Entanglement negativity of a free massless Dirac fermion on 2d torus

Mitsuhiro Nishida (Osaka University D2) [Work in Progress]

Abstract :

- Entanglement negativity is a computable entanglement measure for mixed states.
- We calculate the entanglement negativity by bosonization.

① Entanglement measures

Entanglement is an important nonlocal order.

e.g. topologically ordered phase
confinement/deconfinement phase

separable state $\rho = \sum_i p_i \rho_A^i \otimes \rho_B^i$

entangled state $\rho \neq \sum_i p_i \rho_A^i \otimes \rho_B^i$

In quantum information theory, many entanglement measures are defined to classify the entangled states and the phases.

- Entanglement entropy
- Mutual information
- Entanglement negativity
- Distillable entanglement
- Entanglement cost

⋮

② Entanglement negativity

[G. Vidal, R. F. Werner, 2001] [P. Calabrese, J. Cardy, E. Tonni, 2012]

Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

basis $|e_i^{(A)}\rangle \quad |e_j^{(B)}\rangle$

partial transpose density matrix ρ^{T_B}

$$\langle e_i^{(A)} e_j^{(B)} | \rho^{T_B} | e_k^{(A)} e_l^{(B)} \rangle = \langle e_i^{(A)} e_l^{(B)} | \rho | e_k^{(A)} e_j^{(B)} \rangle$$

logarithmic negativity

$$\mathcal{E} \equiv \ln \mathrm{Tr}|\rho^{T_B}|$$

Example: two-level spin system

2×2 basis $|\uparrow^{(A)}\uparrow^{(B)}\rangle \quad |\uparrow^{(A)}\downarrow^{(B)}\rangle \quad |\downarrow^{(A)}\uparrow^{(B)}\rangle \quad |\downarrow^{(A)}\downarrow^{(B)}\rangle$

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow^{(A)}\uparrow^{(B)}\rangle + |\downarrow^{(A)}\downarrow^{(B)}\rangle\right)$$

$\rho = |\psi\rangle\langle\psi|$

$$\begin{pmatrix} 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{pmatrix}$$

eigenvalues $[1,0,0,0]$

ρ^{T_B}

$$\begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix}$$

eigenvalues $[1/2, 1/2, 1/2, -1/2]$

$$\mathcal{E} \equiv \ln \mathrm{Tr}|\rho^{T_B}| = \ln\left(|1/2| + |1/2| + |1/2| + |-1/2|\right) = \ln 2$$

③ Advantage of entanglement negativity

1. **Good entanglement measure for mixed states**

- Entanglement measure of separable state is zero.
- Entanglement measure doesn't increase under LOCC (local operations and classical communication).

Entanglement entropy of mixed states doesn't satisfy the properties, but entanglement negativity satisfies them.

2. **Computable**

Definition of distillable entanglement

$$E_D(\rho) = \sup\left\{ r : \lim_{n\to\infty}\left[\inf_{\Psi} \mathrm{Tr}|\Psi(\rho^{\otimes n}) - \Phi(2^{rn})|\right] = 0\right\}$$

Computation of entanglement negativity is relatively easy.

④ Entanglement negativity of QFT

In QFT, entanglement negativity can be calculated by path integral.
But, direct calculation of $\mathrm{Tr}|\rho^{T_B}|$ is difficult.
So, we use a trick.

even n_e $\quad \mathrm{Tr}(\rho^{T_B})^{n_e} = \sum_i \lambda_i^{n_e} = \sum_{\lambda_i>0} |\lambda_i|^{n_e} + \sum_{\lambda_i<0} |\lambda_i|^{n_e}$

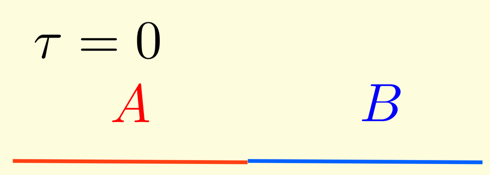
odd n_o $\quad \mathrm{Tr}(\rho^{T_B})^{n_o} = \sum_i \lambda_i^{n_o} = \sum_{\lambda_i>0} |\lambda_i|^{n_o} - \sum_{\lambda_i<0} |\lambda_i|^{n_o}$

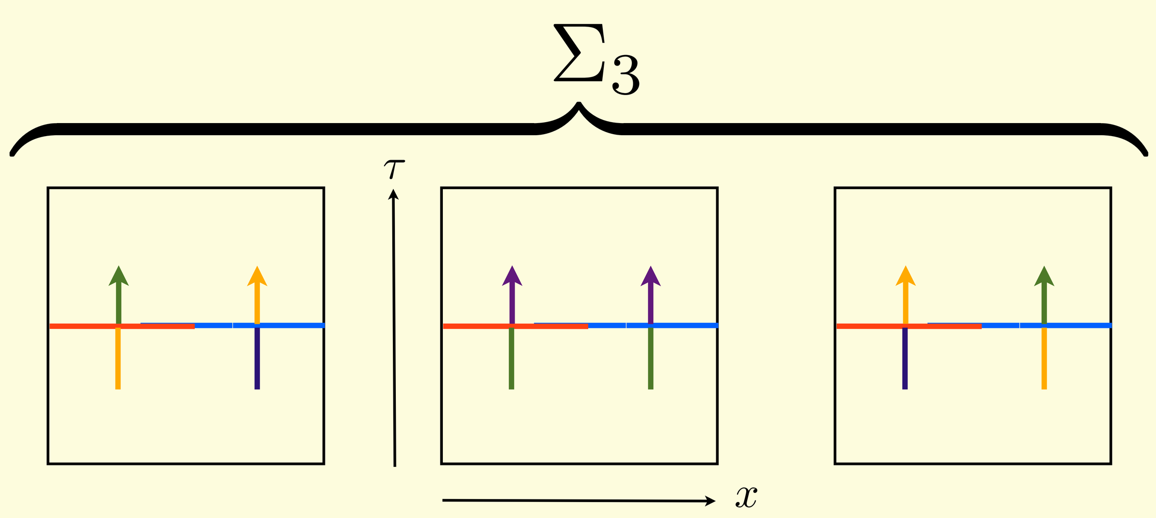
$$\mathcal{E} \equiv \ln \mathrm{Tr}|\rho^{T_B}| = \ln\left(\sum_{\lambda_i>0}|\lambda_i| + \sum_{\lambda_i<0}|\lambda_i|\right) = \lim_{n_e\to 1} \ln \mathrm{Tr}\left(\rho^{T_B}\right)^{n_e}$$

Generally, n dependence of $\mathrm{Tr}\left(\rho^{T_B}\right)^n$ is different in the case of even n_e and odd n_o and we can get nonzero logarithmic negativity.

⑤ Replica trick

Consider two-dimensional field theory.





$\mathrm{Tr}(\rho^{T_B})^3 =$ path integral on Σ_3

Product of ρ^{T_B} corresponds to connecting three plane at $\tau = 0$ and partial transposition corresponds to reversal of the connection way at the red and blue lines.

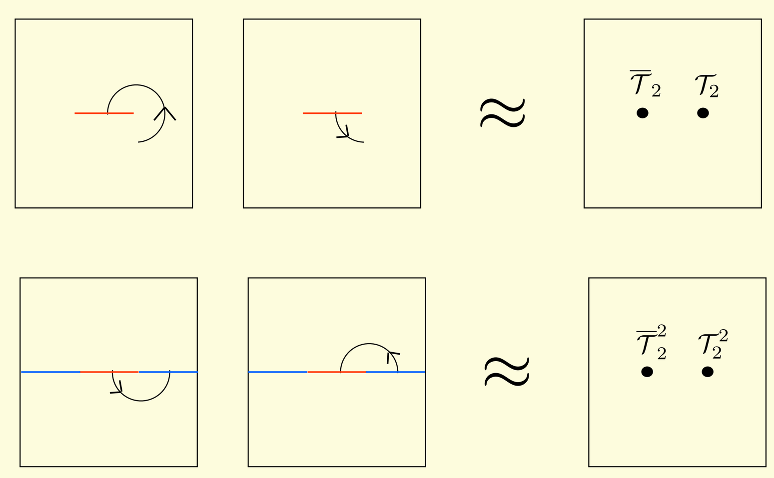
We can calculate $\mathcal{E} = \lim_{n_e\to 1} \ln \mathrm{Tr}\left(\rho^{T_B}\right)^{n_e}$ by path integral on Σ_{n_e}.

⑥ Twist fields

To calculate path integral on Σ_1, we introduce the twist fields.

path integral on Σ_n with ϕ = path integral on Σ_1 with $\phi_1, \phi_2, \cdots, \phi_n$ and the boundary conditions

Example of the twist fields $\mathcal{T}$



From correlation functions of the twist fields, we can calculate entanglement negativity like

$$\mathcal{E} = \lim_{n_e\to 1} \ln\langle \mathcal{T}^2_{n_e}(L)\overline{\mathcal{T}}^2_{n_e}(0)\rangle.$$

Entanglement negativity of a free massless Dirac fermion on 2d torus

Mitsuhiro Nishida (Osaka University D2) [Work in Progress]

Result :

- The result of a pure state is consistent with other results.
- We can get the result of mixed states naively, but its consistency and interpretation are unclear.

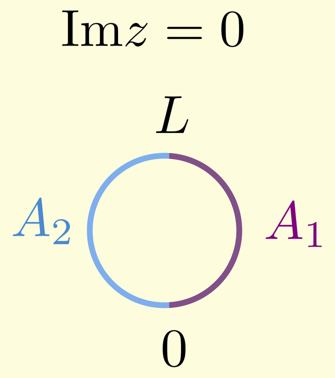
⑦ Free massless Dirac fermion on 2d torus

[T. Azeyanagi, T. Nishioka, T. Takayanagi, 2007]

2d torus

$z \sim z+1$

$z \sim z+i\beta$



bosonization

$$\psi = e^{i\varphi}$$

Dirac fermion ψ scalar φ

By introducing the replica fields $\psi^{(1)}, \psi^{(2)}, \cdots, \psi^{(n)}$,

we calculate the logarithmic negativity

$$\mathcal{E} = \lim_{n_e \to 1} \ln \langle \mathcal{T}^2_{n_e}(L,L) \overline{\mathcal{T}}^2_{n_e}(0,0) \rangle .$$

(We fix the spin structure.)

⑧ Explicit form of the twist fields

Boundary conditions for $\mathcal{T}^2_n(L,L)$

$$\tilde{\psi}^{(m)}(e^{2\pi i}(z-L)) = e^{\frac{2\pi i}{n}(2m-(n-1))} \tilde{\psi}^{(m)}(z-L)$$

In free theory, we can define σ^2_m as $\mathcal{T}^2_n = \prod_m \sigma^2_m$.

$$\sigma^2_m(z,\bar{z}) = \begin{cases} e^{\frac{i}{n}(2m+1)(\varphi^{(m)}(z) - \varphi^{(m)}(\bar{z}))}, & (0 \le m \le \frac{n}{4} - 1), \\ e^{\frac{i}{n}(2m+1-n)(\varphi^{(m)}(z) - \varphi^{(m)}(\bar{z}))}, & (\frac{n}{4} \le m \le \frac{3n}{4} - 1), \\ e^{\frac{i}{n}(2m+1-2n)(\varphi^{(m)}(z) - \varphi^{(m)}(\bar{z}))}, & (\frac{3n}{4} \le m \le n - 1), \end{cases}$$

$$\bar{\sigma}^2_m(z,\bar{z}) = (\sigma^2_m(z,\bar{z}))^{-1}. \qquad (n \in 4\mathbb{N})$$

sum of conformal weight $\Delta_m + \bar{\Delta}_m$ of σ^2_m

$$\sum_{m=0}^{n-1} (\Delta_m + \bar{\Delta}_m) = \frac{2}{n^2} \sum_{m=-\frac{n}{4}}^{\frac{n}{4}-1} (2m+1)^2 = (n/2 - 2/n)/6$$

It is consistent with the result of

[P. Calabrese, J. Cardy, E. Tonni, 2012] .

⑨ Useful formula

[P. D. Francesco, P. Mathieu, D. Senechal, "Conformal Field Theory"]

$$\langle O_{(n,w)}(z,\bar{z}) O_{(-n,-w)}(0,0) \rangle_{\mathbb{T}^2} = \left(\frac{2\pi\eta(\tau)^3}{\theta_1(z|\tau)} \right)^{2\Delta_{n,w}} \overline{\left(\frac{2\pi\eta(\tau)^3}{\theta_1(z|\tau)} \right)}^{2\bar{\Delta}_{n,w}} \frac{\sum_{m,l} q^{\Delta_{m,l}} \bar{q}^{\bar{\Delta}_{m,l}} e^{4\pi i (\alpha_{n,w}\alpha_{m,l} z - \bar{\alpha}_{n,w}\bar{\alpha}_{m,l}\bar{z})}}{\sum_{m,l} q^{\Delta_{m,l}} \bar{q}^{\bar{\Delta}_{m,l}}}$$

$$O_{(n,w)}(z,\bar{z}) = e^{i(n+w/2)\varphi(z) + i(n-w/2)\varphi(\bar{z})}$$

$$\Delta_{n,w} = \frac{1}{2}(n+w/2)^2 \qquad \bar{\Delta}_{n,w} = \frac{1}{2}(n-w/2)^2$$

$$\alpha_{n,w} = \frac{1}{\sqrt{2}}(n+w/2) \qquad \bar{\alpha}_{n,w} = \frac{1}{\sqrt{2}}(n-w/2)$$

$$q = e^{2\pi i \tau}$$

$$\sum_{m,l} q^{\Delta_{m,l}} \bar{q}^{\bar{\Delta}_{m,l}} = \frac{|\theta_2(0|\tau)|^2 + |\theta_3(0|\tau)|^2 + |\theta_4(0|\tau)|^2}{2}$$

To fix the spin structure, we change the summation like

$$\sum_{m,l} q^{\Delta_{m,l}} \bar{q}^{\bar{\Delta}_{m,l}} \to \frac{|\theta_\nu(0|\tau)|^2}{2} .$$

⑩ Entanglement negativity of the pure state

$$\langle \sigma^2_m(L,L) \bar{\sigma}^2_m(0,0) \rangle_\nu = \left| \frac{2\pi\eta(\tau)^3}{\theta_1(L|\tau)} \right|^{4\left(\frac{(2m+1-n)^2}{2n^2}\right)} \cdot \frac{|\theta_\nu(\frac{(2m+1-n)L}{n}|\tau)|^2}{|\theta_\nu(0|\tau)|^2} \qquad \left(\frac{n}{4} \le m \le \frac{3n}{4} - 1 \right)$$

$$\nu = 3$$

$$\begin{aligned} \mathcal{E} &= \lim_{n_e \to 1} \ln \langle \mathcal{T}^2_{n_e}(L,L) \overline{\mathcal{T}}^2_{n_e}(0,0) \rangle_3 \\ &= \lim_{n_e \to 1} \sum_{m=0}^{n_e-1} \ln \langle \sigma^2_m(L,L) \bar{\sigma}^2_m(0,0) \rangle_3 \\ &= \frac{1}{2} \ln[\frac{1}{\pi a} \sin(\pi L)] + \frac{1}{2} \sum_{m=1}^{\infty} \ln \left[\frac{(1 - e^{2\pi i L} e^{-2\pi\beta m})(1 - e^{-2\pi i L} e^{-2\pi\beta m})}{(1 - e^{2\pi\beta m})^2} \right] \\ &\quad + 2 \sum_{l=1}^{\infty} \frac{(-1)^{l-1}}{l} \frac{1}{\sinh(\pi l \beta)} \left(-1 + \frac{1}{\cos(\pi L l)} \right) \end{aligned}$$

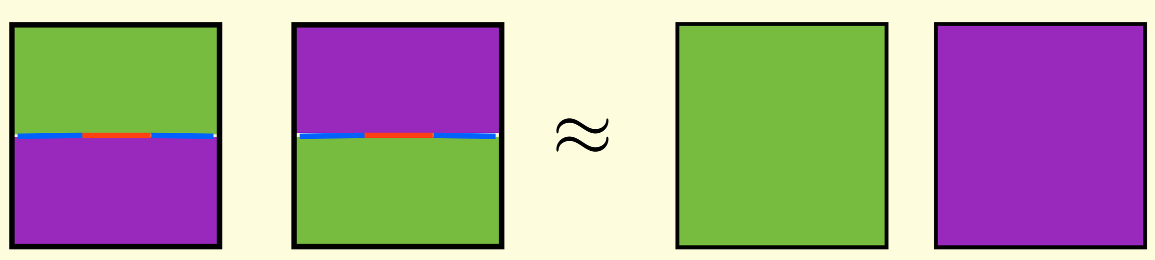
Red part is consistent with the result of $\beta \to \infty$.

(We introduce a cutoff a .)

⑪ Consistency check

From the geometry,

$\langle \mathcal{T}^2_{n_e}(L,L) \overline{\mathcal{T}}^2_{n_e}(0,0) \rangle = (\langle \mathcal{T}_{n_e/2}(L,L) \overline{\mathcal{T}}_{n_e/2}(0,0) \rangle)^2$ must hold.



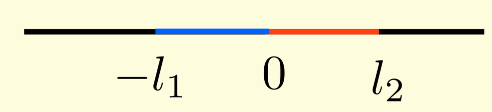
$$\begin{aligned} \langle \mathcal{T}^2_{n_e}(L,L) \overline{\mathcal{T}}^2_{n_e}(0,0) \rangle_\nu &= \prod_{m=0}^{n_e-1} \langle \sigma^2_m(L,L) \bar{\sigma}^2_m(0,0) \rangle_\nu \\ &= \left(\prod_{m=n_e/4}^{3n_e/4-1} \left| \frac{2\pi\eta(\tau)^3}{\theta_1(L|\tau)} \right|^{4\left(\frac{(2m+1-n_e)^2}{2n_e^2}\right)} \cdot \frac{|\theta_\nu(\frac{(2m+1-n_e)L}{n_e}|\tau)|^2}{|\theta_\nu(0|\tau)|^2} \right)^2 \\ &= \left(\prod_{m=0}^{n_e/2-1} \left| \frac{2\pi\eta(\tau)^3}{\theta_1(L|\tau)} \right|^{4\left(\frac{(m+(1-n_e/2)/2)^2}{2(n_e/2)^2}\right)} \cdot \frac{|\theta_\nu(\frac{(m+(1-n_e/2)/2)L}{n_e/2}|\tau)|^2}{|\theta_\nu(0|\tau)|^2} \right)^2 \\ &= \left(\prod_{m=0}^{n_e/2-1} \langle \sigma_m(L,L) \bar{\sigma}_m(0,0) \rangle_\nu \right)^2 = (\langle \mathcal{T}_{n_e/2}(L,L) \overline{\mathcal{T}}_{n_e/2}(0,0) \rangle_\nu)^2 \end{aligned}$$

We can check

$\langle \mathcal{T}^2_{n_e}(L,L) \overline{\mathcal{T}}^2_{n_e}(0,0) \rangle = (\langle \mathcal{T}_{n_e/2}(L,L) \overline{\mathcal{T}}_{n_e/2}(0,0) \rangle)^2$ explicitly.

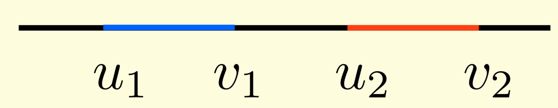
⑫ Naive calculation of mixed states

By assuming that we can use same twist fields for mixed states, we can calculate the entanglement negativity of mixed states naively.



$$\mathcal{E} = \lim_{n_e \to 1} \ln \langle \mathcal{T}_{n_e}(-l_1,-l_1) \overline{\mathcal{T}}^2_{n_e}(0,0) \mathcal{T}_{n_e}(l_2,l_2) \rangle \underset{?}{=} -\infty$$

(non-conservation of charge)



$$\mathcal{E} = \lim_{n_e \to 1} \ln \langle \mathcal{T}_{n_e}(u_1) \overline{\mathcal{T}}_{n_e}(v_1) \overline{\mathcal{T}}_{n_e}(u_2) \mathcal{T}_{n_e}(v_2) \rangle \underset{?}{=} 0$$

Is this result true?

What is the interpretation of this result?