SH^c Realization of Minimal Model CFT:Triality, Poset and Burge Condition

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Introduction and Summary

- In the context of AGT proof, a new algebra appeared!

$$\Rightarrow SH^c$$

  - Rank $\mathbf{N}$ representation: Some combinations of the SH^c generators give $\mathcal{W}_N$ currents.
  - Representation space $|\vec{a}, \vec{Y}\rangle$: Labeled by $\mathbf{N}$-tuple Young diagrams $\vec{Y}$

    ($\vec{a}$ are parameters related to the momentum of Toda Field Theory)

- Conjecture: SH^c describes arbitrary representation of $\mathcal{W}_N$ algebra ($+\mathbf{U(1)}$ factor) for any $\mathbf{N}$.

$$\Downarrow$$

We study the description of minimal models from SH^c.

- For $\mathcal{W}_N$ minimal model, we found explicit correspondence.
  - $\mathbf{N}$-Burge Condition (The condition for states not to have zero norm)

$$Y_{i,R} - Y_{i+1,R+(n_i-1)} \geq -(n'_i - 1).$$

($Y_{i,R}$: the length of the R-th row in the i-th Young diagrams)
n_i, n'_i : positive integer which satisfies $\sum_{i=1}^{N-1} n_i \leq q-1, \sum_{i=1}^{N-1} n'_i \leq p-1$
(p, q : the positive integer which determine the weight of $\mathcal{W}_N$ algebra)

  - Singular Vector

    $\Rightarrow$ The number of null states coincides with that of $\mathcal{W}_N$ algebra at same level!

- Level-rank duality (well-known duality in $\mathcal{W}$ algebra)

$$\mathcal{W}_{N,k} = \frac{\mathfrak{su}(N)_k \oplus \mathfrak{su}(N)_1}{\mathfrak{su}(N)_{k+1}} \sim \frac{\mathfrak{su}(M)_l \oplus \mathfrak{su}(M)_1}{\mathfrak{su}(M)_{l+1}} = \mathcal{W}_{M,l}$$

with

$$k = \frac{N}{M} - N, \quad l = \frac{M}{N} - M.$$

- Duality of states

$$\mathbf{N}\text{-tuple Young diagrams} \overset{?}{\Leftrightarrow} \mathbf{M}\text{-tuple Young diagrams}$$

$\Rightarrow$ Duality of the states is defined by shuffling of lines in Young diagrams!

- In the case of $(N, M) = (2, 3)$, this correspondence reproduces Rogers-Ramanujan identity

SH^c

- SH^c : spherical degenerate double affine Hecke algebra

We focus only on its rank $\mathbf{N}$ representation.
The basis of representation is labeled by $\mathbf{N}$-tuple Young diagrams and some parameter $\vec{a}$.

- The generators of SH^c : $D_{r,l}$ with $r \in \mathbb{Z}$ and $l \in \mathbb{Z}_{\geq 0}$.

$$\begin{aligned} [D_{0,l}, D_{1,k}] &= D_{1,l+k-1}, & l &\geq 1 \\ [D_{0,l}, D_{-1,k}] &= -D_{-1,l+k-1}, & l &\geq 1 \\ [D_{-1,k}, D_{1,l}] &= E_{k+l}, & l, k &\geq 1 \\ [D_{0,l}, D_{0,k}] &= 0, & k, l &\geq 0 \end{aligned}$$

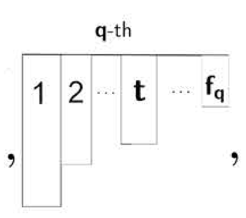
with

$$E(\zeta) = 1 + (1-\beta)\sum_{l\geq 0} E_l \zeta^{l+1} = \mathcal{C}(\zeta)\mathcal{D}(\zeta)$$

$$\mathcal{C}(\zeta) = \exp\left(\sum_{l\geq 0}(-1)^{l+1} c_l \pi_l(\zeta)\right) \equiv \prod_{q=1}^{N} T(\zeta, a_q), \quad T(\zeta, a) \equiv \frac{1+\zeta a}{1+\zeta(a-\xi)}$$

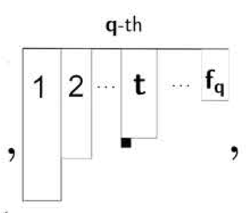
(c_ls are central charges) (rank $\mathbf{N}$ representation)

- $\mathbf{N}$-Burge Condition



$$D_{1,l}\left|\left(\cdots, \quad , \cdots\right)\right\rangle =$$

$$\sum_{q=1}^{N}\sum_{t=1}^{f_q+1}(-1)^l (a_q + A_t(Y_q))^l \boxed{\Lambda_q^{(t,+)}(\vec{Y})}\left|\left(\cdots, \quad , \cdots\right)\right\rangle$$



$$= 0$$

$\Rightarrow$ Then this state can not be generated!

$\Rightarrow$ This coincides the $\mathbf{N}$-Burge condition known for $\mathcal{W}_N$ algebra!

- Singular Vector
  - The condition for a state to be singular, *i.e.*

$$D_{-1,\ell}|\vec{a}, \vec{Y}\rangle = 0, \quad \ell \geq 0 \quad \text{(the singular state condition)}$$

the highest states of null states tower

is

$$\Lambda_p^{(k,-)}(\vec{a}, \vec{Y}) = 0 \quad \text{for all } p, k$$

($\Lambda_p^{(k,-)}$ is defined similarly as $\Lambda_p^{(k,+)}$.)

$\Rightarrow$ This reproduces the null states in $\mathcal{W}_N$ algebra!

Level-Rank Duality in SH^c

- Triality
  - Two symmetries which keep the central charge unchanged.

$$c = (N-1)(1 - Q^2 N(N+1)), \quad Q = \sqrt{\beta} - \frac{1}{\sqrt{\beta}} \quad (\text{here } \beta = \tfrac{N+M}{N})$$

Trivial symmetry:	Level-rank duality:
$\sigma_1 : \beta \mapsto \frac{1}{\beta}, \ N \mapsto N$ $D'_{0,l+1} = (-\beta)^{-l} D_{0,l+1}$	$\sigma_2 : \beta \mapsto \frac{\beta}{\beta - 1}, \ N \mapsto M$ $D'_{0,l+1} = (\beta - 1)^{-l} D_{0,l+1}$

$$\searrow \mathfrak{S}_3 \swarrow$$

$\Rightarrow$ Level-rank Duality in SH^c

- Duality for central charges

$$\mathcal{C}_N(\zeta, \vec{a}) = \mathcal{C}_M(\zeta', \vec{a}').$$

(We can also determine $\mathcal{W}_M$ central charges c'_l from the finite combination of c_l (infinite many).)

When this duality holds,

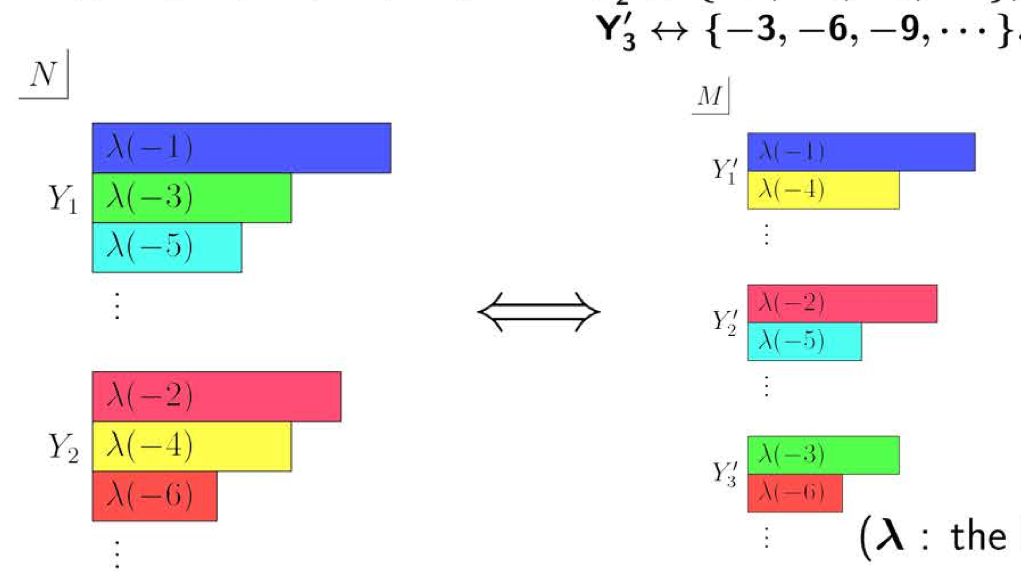
- Duality for states
  - Duality of the states is defined by shuffling of lines in Young diagrams!
- Example in the case of $(N, M) = (2, 3)$

  For $\Delta = -1/5$ case, (X (X') : the set of labels of rows in $\vec{Y}$ ($\vec{Y}'$))

  $X = X' = \{-1, -2, -3, -4, -5, -6, \cdots\}$ (X, X' are determined by the eigenvalues of $D_{0,1}$.)

$N = 2$: $Y_1 \leftrightarrow \{-1, -3, -5, \cdots\}$, $Y_2 \leftrightarrow \{-2, -4, -6, \cdots\}$

$M = 3$: $Y'_1 \leftrightarrow \{-2, -5, -8, \cdots\}$, $Y'_2 \leftrightarrow \{-1, -4, -7, \cdots\}$, $Y'_3 \leftrightarrow \{-3, -6, -9, \cdots\}$.



(λ : the length of the row)

- All the states are generated by the partition,

$$\lambda(x - N), \lambda(x - M) \leq \lambda(x) \quad \text{for each } x \in X$$

Poset and Partition Function

- Actually, the conditions

$$\lambda(x - N), \lambda(x - M) \leq \lambda(x) \quad \text{for each } x \in X$$

are what called the "Partition of POSET (Partially Ordered SET)".
(Here POSET is the set of labels of the rows of Young diagrams.)

$\Rightarrow$ Partition Function is acquired by the mathematical technique.

- In the case $(N, M) = (2, 3)$, we get

  ($(a; q)_j = \prod_{k=0}^{j-1}(1 - aq^k)$: q-Pochhammer symbol)

$$Z_X^{(\Delta=-\frac{1}{5})}(q) = \frac{1}{(q;q)_\infty}\sum_{k=0}^{\infty}\frac{q^{k^2}}{(q;q)_k}.$$

- On the other hand, by the character formula for $\mathcal{W}_N$ minimal model, we know

$$Z_X^{(\Delta=-\frac{1}{5})}(q) = \frac{(q^2;q^5)_\infty (q^3;q^5)_\infty (q^5;q^5)_\infty}{(q;q)_\infty^2}$$

- Therefore, we get the identity,

$$\frac{(q^2;q^5)_\infty (q^3;q^5)_\infty (q^5;q^5)_\infty}{(q;q)_\infty^2} = \frac{1}{(q;q)_\infty}\sum_{k=0}^{\infty}\frac{q^{k^2}}{(q;q)_k}.$$

$\Rightarrow$ This is the first Rogers-Ramanujan identity!

- We may be able to general this to (N, M) case

  $\rightarrow$ Get some generalization of Rogers-Ramanujan identity!

Further Discussion

- Physical Interpretation
  - AGT correspondence

2-dim CFT: conformal block of $\mathcal{W}_N$ algebra

$$\Updownarrow$$

4-dim $\mathcal{N} = 2$ gauge theory: partition function of $SU(N)$ SYM

  - Level-rank duality is well-known.

$$\mathcal{W}_N \Leftrightarrow \mathcal{W}_M$$

$$\Downarrow$$

  - Is there physical meaning of level-rank duality in 4-dim gauge theory?
  - What is the meaning of the null states?