_0 - p_1) & 0 & -p_2 \end{pmatrix},$$

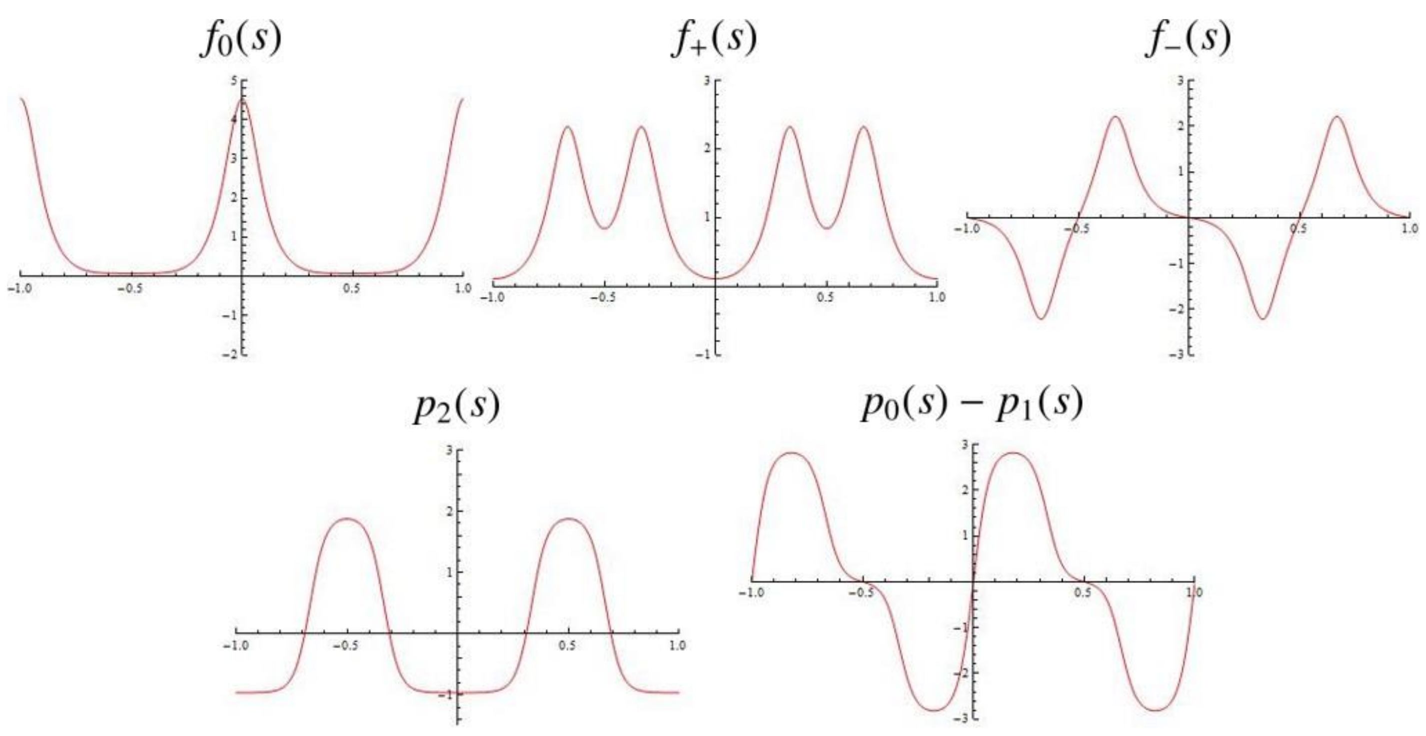
where $f_\pm := (f_1 \pm f_2)/2$.

$$f_0(s) := iC\frac{\sqrt{\vartheta_\nu(s_1)\vartheta_\nu(s_2)}}{\vartheta_\nu(s_0)},\ f_1(s) := iC\frac{\sqrt{\vartheta_\nu(s_2)\vartheta_\nu(s_0)}}{\vartheta_\nu(s_1)},\qquad p_2(s) := \frac{d}{ds}\log\frac{\vartheta_\nu(s_2)}{\vartheta_\nu(s_1)},$$

$$f_2(s) := iC\frac{\sqrt{\vartheta_\nu(s_0)\vartheta_\nu(s_1)}}{\vartheta_\nu(s_2)}.\qquad p_0(s) := \frac{d}{ds}\log\frac{\vartheta_\nu(s_0)}{\vartheta_\nu(s_2)},\ p_1(s) := \frac{d}{ds}\log\frac{\vartheta_\nu(s_1)}{\vartheta_\nu(s_0)}.$$

$\vartheta_\nu(s)$ are the Jacobi theta functions (and now, we omit modulus parameter q).
And $\nu = 0$ or 3. $\quad 0 < q < 1$

Here, $C := \vartheta_1'(0)/\vartheta_1(1/3) \in i\mathbb{R}$ is pure imaginary constant and $s_j := s + j/3$.



Figs: functions of the cyclic 3-caloron Nahm data for ϑ_0.

- Cyclic 3-caloron's Nahm data does not have simple poles at $s = \pm 1$.

- Boundary Nahm data: $W(\mu_0) = (\lambda, \rho, \chi)$.

$$\lambda = i\lambda_1(\sigma_1 + \sigma_2), \rho = -\frac{2}{\lambda_1} g(\mu_0)\mathbf{1}_2, \quad \chi = -i\lambda_1(\sigma_1 - \sigma_2),$$

$$g(\mu_0) := -f_-(\mu_0), \quad \lambda_1^2 = h(\mu_0) := -\frac{1}{2}(p_0(\mu_0) - p_1(\mu_0)).$$

Conditions that the boundary data are well-defined.

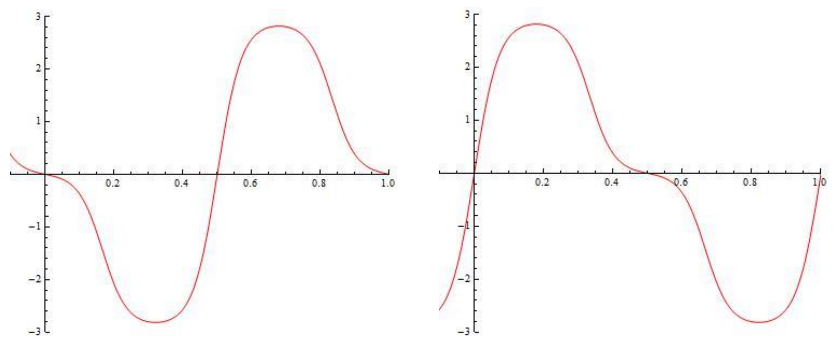
1.Theta functions condition

Quaternion's component

$$\lambda_1^2 = h(\mu_0) \ \wedge \ \lambda_1 \in \mathbb{R} \Longrightarrow h(\mu_0) > 0.$$

Form Nahm data

Figs plot function $p_0(\mu_0) - p_1(\mu_0)$



In the case of ϑ_3 In the case of ϑ_0

Thus we require the condition: $p_0(\mu_0) - p_1(\mu_0) < 0 \iff \vartheta_\nu = \begin{cases} \vartheta_3, & \mu_0 \in (0.0,\ 0.5) \\ \vartheta_0, & \mu_0 \in (0.5,\ 1.0) \end{cases}$

2.Boundary data condition

We require a condition : $^\forall \mu_0 \in \mathbb{R}_+,\ W(\mu_0) \neq 0.$

$\because\ W(\mu_0) = 0 \Rightarrow$ boundary zero mode $\mathbf{v}_l$ are not well-defined at μ_0.

But now, we find that boundary data $W(0.5) = W(1.0) = 0$. Consequently, the cyclic 3-caloron Nahm data is not well-defined at $\mu_0 = 0.5,\ 1.0$.

Figs plot functions of the boundary Nahm data

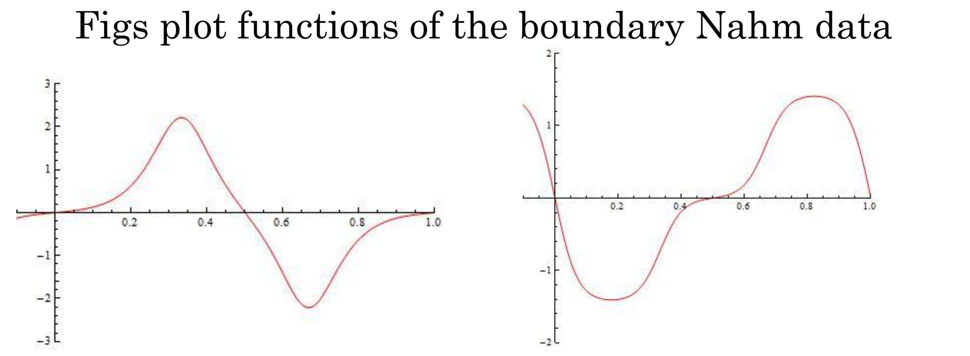


Fig: the $g(\mu_0)$ as a function of μ_0.   Fig: the $h(\mu_0)$ as a function of μ_0.

III-4: Instanton limit

We discuss the limiting behavior of the calorons from viewpoint of the difference of Nahm data of symmetric 3-caloron and cyclic 3-caloron.

The instanton limit is a limit that the scale parameter μ_0 and s simultaneously go to zero. In this limit, the Nahm data of caloron equal an ADHM data of the instanton.

Nahm data of the caloron	$\mu_0, s \to 0$	ADHM data of the instanton
bulk data $T_i(s)$, boundary data $W(\mu_0)$	$\longrightarrow$	ADHM data Δ

ADHM data: $\Delta = \underbrace{\begin{pmatrix} \tilde{W} \\ iT_i(0) \otimes \sigma_i \end{pmatrix}}_{=:\tilde{\Delta}} + \begin{pmatrix} 0 \\ \mathbf{1}_k \otimes x^\mu e_\mu \end{pmatrix}.$

ADHM data correspond to the Nahm data of the Instanton.

$\tilde{W}$ is the lowest order in μ_0 for the boundary data.

Symmetric 3-caloron

$$\tilde{\Delta} = \frac{\omega}{\sqrt{3}} \begin{pmatrix} \mathbf{1}_2 & i\sigma_3 & -i\sigma_2 \\ 0 & i\sigma_3 & i\sigma_2 \\ i\sigma_3 & 0 & i\sigma_1 \\ i\sigma_2 & i\sigma_1 & 0 \end{pmatrix}, \quad \omega = \frac{\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{1}{3}\right)}{4\sqrt{\pi}}.$$

Cyclic 3-caloron

$$\tilde{\Delta} = \frac{1}{2} \begin{pmatrix} 2i\tilde{\lambda}_1(\sigma_1 + \sigma_2) & 2\tilde{\rho}_0 & -2i\tilde{\lambda}_1(\sigma_1 - \sigma_2) \\ i(f_0^0\sigma_2 - \frac{1}{2}p_2^0\sigma_3) & if_+^0(\sigma_1 - \sigma_2) & if_0^0\sigma_1 \\ if_+^0(\sigma_1 - \sigma_2) & ip_2^0\sigma_3 & if_+^0(-\sigma_1 + \sigma_2) \\ if_0^0\sigma_1 & if_+^0(\sigma_1 + \sigma_2) & -i(f_0^0\sigma_2 + \frac{1}{2}p_2^0\sigma_3) \end{pmatrix}.$$

$$4\tilde{\lambda}_1^2 := -C^2\left\{\left(\frac{\vartheta_\nu(\kappa)}{\vartheta_\nu(0)}\right)^2 - \frac{\vartheta_\nu(0)}{\vartheta_\nu(\kappa)}\right\}, \quad \tilde{\rho}_0 := \frac{iC}{4\tilde{\lambda}_1}(-3\vartheta_\nu'(\kappa))\sqrt{\frac{\vartheta_\nu(0)}{\vartheta_\nu^3(\kappa)}}$$

$$f_+^0 := f_+(0), \quad f_0^0 := f_0(0), \quad p_2^0 := p_2(0).$$

In the both cases, the ADHM data exist. Therefore, from viewpoint of the Nahm data, we find that the both calorons have the instanton limits.

III-5: Monopole limit

The monopole limit is a limit that scale parameter μ_0 goes to one. It is needed that the bulk Nahm data of the caloron should coincide with the monopole Nahm data in this limit.

Nahm data of the caloron	$\mu_0 \to 1$	Nahm data of the monopole
bulk data $T_i(s)$, boundary data $W(\mu_0)$	$\longrightarrow$	Nahm data $T_i(s)$ ($W(1)$ is omitted.)

The condition of bulk Nahm data at the monopole limit

- $T_i(s)$ has simple poles at $s = \pm 1$.
- The matrix residues of (T_1, T_2, T_3) at each pole form the irreducible k-dimensional representation of su(2).

Symmetric 3-caloron

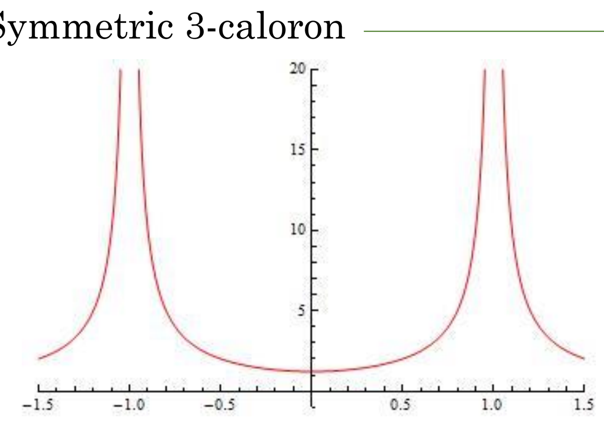
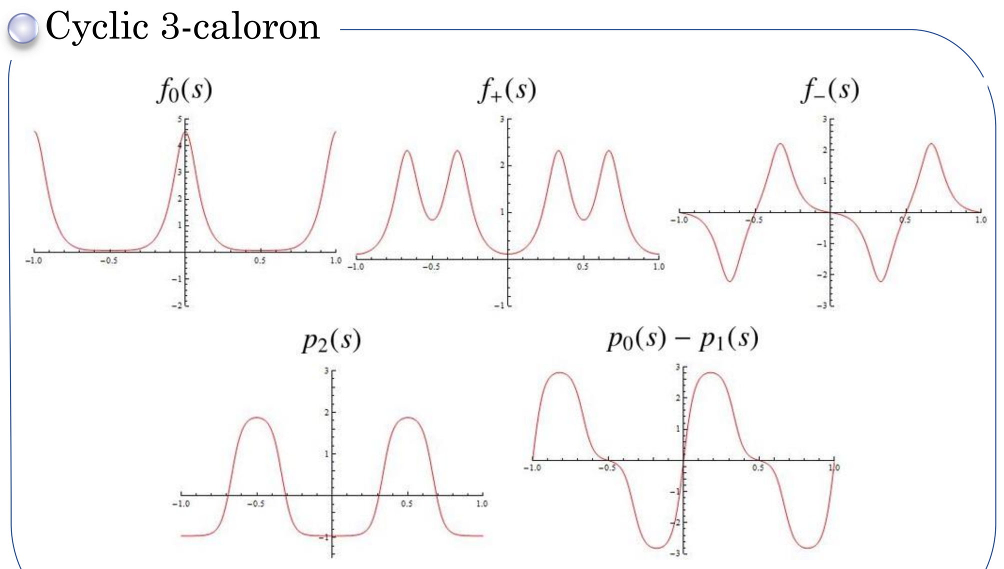


Fig: function of the symmetric 3-caloron Nahm data

Cyclic 3-caloron



Figs: functions of the cyclic 3-caloron Nahm data

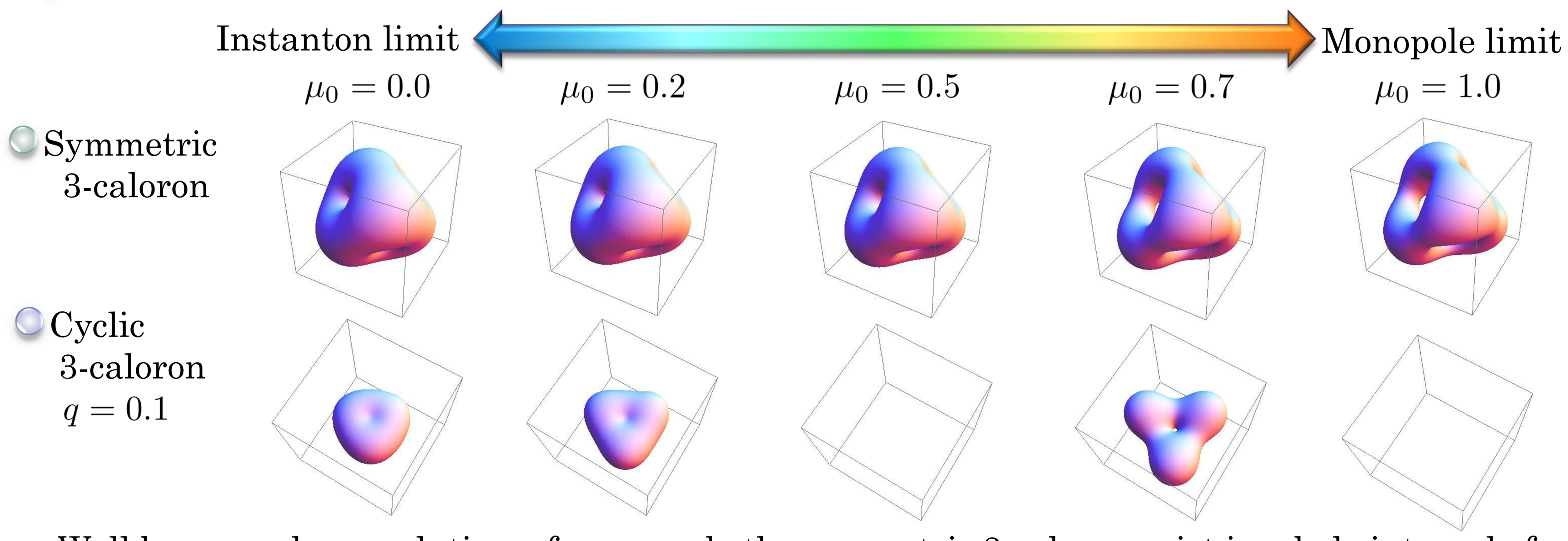
The symmetric 3-caloron Nahm data have the simple pole. So we find that the symmetric 3-caloron have the monopole limit, too.

On the other hand, one can easily see that Nahm data of cyclic 3-caloron have no pole. Thus case of the cyclic 3-caloron, we conclude that the monopole limit does not exist.

IV: ACTION DENSITIES

We will discuss the behavior of limits on the configuration space by performing the numerical Nahm transform for the Nahm data.
It is more easy and straightforward to see the existence of the limits of the solutions when we go to analysis in the configuration space, in addition to the dual space discussion.

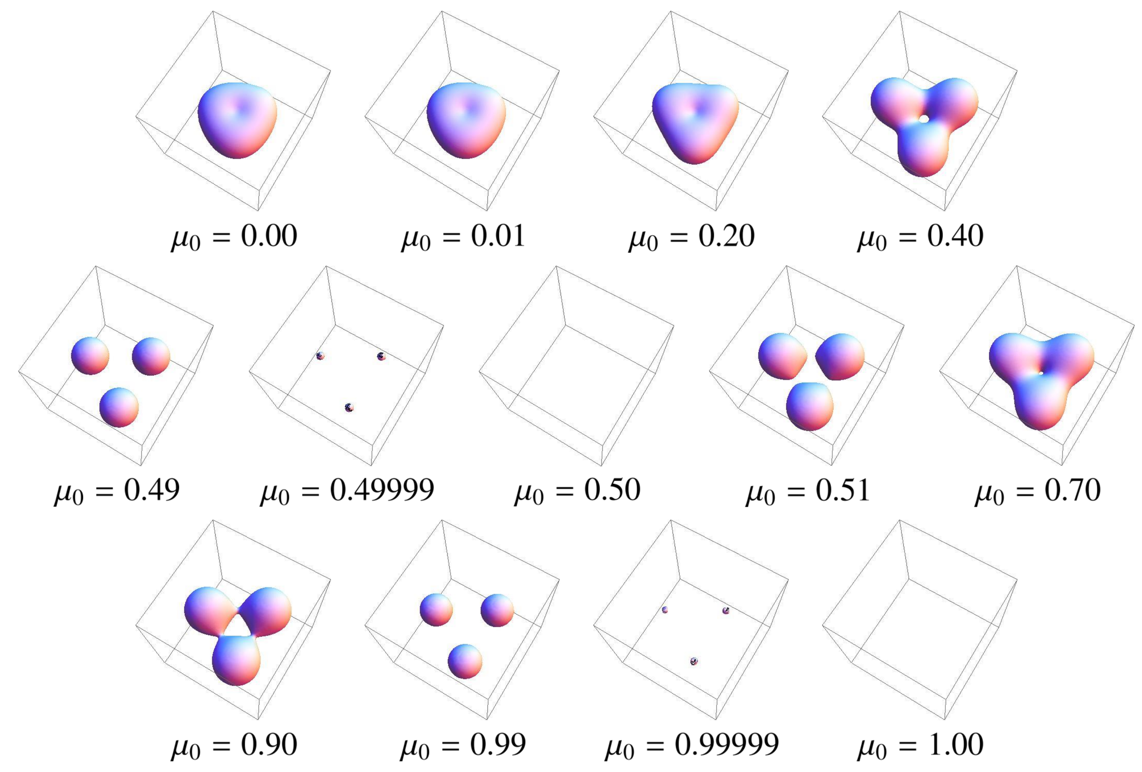
Action isosurface at $x^4 = t = 0.0$



Well known caloron solutions, for example the symmetric 3-caloron exist in whole interval of μ_0. On the other hand, the cyclic 3-caloron vanishes at μ_0 equal 0.5 and also 1.0. So this property is the notable feature of the cyclic 3-calorons.

Cyclic 3-calorons in more detail

parameter : $q = 0.1, \quad s = 7.0$

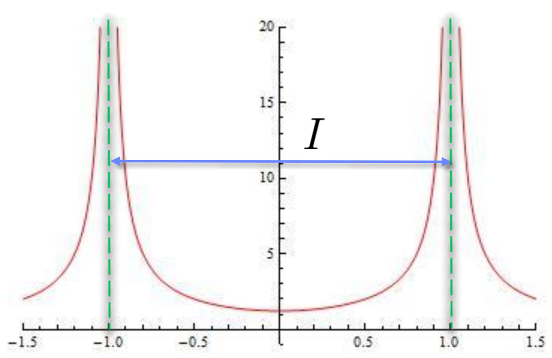


We find that the action density gradually shrinks as μ_0 grows and finally disappears at $\mu_0 = 0.5$. After passing $\mu_0 = 0.5$, it appears again and increases the size as μ_0 grows, and repeatedly it reduces and finally vanishes at $\mu_0 = 1.0$.

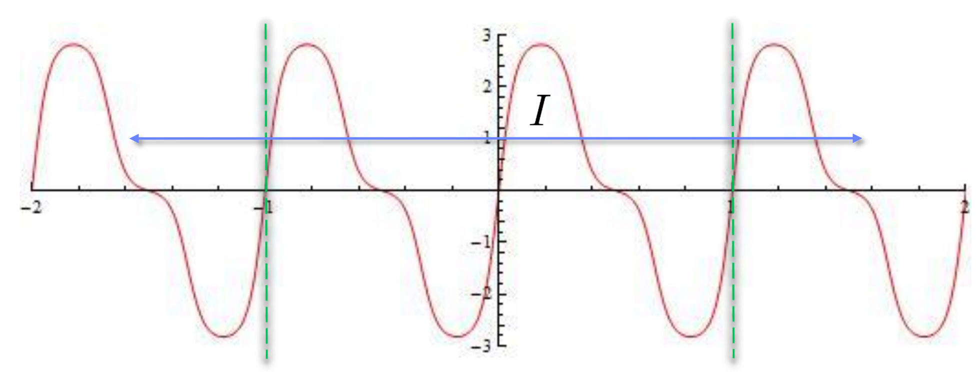
IV-1: "exterior sectors"

The range of regular on the fundamental interval $I = (-\mu_0, \mu_0)$.

Symmetric 3-caloron: $0 < \mu_0 < 1.0$



Cyclic 3-caloron: $0 < \mu_0 < \infty$



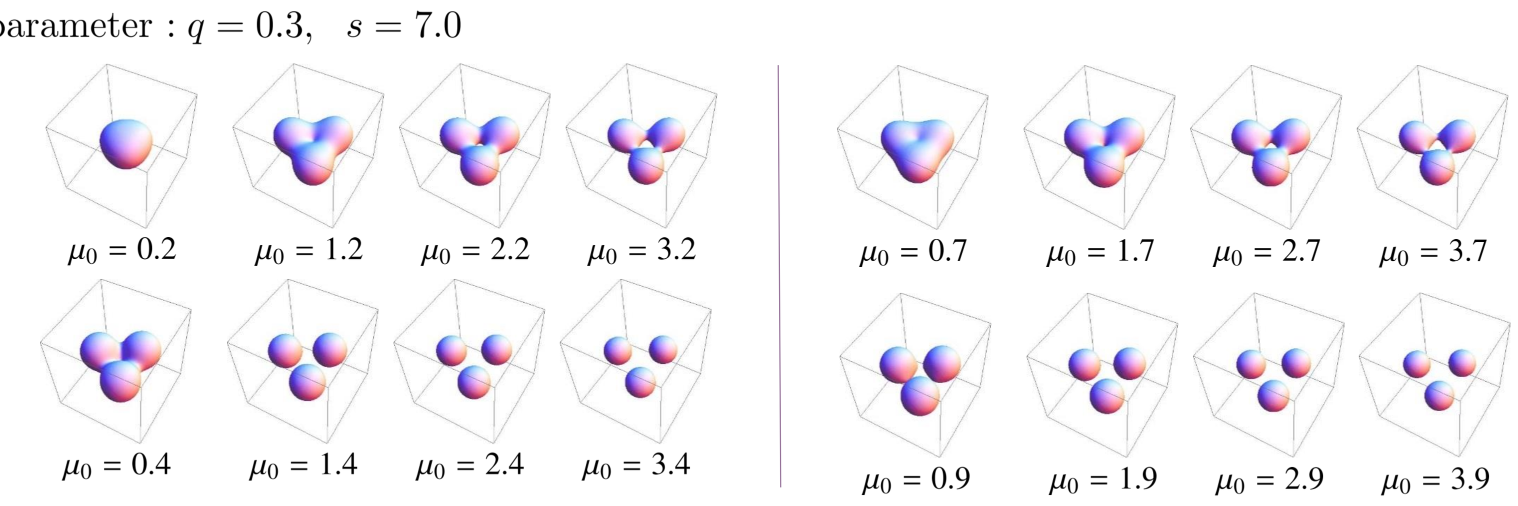
As a concrete example of which the solution has a monopole limit, first we consdier the symmetric 3-caloron. The Nahm data have simple pole at $s = \pm 1$. Hence, the scale parameter μ_0 can be taken the values in the range from zero to one. In other words, it has upper bound which corresponds to the monopole limit.
For the cyclic 3-calorons considering here, however, the Nahm data is regular in dual space from zero to infinity because they do not have the pole in whole dual space.
Consequently, μ_0 can be taken the values from zero to infinity. So we can consider an external region where the μ_0 has the value greater than 1. We call it as "exterior sectors".
Now we summarize this issue in the table.

	Bulk data with pole	Bulk data without pole
Does the caloron have the monopole limit?	Yes	No
Does the caloron have the exterior sector? $\mu_0 > 1.0$	No	Yes
Range of the scale parameter: μ_0	$\mu_0 \in (0, 1.0)$	$\mu_0 \in (0, \infty)$
Examples of the caloron	Symmetric 3-caloron etc...	Cyclic 3-caloron

This is a kind of mutation from the known Nahm data which have the monopole limit. Because the behavior of the caloron on the exterior sectors wasn't known, so it is quite important to study such a new class of solutions.
Next, we are able to discuss the behavior of the caloron on the exterior sectors in configuration space by performing the numerical Nahm transform for cyclic 3-caloron Nahm data.

IV-2: Action densities on exterior sectors

parameter : $q = 0.3, \quad s = 7.0$



- We see a periodic behavior of the isosurfaces with fixed μ_0 (without $(0.0, 0.5)$) and $\mu_0 + n, \ (n = 1, 2, \ldots)$.
- The action densities reduce gradually as n increases.
- We expect the action density tends to vanish for $\mu_0 \to \infty$.

V: SUMMARY

- We have constructed a general scheme for numerical Nahm transform of k-caloron.
- We applied the numerical Nahm transform to cyclic 3-caloron which doesn't have monopole limit and got some interesting result.
  - We showed that action density of the cyclic 3-caloron disappears at $\mu_0 = n/2$.
  - We plotted of the action density for larger values of μ_0, called "exterior sector", and found the quasi-periodic behavior of the density as μ_0 varies.