Integrable sectors of multi-vortices in the Skyrme-Faddeev type model

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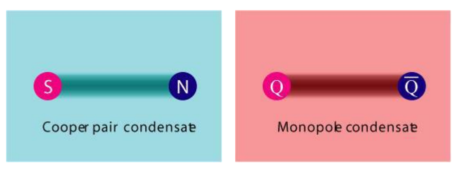
abstract

Integrable, molecular-type vortex solutions in the extended Skyrme-Faddeev (ESF) model are constructed.The solutions are a holomorphic type which satisfies the zero curvature condition and then they necessarily have an infinite number of conserved current. We propose a new potential which supports the existence of the solutions. Numerically it is checked employing the simulated annealing method.

Background

Introduction

Mechanism of the confinement ⟷ Topological excitations



Vortex in a superconductor

Katsuya Ishiguro, http://www.nt.phys.kyushu-u.ac.jp/workshop/hadron2008/pdf/ishiguro.pdf

Vortices in the extended Skyrme-Faddeev model

L.A.Ferreira, JHEP05(2009)001
L.A.Ferreira et al. Phys.Rev. D85 (2012) 105006

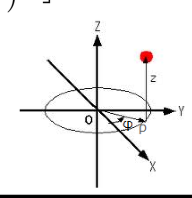
$$\mathcal{L} = 4\mathcal{M}^2 \frac{\partial_\mu u \partial^\mu u^*}{(1+|u|^2)^2} + \frac{8}{e^2}\left[\frac{(\partial_\mu u)^2(\partial_\nu u^*)^2}{(1+|u|^2)^4} + (\beta e^2 - 1)\frac{(\partial_\mu u \partial^\mu u^*)^2}{(1+|u|^2)^4}\right] - V(u,u^*)$$

I) $\beta e^2 = 1 \quad V(u,u^*) = 0$

$$(1+|u|^2)\partial^\mu \mathcal{K}_\mu - 2u^* \mathcal{K}_\mu \partial^\mu u = 0 \qquad \mathcal{K}_\mu \equiv \mathcal{M}^2 \partial_\mu u + \frac{4}{e^2}\left[\frac{(\partial_\nu u)^2 \partial_\mu u^*}{(1+|u|^2)^2}\right]$$

$(\partial_\mu u)^2 = 0$ ⇨ The equation of motion $\partial_\mu \partial^\mu u = 0$

The Ferreira's exact solution $u = \left(\frac{\rho}{a}\right)^N e^{i[N\varphi + k(t+z)]}$



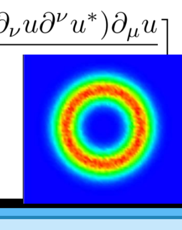
II) $\beta e^2 \neq 1 \quad V(u,u^*) = V(|u|^2)$

$$(1+|u|^2)\partial^\mu \mathcal{K}_\mu - 2u^* \mathcal{K}_\mu \partial^\mu u = -\frac{1}{4}(1+|u|^2)^3 \frac{\partial V}{\partial u^*}$$

$$\mathcal{K}_\mu \equiv \mathcal{M}^2 \partial_\mu u + \frac{4}{e^2}\left[\frac{(\partial_\nu u)^2 \partial_\mu u^* + (\beta e^2 - 1)(\partial_\nu u \partial^\nu u^*)\partial_\mu u}{(1+|u|^2)^2}\right]$$

$(\partial_\mu u)^2 = 0$

The potential of the Ferreira's solution $V = \frac{\lambda}{(1+|u|^2)^4}|u|^{2-\frac{2}{N}}$



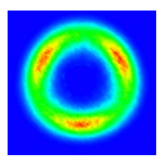
O(3)($\boldsymbol{CP^1}$) extended Skyrme-Faddeev model

Motivation

We constructed multi-vortices solutions which satisfy zero curvature condition.

Some features of the solutions

- They satisfy zero curvature condition.
- They have multi-centered structure.



How to construct and check these solution.

- We chose the suitable form of potential.
- We checked the stability of these solution numerically.

Step1:
- We construct a N-centered solution which satisfy the zero curvature condition.

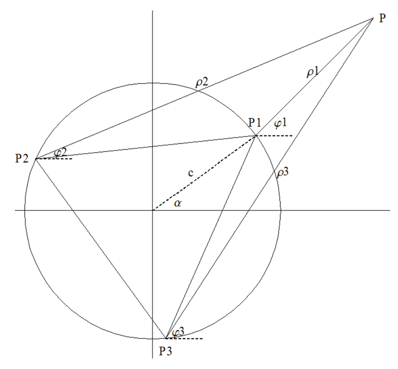
Step2
- Substituting the solution into the equation of motion, we obtain the derivative of potential (①).

Step3:
- We rewrite the derivative of potential in terms of the field.

Step4:
- We assume a candidate of the potential for field and compute its derivative (②).

Step5
- We compare the results of ① and ②, we find a form of the potential.

$$u_1 = \left(\frac{\rho}{a}\right) e^{i[\varphi+k(t+z)]}$$
$$u_2 = \left(\frac{\rho}{a}\right)^2 e^{i[2\varphi+k(t+z)]} - \left(\frac{c}{a}\right)^2 e^{i[2\alpha+k(t+z)]}$$
$$u_3 = \left(\frac{\rho}{a}\right)^3 e^{i[3\varphi+k(t+z)]} - \left(\frac{c}{a}\right)^3 e^{i[3\alpha+k(t+z)]}$$
$$\vdots$$
$$u_N = \left(\frac{\rho}{a}\right)^N e^{i[N\varphi+k(t+z)]} - \left(\frac{c}{a}\right)^N e^{i[N\alpha+k(t+z)]}$$



These ansatz satisfy following conditions $(\partial_\mu u)^2 = 0 \quad \partial_\mu \partial^\mu u \neq 0$

$$(1+|u_N|^2)\partial^\mu \mathcal{K}_\mu - 2u^* \mathcal{K}_\mu \partial^\mu u_N = -\frac{1}{4}(1+|u_N|^2)^3 \frac{\partial V}{\partial u_N^*}$$

$$\mathcal{K}_\mu \equiv \mathcal{M}^2 \partial_\mu u_N + \frac{4}{e^2}\left[\frac{(\partial_\mu u_N)^2 \partial_\mu u_N^* + (\beta e^2 - 1)(\partial_\nu u_N \partial^\nu u_N^*)\partial_\mu u_N}{(1+|u_N|^2)^2}\right]$$

$$\boxed{(\partial_\mu u_N)^2 = 0 \qquad \partial_\mu \partial^\mu u_N = 0}$$

$$\frac{\partial V}{\partial u_N^*} = \frac{1}{(1+|u_N|^2)^2}\partial^\mu\left[\frac{16(\beta e^2 - 1)(\partial_\nu u_N \partial^\nu u_N^*)\partial_\mu u_N}{e^2(1+|u_N|^2)^2}\right]$$

$$\frac{1}{(1+|u_N|^2)^2}\partial^\mu\left[\frac{16(\beta e^2 - 1)(\partial_\nu u_N \partial^\nu u_N^*)\partial_\mu u_N}{e^2(1+|u_N|^2)^2}\right]$$

$$= -\frac{64N^3(\beta e^2-1)\left(\frac{\rho}{a}\right)^N \left(\left(\frac{\rho}{a}\right)^N e^{-iN\varphi}\right)^{1-\frac{2}{N}} \left(\left(\frac{\rho}{a}\right)^N e^{iN\varphi}\right)^{2-\frac{2}{N}}}{a^4 e^2 \left(1 + \left(\left(\frac{\rho}{a}\right)^N e^{-iN\varphi} - \left(\frac{c}{a}\right)^N\right)\left(\left(\frac{\rho}{a}\right)^N e^{iN\varphi} - \left(\frac{c}{a}\right)^N\right)\right)^5} \left((N-1)e^{2iN\varphi}\left(\frac{c}{a}\right)^N\left(\frac{\rho}{a}\right)^N + e^{iN\varphi}\left(N\left(-\left(\frac{c}{a}\right)^{2N} + \left(\frac{\rho}{a}\right)^{2N} - 1\right) + \left(\frac{c}{a}\right)^{2N} + \left(\frac{\rho}{a}\right)^{2N} + 1\right) - (N+1)\left(\frac{c}{a}\right)^N\left(\frac{\rho}{a}\right)^N\right)$$

(Example that we rewrite the derivative of potential in terms of the field.)

$$\left(\frac{\rho}{a}\right)^N e^{-iN\varphi} \to \left(\left(\frac{c}{a}\right)^N + u_N^*\right)e^{k(z+t)} \qquad \left(\frac{\rho}{a}\right)^N e^{iN\varphi} \to \left(\left(\frac{c}{a}\right)^N + u_N\right)e^{-k(z+t)} \qquad \left(1 + \left(\left(\frac{\rho}{a}\right)^N e^{-iN\varphi} - \left(\frac{c}{a}\right)^N\right)\left(\left(\frac{\rho}{a}\right)^N e^{iN\varphi} - \left(\frac{c}{a}\right)^N\right)\right)^5 \to (1+|u_N|^2)^5$$

$$= -\frac{64(\beta e^2-1)N^3\left(\left(\frac{c}{a}\right)^N + u_N^*\right)^{1-\frac{2}{N}}\left(\left(\frac{c}{a}\right)^N + u_N\right)^{2-\frac{2}{N}}\left(N\left(u_N\left(2\left(\frac{c}{a}\right)^N + u_N^*\right) - 1\right) + u_N^* u_N + 1\right)}{a^4 e^2 (1+|u_N|^2)^5}$$

$$\frac{\partial V_N}{\partial u_N^*} = -\frac{64(\beta e^2-1)N^3\left(\left(\frac{c}{a}\right)^N + u_N^*\right)^{1-\frac{2}{N}}\left(\left(\frac{c}{a}\right)^N + u_N\right)^{2-\frac{2}{N}}\left(N\left(u_N\left(2\left(\frac{c}{a}\right)^N + u_N^*\right) - 1\right) + u_N^* u_N + 1\right)}{a^4 e^2 (1+|u_N|^2)^5}$$

So, we set following the form of the potential,

$$V_N \sim \frac{\lambda\left\{u_N + \tilde{c}^{\,N} e^{i[N\alpha + k(t+z)]}\right\}^{2-2/N}\left\{u_N^* + \tilde{c}^{\,N} e^{-i[N\alpha + k(t+z)]}\right\}^{2-2/N}}{(1+|u_N|^2)^4}$$

$$\frac{1}{(1+|u_N|^2)^2}\partial^\mu\left[\frac{16(\beta e^2 - 1)(\partial_\nu u_N \partial^\nu u_N^*)\partial_\mu u_N}{e^2(1+|u_N|^2)^2}\right]$$

$$= -\frac{64N^3(\beta e^2-1)\left(\frac{\rho}{a}\right)^N\left(e^{-iN\varphi}\left(\frac{\rho}{a}\right)^N\right)^{\frac{N-2}{N}}\left(e^{iN\varphi}\left(\frac{\rho}{a}\right)^N\right)^{\frac{N-2}{N}}\left((N-1)e^{2iN\varphi}\left(\frac{c}{a}\right)^N\left(\frac{\rho}{a}\right)^N + e^{iN\varphi}\left(N\left(-\left(\frac{c}{a}\right)^{2N} + \left(\frac{\rho}{a}\right)^{2N} - 1\right) + \left(\frac{c}{a}\right)^{2AN} + \left(\frac{\rho}{a}\right)^{2N} + 1\right) - (N+1)\left(\frac{c}{a}\right)^N\left(\frac{\rho}{a}\right)^N\right)}{a^4 e^2\left(1 + \left(-\left(\frac{c}{a}\right)^N + e^{-iN\varphi}\left(\frac{\rho}{a}\right)^N\right)\left(-\left(\frac{c}{a}\right)^N + e^{iN\varphi}\left(\frac{\rho}{a}\right)^N\right)\right)^5}$$

$$\frac{\partial V_N}{\partial u_N^*} = \frac{2\lambda\left(\left(\frac{c}{a}\right)^N + u_N^*\right)^{1-\frac{2}{N}}\left(\left(\frac{c}{a}\right)^N + u_N\right)^{2-\frac{2}{N}}\left(N\left(u_N\left(2\left(\frac{c}{a}\right)^N + u_N^*\right) - 1\right) + |u_N|^2 + 1\right)}{(1+|u_N|^2)^5}$$

$$= \frac{2\lambda\left(\frac{\rho}{a}\right)^N\left(e^{-iN\varphi}\left(\frac{\rho}{a}\right)^N\right)^{\frac{N-2}{N}}\left(e^{iN\varphi}\left(\frac{\rho}{a}\right)^N\right)^{\frac{N-2}{N}}\left((N-1)e^{2iN\varphi}\left(\frac{c}{a}\right)^N\left(\frac{\rho}{a}\right)^N + e^{iN\varphi}\left(N\left(-\left(\frac{c}{a}\right)^{2N} + \left(\frac{\rho}{a}\right)^{2N} - 1\right) + \left(\frac{c}{a}\right)^{2N} + \left(\frac{\rho}{a}\right)^{2N} + 1\right) - (N+1)\left(\frac{c}{a}\right)^N\left(\frac{\rho}{a}\right)^N\right)}{N\left(1 + \left(-\left(\frac{c}{a}\right)^N + e^{-iN\varphi}\left(\frac{\rho}{a}\right)^N\right)\left(-\left(\frac{c}{a}\right)^N + e^{iN\varphi}\left(\frac{\rho}{a}\right)^N\right)\right)^5}$$

$$\lambda = \frac{-32N^4(\beta e^2 - 1)}{a^4 e^2}$$

Numerical study

$$\mathcal{H} = \frac{4\mathcal{M}^2}{(1+|u|^2)^2}(|\dot{u}|^2 + \vec{\nabla}u\cdot\vec{\nabla}u^*) + \frac{16}{e^2(1+|u|^2)^4}\left[\dot{u}^2(\dot{u^*} - \vec{\nabla}u^*)^2 + \dot{u^*}^2(\dot{u} - \vec{\nabla}u)^2 - \frac{1}{2}(\dot{u} - \vec{\nabla}u)^2(\dot{u^*} - \vec{\nabla}u^*)^2\right]$$
$$+ \frac{24(\beta e^2 - 1)}{e^2(1+|u|^2)^4}\left(|\dot{u}|^2 - \vec{\nabla}u\cdot\vec{\nabla}u^*\right)\left(|\dot{u}|^2 + \frac{1}{3}\vec{\nabla}u\cdot\vec{\nabla}u^*\right) + V_N$$

Ansatz for the simulation $u = f_{(x,y)}\exp[i\Theta_{(x,y)}]\exp[ik(z+t)]$

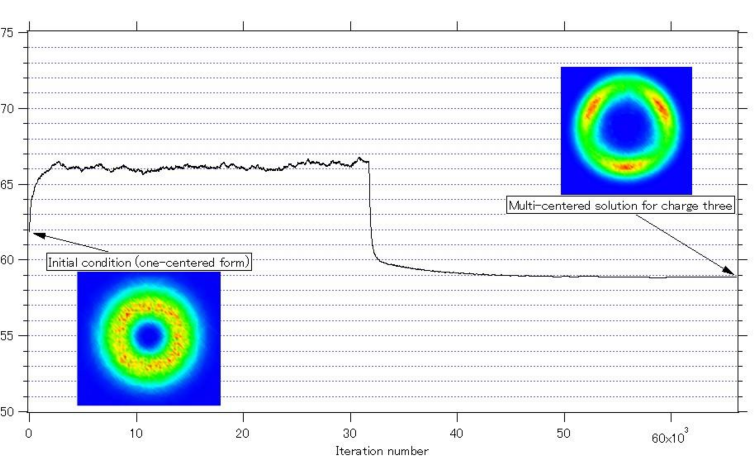
The energy density

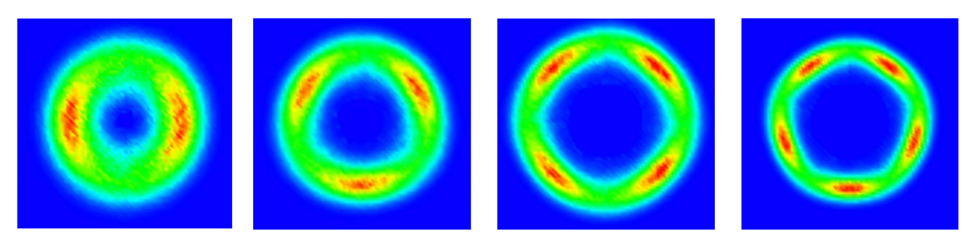
$$\mathcal{E}(x,y) = 4\mathcal{M}^2\frac{(2k^2|u|^2 + \partial_x u\partial_x u^* + \partial_y u \partial_y u^*)}{(1+|u|^2)^2} - \frac{24(\beta e^2 - 1)}{e^2}\frac{(\partial_x u \partial_x u^* + \partial_y u \partial_y u^*)\left(\frac{4}{3}k^2|u|^2 + \frac{1}{3}(\partial_x u \partial_x u^* + \partial_y u \partial_y u^*)\right)}{(1+|u|^2)^4}$$
$$- \frac{8}{e^2}\frac{\left((\partial_x u)^2 + (\partial_y u)^2\right)\left((\partial_x u^*)^2 + (\partial_y u^*)^2\right) - 2k^2u^2\left((\partial_x u^*)^2 + (\partial_y u^*)^2\right) + k^2 u^{*2}\left((\partial_x u)^2 + (\partial_y u)^2\right)}{(1+|u|^2)^4} + V_N(x,y)$$

$$V_N = \frac{\lambda\left\{u_N + \left(\frac{c}{a}\right)^N e^{i[N\alpha]}\right\}^{2-2/N}\left\{u_N^* + \left(\frac{c}{a}\right)^N e^{-i[N\alpha]}\right\}^{2-2/N}}{(1+|u_N|^2)^4}$$

Results

The energy density





$N=2$ $N=3$ $N=4$ $N=5$

$c/a = 1.0, \beta = -2.0, e^2 = -1.0, \mathcal{M} = 1.0, k = 0$

N.Sawado and Y.Tamaki, arXiv:1309.6004 [hep-th].

$\boldsymbol{CP^2}$ Skyrme-Faddeev model

L.A. Ferreira,P.Klimas,JHEP1007.1667(2010)
P.Klimas,N.Sawado,arXiV:1210.7523

Lagrangian for the $\boldsymbol{CP^2}$ model

$$\mathcal{L} = -\frac{1}{2}\left[M^2\eta_{\mu\nu} + C_{\mu\nu}\right]\tau^{\nu\mu} - \mu^2 V$$

$$C_{\mu\nu} \equiv M^2\eta_{\mu\nu} - \frac{4}{e^2}\left[(\beta e^2 - 1)\tau_\rho^\rho \eta_{\mu\nu} + (\gamma e^2 - 1)\tau_{\mu\nu} + (\gamma e^2 + 2)\tau_{\nu\mu}\right]$$
$$\tau_{\mu\nu} \equiv -\frac{4}{\theta^4}\left[\theta^2 \partial_\nu u^\dagger \cdot \partial_\mu u - (\partial_\nu u^\dagger \cdot u)(u^\dagger \cdot \partial_\mu u)\right]$$

The equations

$$(1 + u^\dagger \cdot u)\partial^\mu(C_{\mu\nu}\partial^\nu u_i) - C_{\mu\nu}\left[(u^\dagger \cdot \partial^\mu u)\partial^\nu u_i + (u^\dagger \cdot \partial^\nu u)\partial^\mu u_i\right] + \mu^2 \frac{u_i}{4}(1 + u^\dagger \cdot u)^2\left[\frac{\delta V}{\delta |u_i|^2} + \sum_{k=1}^{N}|u_k|^2 \frac{\delta V}{\delta |u_k|^2}\right] = 0$$

A vortex solution

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} c_1\left(\frac{\rho}{a}\right)^{n_1} e^{in_1\varphi}e^{ik_1(z+t)} \\ c_2\left(\frac{\rho}{a}\right)^{n_2} e^{in_2\varphi}e^{ik_2(z+t)} \end{pmatrix}, u^* = \begin{pmatrix} u_1^* \\ u_2^* \end{pmatrix} = \begin{pmatrix} c_1\left(\frac{\rho}{a}\right)^{n_1} e^{-in_1\varphi}e^{-ik_1(z+t)} \\ c_2\left(\frac{\rho}{a}\right)^{n_2} e^{-in_2\varphi}e^{-ik_2(z+t)} \end{pmatrix}$$

Zero curvature condition for the $\boldsymbol{CP^2}$ model

$$\partial_\mu u_i \partial^\mu u_j = 0 \qquad \text{for any } i,j = 1,2$$

where the coupling constants satisfy following condition
$$\beta e^2 + \gamma e^2 = 2.$$

Ferreira and Klimas set the vortex solution which satisfy zero curvature condition .For a constraint of the model parameter $\beta e^2 + \gamma e^2 = 2$,these solutions exist.
For $\beta e^2 + \gamma e^2 \neq 2$, special form of the potential are employed for the stabirity of these solutions.

A special case: $(n_1, n_2) = (n, 0) \quad \beta e^2 + \gamma e^2 \neq 2$

$$V = -\frac{128n^2|u_1|^4\left(\beta e^2 + \gamma e^2 - 2\right)c_1^{4/n}c_2^{\frac{4}{n}-4}\left(c_2^2\left(2c_2^2 - n + 1\right) - |u_2|^2\left(c_2^2 n + 2c_2^2 + 1\right)\right)^2\left(|u_1|^2|u_2|^2\right)^{-\frac{1}{n}}}{e^2(|u_1|^2 + |u_2|^2 + 1)^4}$$

A special case: $(n_1, n_2) = (n, n) \quad \beta e^2 + \gamma e^2 \neq 2$

$$V = -\frac{128\left(\beta e^2 + \gamma e^2 - 2\right)c_1^{2/n}c_2^{2/n}\left(|u_1|^2|u_2|^2\right)^{-\frac{2}{n_1+n_2}}\left(n_1^2|u_1|^2(|u_2|^2+1) - 2n_1n_2|u_1|^2|u_2|^2 + n_2^2(|u_1|^2+1)|u_2|^2\right)^2}{e^2(|u_1|^2 + |u_2|^2 + 1)^4}$$

Conclusion and outlook

- We introduced one-centered vortex solution in the ESF model.
- We found forms of the potential for our ansatz of the multi-vortices solutions in the ESF model.
- We confirmed how the potentials work by examining the field relaxations which coincide with the assumed analytical solutions.
- We found forms of the potential for vortex ansatz for the special case $(n_1, n_2) = (n, 0)$ and $(n_1, n_2) = (n, n)$.
- Our scheme is quite general and is easily applicable to the related solitonic models such as the baby-Skyrme model, the Skyrme model and so on.

Thank you!!