Quantized states of vertex in a CP^2 Skyrme-Faddeev type model

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Introduction

In this presentation we consider quantization of the vortex in the CP^2 Skyrme-Faddeev model.

- ✓ It has been conjectured that the model can be seen as a low-energy effective classical model of $SU(3)$ Yang-Mills theory. And it has also several physical applications in condensed matter physics.
- ✓ For phenomenological point of view, it is worth to study several quantum excitations concerning with the spin, angular momentum, so on.
- ✓ There are two methods for quantizing solitons, the semiclassical and canonical quantization.
  - ・the semiclassical approach: solitons are considered as a classical rigid rotator and the angular momenta are quantized in Bohr-Sommerfeld framework.
  - ・the canonical approach: in addition to the semiclassecal procedure the canonical commutation relation is taken into account. Then new quantum correction to the mass spectrum appear.

By employing these two methods we derive the formula of mass spectrum of the quantized vortex.

The CP^N Skyrme-Faddeev model and vortex solutions

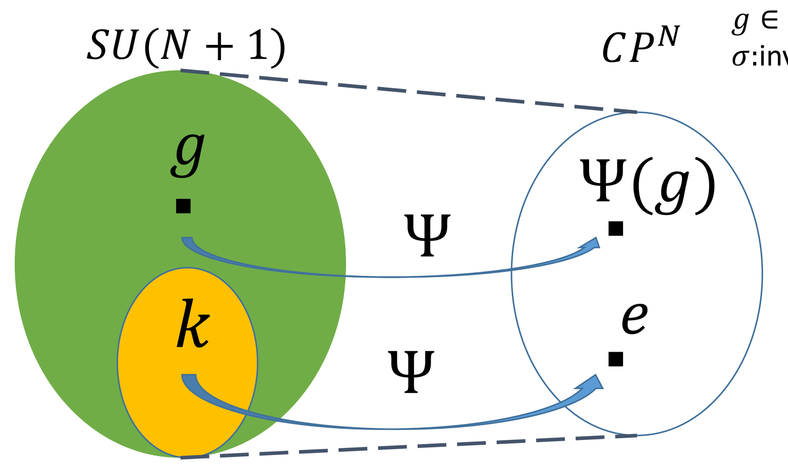
The space $CP^N = \frac{SU(N+1)}{SU(N)\times U(1)}$ can be parametrized by ***Principal variable*** $\Psi(g) \equiv g\sigma(g)^{-1}$

$\sigma(k) = k, \quad k \in SU(N)\otimes U(1)$

$\Psi(k) = k\sigma(k)^{-1} = kk^{-1} = e$

$$\Psi(gk) = gk\sigma(gk)^{-1} = gk\sigma(k^{-1}g^{-1}) = gk\sigma(k^{-1})\sigma(g^{-1}) = g\sigma(g)^{-1} = \Psi(g)$$

$g \in SU(N+1)$
σ:involutive automorphism



The Lagrangian density

$$\mathcal{L} = -\frac{M^2}{2}\mathrm{Tr}(\Psi^{-1}\partial_\mu\Psi)^2 + \frac{1}{e^2}\mathrm{Tr}\left([\Psi^{-1}\partial_\mu\Psi, \Psi^{-1}\partial_\nu\Psi]\right)^2 + \frac{\beta}{2}\left[\mathrm{Tr}\left(\Psi^{-1}\partial_\mu\Psi\right)^2\right]^2 + \gamma\left[\mathrm{Tr}(\Psi^{-1}\partial_\mu\Psi\,\Psi^{-1}\partial_\nu\Psi)\right]^2 - \mu^2 V$$

In $(N+1)$-dim representation of $SU(N+1)$, σ and g are defined as

$$\sigma(T) \equiv \begin{pmatrix}\mathbb{1}_{N\times N} & 0\\ 0 & -1\end{pmatrix} T \begin{pmatrix}\mathbb{1}_{N\times N} & 0\\ 0 & -1\end{pmatrix}, \quad g \equiv \frac{1}{\sqrt{1+\boldsymbol{u}^\dagger\cdot\boldsymbol{u}}}\begin{pmatrix}\Delta & i\boldsymbol{u}\\ i\boldsymbol{u}^\dagger & 1\end{pmatrix}$$

where $\boldsymbol{u}$ is N-dim complex field and $\Delta_{ij} = \sqrt{1+\boldsymbol{u}^\dagger\cdot\boldsymbol{u}}\,\delta_{ij} - \frac{u_i u_j^*}{1+\sqrt{1+\boldsymbol{u}^\dagger\cdot\boldsymbol{u}}}$

$$\Psi = \begin{pmatrix}\mathbb{1}_{N\times N} & 0\\ 0 & -1\end{pmatrix} + \frac{2}{1+\boldsymbol{u}^\dagger\cdot\boldsymbol{u}}\begin{pmatrix}-\boldsymbol{u}\otimes\boldsymbol{u}^\dagger & i\boldsymbol{u}\\ i\boldsymbol{u}^\dagger & 1\end{pmatrix}$$

Note that the Lagrangian has a global symmetry $\Psi \to A\Psi B^\dagger$ where $A, B \in SU(N+1)$.
On account of this symmetry we can translate the variable into Hermitian such as

$$X := \begin{pmatrix}\mathbb{1}_{N\times N} & 0\\ 0 & 1\end{pmatrix} + \frac{2}{1+\boldsymbol{u}^\dagger\cdot\boldsymbol{u}}\begin{pmatrix}-\boldsymbol{u}\otimes\boldsymbol{u}^\dagger & i\boldsymbol{u}\\ -i\boldsymbol{u}^\dagger & -1\end{pmatrix}$$

X makes the manipulation of quantization much easier

We introduce dimensionless cylindrical coordinates (t, ρ, φ, z)

$$x^0 = r_0 t,\ x^1 = r_0\,\rho\cos\varphi,\ x^2 = r_0\rho\sin\varphi,\ x^3 = r_0 z \quad \text{where} \quad r_0^2 = -\frac{4}{M^2e^2}$$

and adopt an axial symmetric ansatz $u_j = f_j(\rho)e^{in_j\varphi}$.

Classical solutions

(i) Integrable sector

- Zero curvature condition $\quad \partial_\mu u_i \partial^\mu u_j = 0$
- Constraints for parameters $\quad \beta e^2 + \gamma e^2 = 2,\ \mu^2 = 0$

➡ The scale invariant solution $u_j = c_j\rho^{n_j}e^{in_j\varphi}$

Since the solutions satisfy the zero curvature condition,
they possess infinite number of conserved currents!

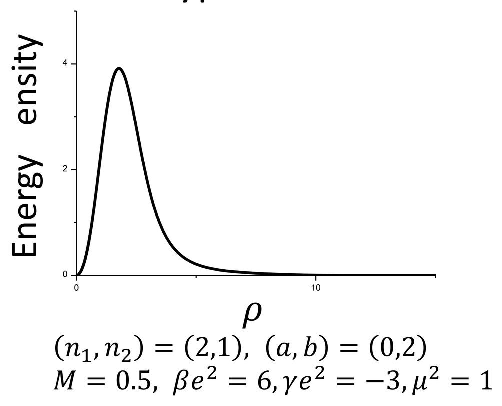
(ii) Outside the integrable sector

In order to break scale invariance, we introduce a potential term

$$V \propto \underbrace{\mathrm{Tr}(I - X_0^{-1}X)^a}_{\text{new-baby type}}\ \overbrace{\mathrm{Tr}(I - X_\infty^{-1}X)^b}^{\text{old-baby type}} \quad a \geq 0, b > 0$$

$$X_0 \equiv X|_{\rho=0} \quad X_\infty \equiv X|_{\rho\to\infty}$$

The typical solution



$(n_1, n_2) = (2,1),\ (a,b) = (0,2)$
$M = 0.5,\ \beta e^2 = 6, \gamma e^2 = -3, \mu^2 = 1$

Topological charge $\quad Q_{\mathrm{top}} = \frac{1}{8\pi}\int d^2x\varepsilon_{ij}\mathrm{Tr}[X\partial_i X\partial_j X] = \left[\frac{\sum_{k=1}^N n_k f_k^2}{1+\sum_{k=1}^N f_k^2}\right]_0^\infty = n_{\max} + |n_{\min}|$

$n_{\max}$: the highest positive integer in the set n_j $\quad n_{\min}$: the lowest negative integer in the same set

Semiclassical quantization

We shall quantize rotational zeromodes of the vortex in the CP^2 model by applying the standard collective coordinate quantization method.

For standard Hamiltonian (quadratic in time derivatives), we set $\beta e^2 + 2\gamma e^2 = 0$.

Symmetry of classical Lagrangian $\quad X(\boldsymbol{r}) \to AX(\boldsymbol{r})B^\dagger,\ \ A, B \in SU(3)$

This symmetry is spontaneously broken. Therefore we have to extract the proper rotational degree of freedom. The rotational matrix A and B which correspond to symmetry of the solution should satisfy

$$\begin{cases} Q_{\mathrm{top}} = Q'_{\mathrm{top}} \equiv \frac{1}{8\pi}\int d^2x\varepsilon_{ij}\mathrm{Tr}[AXB^\dagger A\partial_i XB^\dagger A\partial_j XB^\dagger] \\ AX_\infty B^\dagger = X_\infty \end{cases}$$

X_∞ depends on the combination of winding numbers (n_1, n_2). Therefore we need case analysis. In this presentation, however, we concentrate on the case $\{n_1 > 0\ \cap n_1 > n_2\}$. From the conditions, one can find

$$A = B = e^{-i\lambda_u\alpha_1/2}e^{-i\lambda_7\alpha_2/2}e^{-i\lambda_u\alpha_3/2}e^{-i\lambda_v\alpha_4/2} \qquad \lambda_u = -\frac{1}{2}(\lambda_3 - \sqrt{3}\lambda_8),\quad \lambda_v = -\left(\lambda_3 + \frac{\lambda_8}{\sqrt{3}}\right)$$

Commutation relations of the generators

$$[\lambda_i, \lambda_j] = 2\epsilon_{ijk}\lambda_k \quad [\lambda_i, \lambda_v] = 0 \qquad i,j,k = 6,7,\mathrm{u}$$

➡ $A = B \in SU(2)\times U(1)$

In order to remove degeneracy of the classical configuration, we consider time dependent rotation.

Dynamical ansatz $\quad X(\boldsymbol{r}; A(t)) = A(t)X(\boldsymbol{r})A^\dagger(t),$

The angular velocities $\quad A^\dagger\dot{A} = -\frac{i}{2}\lambda_P\Omega^P \qquad \dot{A} = \frac{1}{r_0}\frac{\partial A}{\partial t}$

The effective Lagrangian

$$L_{\mathrm{eff}} = \frac{1}{2}I_{PQ}\Omega^P\Omega^Q - M_{\mathrm{cl}} \qquad P, Q = 6,7,\mathrm{u},\mathrm{v}.$$

The inertia tensor

$$I_{PQ} = \frac{2\pi}{e^2}\int \rho\mathrm{d}\rho\left[\mathrm{Tr}([\lambda_P, X][\lambda_Q, X]) + \mathrm{Tr}\left([[\lambda_P, X], \partial_k X]\left[[\lambda_Q, X], \partial_k X\right]\right) + \frac{\beta e^2}{2}\left\{\mathrm{Tr}([\lambda_P, X][\lambda_Q, X])\mathrm{Tr}(\partial_k X\partial_k X) - 2\mathrm{Tr}([\lambda_P, X]\partial_k X)\mathrm{Tr}([\lambda_Q, X]\partial_k X)\right\}\right]$$

By virtue of the axial symmetry of the ansatz, several components have notable feature such as $I_{66} = I_{77}$, $I_{\mathrm{uv}} = I_{\mathrm{vu}}$, and off diagonal components vanish except I_{uv}.

Legendre transf. $H_{\mathrm{q}} = \mathcal{J}_P\Omega^P - L_{\mathrm{eff}} \quad \mathcal{J}_P = \frac{\partial L_{\mathrm{eff}}}{\partial\Omega^P}$

The quantum Hamiltonian

$$H_{\mathrm{q}} = M_{\mathrm{cl}} + \frac{1}{2I_{66}}(\mathcal{J}_6^2 + \mathcal{J}_7^2) + \frac{1}{I_{\mathrm{uu}}I_{\mathrm{vv}} - I_{\mathrm{uv}}^2}\left(\frac{I_{\mathrm{uu}}}{2}\mathcal{J}_{\mathrm{v}}^2 - I_{\mathrm{uv}}\mathcal{J}_{\mathrm{u}}\mathcal{J}_{\mathrm{v}} + \frac{I_{\mathrm{vv}}}{2}\mathcal{J}_{\mathrm{u}}^2\right)$$

We promote the angular momenta to operators defined as $\quad [\mathcal{J}_P, A] = -\frac{1}{2}\lambda_P A$

$$\mathcal{J}_6 = i\left(\cos\alpha_1\cot\alpha_2\frac{\partial}{\partial\alpha_1} + \sin\alpha_1\frac{\partial}{\partial\alpha_2} - \frac{\cos\alpha_1}{\sin\alpha_2}\frac{\partial}{\partial\alpha_3}\right), \quad \mathcal{J}_{\mathrm{u}} = -i\frac{\partial}{\partial\alpha_1}$$

$$\mathcal{J}_7 = i\left(\sin\alpha_1\cot\alpha_2\frac{\partial}{\partial\alpha_1} - \cos\alpha_1\frac{\partial}{\partial\alpha_2} - \frac{\sin\alpha_1}{\sin\alpha_2}\frac{\partial}{\partial\alpha_3}\right), \quad \mathcal{J}_{\mathrm{v}} = -i\frac{\partial}{\partial\alpha_4}$$

From this definition we find the set $\{\mathcal{J}_{\mathrm{u}}, \mathcal{J}_{\mathrm{v}}, \mathcal{J}_6^2 + \mathcal{J}_7^2 + \mathcal{J}_{\mathrm{u}}^2\}$ are simultaneously diagonalizable.

The eigenfunction $\quad \psi_{m,k,Y}^j \propto \mathcal{D}_{m\,k}^j(\alpha_1, \alpha_2, \alpha_3)e^{iY\alpha_4}$

The mass spectrum

$$E = M_{\mathrm{cl}} + \frac{1}{2I_{66}}\{j(j+1) - m^2\} + \frac{1}{I_{\mathrm{uu}}I_{\mathrm{vv}} - I_{\mathrm{uv}}^2}\left(\frac{I_{\mathrm{uu}}}{2}m^2 - I_{\mathrm{uv}}mY + \frac{I_{\mathrm{vv}}}{2}Y^2\right)$$

Canonical quantization

=**Semiclassical quantization + the commutation relation** $[\dot{\alpha}^a, \alpha^b] = -if^{ab}(\alpha)$ $\quad \alpha$:Euler angles. $a, b = 1\sim4$.

We consider the problem in quantum mechanical way *ab initio*, of which we properly treat the commutation relation of collective variables. Non-zero value of the variables induces a Goldstone boson which was absent in the previous semiclassical analysis.

Main points of the modification

	Semiclassical	***Canonical***
Ω^P	$\dot{\alpha}^a C_a^P(\alpha)$	$\{\dot{\alpha}^a, C_a^P(\alpha)\}/2$
$\dot{A}$	$\dot{\alpha}^a\partial_a A(\alpha)$	$\{\dot{\alpha}^a, \partial_a A(\alpha)\}/2$
$A^\dagger\dot{A}$	$\frac{i}{2}\lambda_P\Omega^P$	$\frac{i}{2}\lambda_P\Omega^P + \frac{i}{8}g^{PQ}\lambda_P\lambda_Q$

where $g^{PQ} = f^{ab}C_a^P C_b^Q$,

$$g^{66} = g^{77} = \frac{1}{I_{66}},\ g^{\mathrm{uu}} = \frac{I_{\mathrm{vv}}}{I_{\mathrm{uu}}I_{\mathrm{vv}} - I_{\mathrm{uv}}^2},\ g^{\mathrm{uv}} = g^{\mathrm{vu}} = \frac{-I_{\mathrm{uv}}}{I_{\mathrm{uu}}I_{\mathrm{vv}} - I_{\mathrm{uv}}^2},\ g^{\mathrm{vv}} = \frac{I_{\mathrm{uu}}}{I_{\mathrm{uu}}I_{\mathrm{vv}} - I_{\mathrm{uv}}^2}$$

After lengthy calculation, we derive the mass spectrum of the form

$$E = M_{\mathrm{cl}} + \Delta M + \frac{1}{2I_{66}}\{j(j+1) - m^2\} + \frac{1}{I_{\mathrm{uu}}I_{\mathrm{vv}} - I_{\mathrm{uv}}^2}\left(\frac{I_{\mathrm{uu}}}{2}m^2 - I_{\mathrm{uv}}mY + \frac{I_{\mathrm{vv}}}{2}Y^2\right)$$

The effective Goldstone boson mass term

$$\Delta M = \frac{2\pi}{e^2}\int \rho d\rho\left[\{2Z_{PR}Z_{QS} - 2W_{kPR}W_{kQS} + \beta e^2 Z_{PR}Z_{QS}\mathrm{Tr}(X^{-1}\partial_k X)^2\}\times\left\{\Theta^P\Theta^Q\mathrm{Tr}(\lambda_R\lambda_S) - (\Theta^P g^{QT})\mathrm{Tr}(\lambda_R\lambda_S\lambda_T) + \frac{1}{4}g^{PT}g^{QU}\mathrm{Tr}(\lambda_T\lambda_R\lambda_S\lambda_U)\right\} + \frac{\beta e^2}{2}\left(V_{kPR}V_{kQS} + U_{kPR}U_{kQS}\right)g^{PT}g^{QS}\mathrm{Tr}(\lambda_R\lambda_S)\mathrm{Tr}(\lambda_R\lambda_S)\right]$$

$$\left[\frac{\lambda_P}{2}, X\right] \equiv Z_{PQ}\lambda_R,\ \left[\left[\frac{\lambda_P}{2}, X\right], \partial_k X\right] \equiv W_{kPQ}\lambda_Q \quad \left[\frac{\lambda_P}{2}, X\right]\partial_k X \equiv V_{kPQ}\lambda_Q,\ \partial_k X\left[\frac{\lambda_P}{2}, X\right] \equiv U_{kPQ}\lambda_Q$$

$$\Theta^6 = \Theta^7 = 0,\ \Theta^{\mathrm{u}} = \frac{1}{3}g^{\mathrm{uv}},\quad \Theta^{\mathrm{v}} = \frac{1}{8}\left(2g^{66} + g^{\mathrm{uu}} - \frac{4}{3}g^{\mathrm{vv}}\right)$$

The vortices might be stable in terms of existence of such mass term, **without any potential**.