Novel construction and monodromy relation for 3pt. function@ weak copling

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Based on collaboration with
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Brief summary

What?

1. Developed a new method to construct 3pt. functions in N= SYM @weak coupling.
2. Derived non-trivial identities (monodromy relations), which is a manifestation of integrability.

How?

1. Map the theory to spin chain problem.
2. Construct a vertex which correctly produces the Wick contraction using PSU(2,2|4) symmetry.

1. Introduction

Correlation functions are fundamental observables in AdS/CFT.

$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\rangle$ $\quad \Delta_i \quad \longleftrightarrow$ Spectrum

$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3)\rangle$ $\quad C_{ijk} \quad \longleftrightarrow$ Interaction or dynamic

To reveal the underlying mechanism of AdS/CFT, it is of importance to study these observable in detail.

In particular, in AdS_5/CFT_4, integrablity plays a quite important role.
[Beisert et al'10]

➡ *Study 3pt. functions using integrability!*

2. Spectrum and spin chain

PSU(2,2|4) symmetry

N=SYM has superconformal PSU(2,2|4) symmetry:

$$J^A{}_B := \left(\begin{array}{cc|c} Y_\alpha{}^\beta & iP_{\alpha\dot\beta} & Q^b_\alpha \\ iK^{\dot\alpha\beta} & Y^{\dot\alpha}{}_{\dot\beta} & i\bar S^{\dot\alpha b} \\ \hline S^\beta_a & i\bar Q_{\dot\beta a} & W_a{}^b \end{array}\right)_{AB} \qquad \begin{array}{l} Y_\alpha{}^\beta = M_\alpha{}^\beta + \frac{1}{2}\delta_\alpha{}^\beta(-iD+C-B) \\ Y^{\dot\alpha}{}_{\dot\beta} = \bar M^{\dot\alpha}{}_{\dot\beta} + \frac{1}{2}\delta^{\dot\alpha}{}_{\dot\beta}(iD+C-B) \\ W_a{}^b = R_a{}^b + \frac{1}{2}\delta_a{}^b B \end{array}$$

$$[J^A{}_B, J^C{}_D] = \delta^C{}_B J^A{}_D - (-1)^{(|A|+|B|)(|C|+|D|)}\delta^A{}_D J^C{}_D$$

Oscillator representation

We can express them using the following bosonic/fermionic oscillators:

$$\underbrace{[a^\alpha, \bar a_\beta] = \delta^\alpha_\beta\ ,\ [b^{\dot\alpha}, \bar b_{\dot\beta}] = \delta^{\dot\alpha}_{\dot\beta}}_{SL(2,\mathbb{C})\times SL(2,\mathbb{C})}\ ,\ \underbrace{\{c^a, \bar c_b\} = \delta^a{}_b}_{SU(4)}$$

$$J^A{}_B = \bar\zeta^A\zeta_B\ ,\quad \bar\zeta^A = \begin{pmatrix} \bar a_\alpha \\ i b^{\dot\alpha} \\ \bar c_a \end{pmatrix}^A\ ,\quad \zeta_A = \begin{pmatrix} a^\alpha \\ i\bar b_{\dot\alpha} \\ c^a \end{pmatrix}_A$$

The field of N=4 SYM can be represented by oscillators as well:

$$a^\alpha|0\rangle = b^{\dot\alpha}|0\rangle = c^a|0\rangle = 0$$

$$F_{\alpha\beta} \leftrightarrow \bar a_\alpha \bar a_\beta|0\rangle\ ,\quad \bar\psi^a_{\dot\alpha} \leftrightarrow \frac{1}{3!}\epsilon^{abcd}\bar b_{\dot\alpha}\bar c_b \bar c_c \bar c_d|0\rangle\ ,$$

$$\psi_{\alpha a} \leftrightarrow \bar a_\alpha \bar c_a|0\rangle\ ,$$

$$\phi_{ab} \leftrightarrow \bar c_a \bar c_b|0\rangle\ ,\quad \bar F_{\dot\alpha\dot\beta} \leftrightarrow \frac{1}{4!}\epsilon^{abcd}\bar b_{\dot\alpha}\bar b_{\dot\beta}\bar c_a\bar c_b\bar c_c\bar c_d|0\rangle\ ,$$

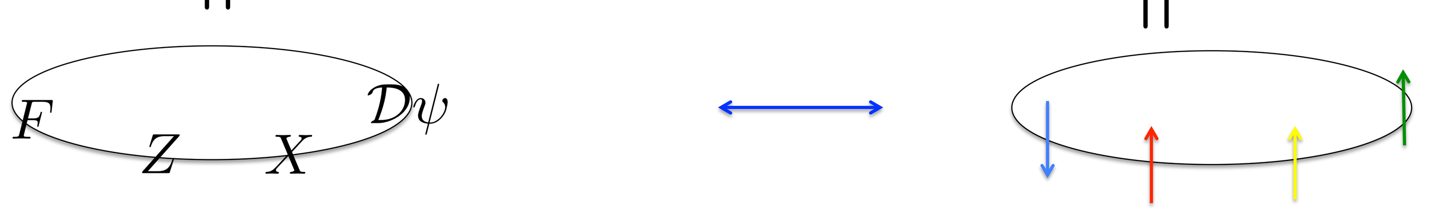
All the fields carry the zero central charge: $C = \frac{1}{2}(N_a - N_b + N_c - 2)$

Dilatation op. and spin chain Hamiltonian

1-loop dilatation op. was identified to a integrable spin chain Hamitonian.
[Minahan,Zarembo'02, Beisert'02]

$$\mathcal{D}_{1-\text{loop}} \leftrightarrow H_{XXX}$$

$\mathcal{O}(x) = \text{Tr}[Z\ldots X\ldots \mathcal{D}X\ldots F]$ $\longleftrightarrow$ $|\mathcal{O}(x)\rangle$



We can map the single tr. op.s to eigenstates of the Hamiltonian!

3. Tailoring of three-point functions

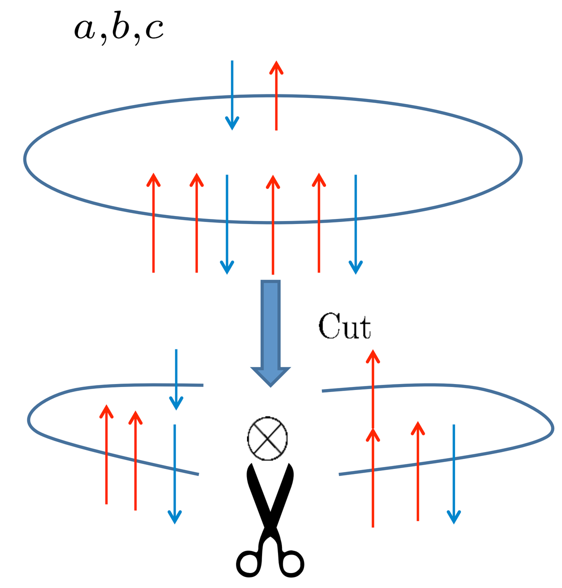
● Tree-level 3pt. functions are obtained by summing all possible planar Wick contractions.

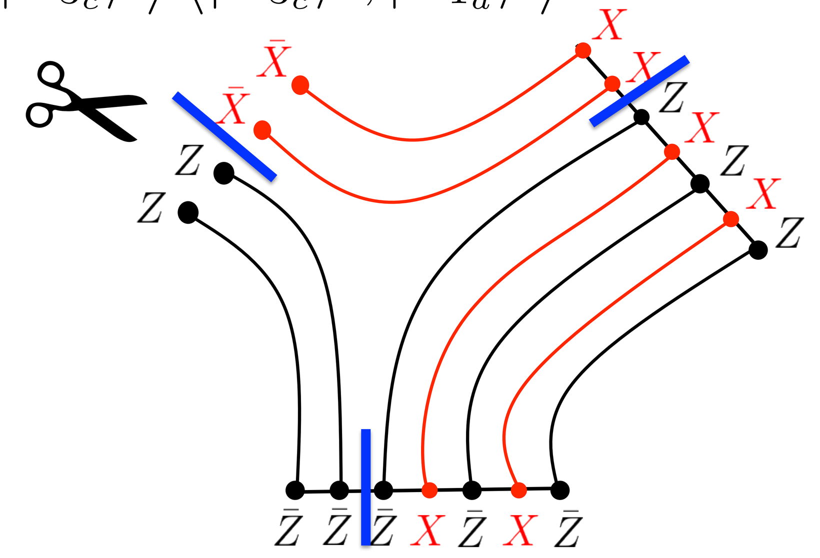
● It involves complicated combinatorics since we need to prepare 1-loop eigenstates of the dilatation operator. (*degenerate perturbation theory*)

➡ *``Tailoring'' gives an efficient method.* [Escobedo,Gromov,Sever,Vieira'09]

1. Cut $|\mathcal{O}_i\rangle \to \sum_a |\mathcal{O}_{i_a}\rangle^l \otimes |\mathcal{O}_{i_a}\rangle^r$

2. Sew $\sum_{a,b,c} \langle |\mathcal{O}_{1_a}\rangle^r, |\mathcal{O}_{2_b}\rangle^l\rangle \langle |\mathcal{O}_{2_b}\rangle^r, |\mathcal{O}_{3_c}\rangle^l\rangle \langle |\mathcal{O}_{3_c}\rangle^r, |\mathcal{O}_{1_a}\rangle^l\rangle$





4. Construction of vertex

We wish to find the tree-level 3pt. vertex of the form:

$$\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle = \langle V_{123}|(|\mathcal{O}_1\rangle \otimes |\mathcal{O}_2\rangle \otimes |\mathcal{O}_3\rangle)$$

Wick contraction and singlet

Since the building block is the Wick contraction at tree-level, we first consider an elementary vertex.

$$\overbrace{\mathcal{F}_1\mathcal{F}_2} = \langle\circ|(|\mathcal{F}_1\rangle\otimes|\mathcal{F}_2\rangle) \qquad \mathcal{F}_i :$$ fundamental fields of N=4 SYM

Idea: Use the Ward identity of PSU(2,2|4)

$$0 = \langle (J\mathcal{F}_1)\mathcal{F}_2\rangle + \langle \mathcal{F}_1(J\mathcal{F}_2)\rangle$$

$$\Leftrightarrow\ 0 = \langle\circ|(J_1 + J_2)(|\mathcal{F}_1\rangle\otimes|\mathcal{F}_2\rangle)$$

J_i : Generator of PSU(2,2|4)

It must be a singlet of PSU(2,2|4): $\langle\circ| = \langle \mathbf{1}|$

Using the oscillator representation, it turns out that the singlet is given by the following form: Similar expression is given in [Jiang,Kostov,Petrovskii,Serban'14]

$$|\mathbf{1}_{12}\rangle = \exp\left(\bar a^1_\alpha \otimes a^\alpha_2 - \bar b^1_{\dot\alpha}\otimes b^{\dot\alpha}_2 + \bar c^1_i \otimes c^i_2 - \bar d^1_j \otimes d^j_2\right)|Z\rangle\otimes|\bar Z\rangle$$

$$|Z\rangle = |0\rangle_B \otimes \bar c_3\bar c_4|0\rangle_F\ ,\quad |\bar Z\rangle = |\bar 0\rangle_B\otimes \bar c_1\bar c_2|0\rangle_F \qquad \begin{array}{l} \bar a_\alpha|\bar 0\rangle_B = \bar b_{\dot\alpha}|\bar 0\rangle_B = 0 \\ \bar d_i = c^{i+2}\ ,\ d^i = \bar c_{i+2} \end{array}$$

Note: It is an element of tensor product of HW and LW module.

Crossing relation

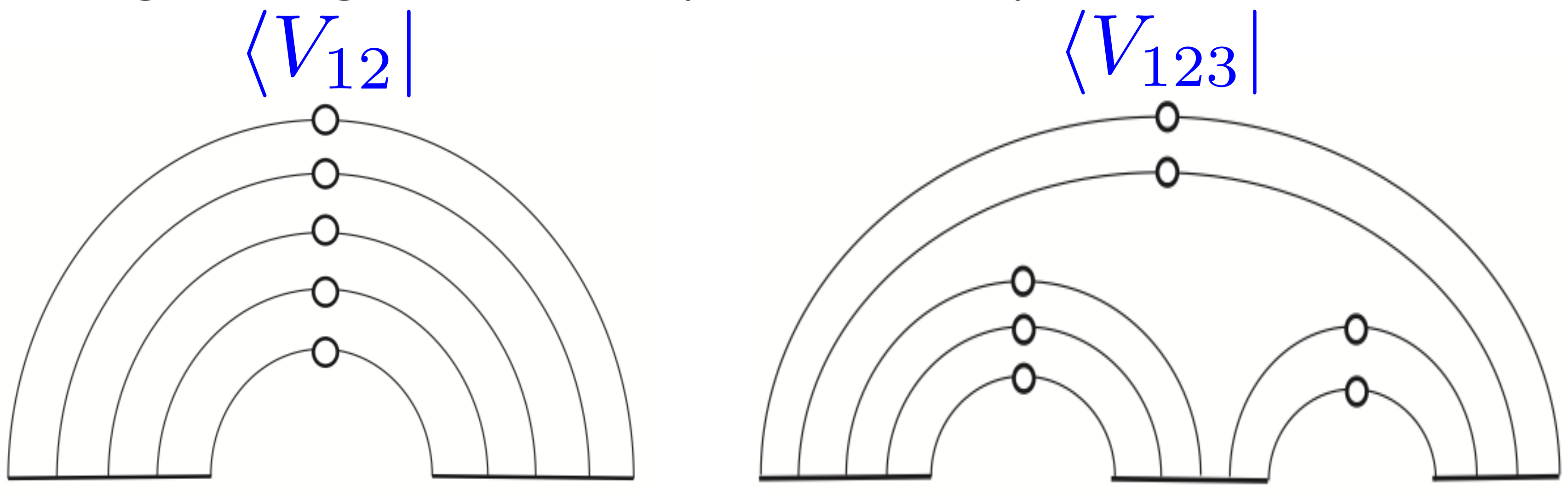
Using the following relation for *each oscillator,* we can see that the correct Wick contractions are reproduced.

$$\langle \mathbf{1}_{12}|\zeta_A \otimes 1 = -\langle \mathbf{1}_{12}|1\otimes\zeta_A$$

$$\langle \mathbf{1}_{12}|\bar\zeta^A \otimes 1 = \langle \mathbf{1}_{12}|1\otimes\bar\zeta^A$$

Three-point vertex

Using the singlet, we can express 2 and 3-point vertex.



By construction, they satisfy the Ward identity.

$$0=\langle V_{12}|(J_1+J_2)\quad 0=\langle V_{123}|(J_1+J_2+J_3)$$

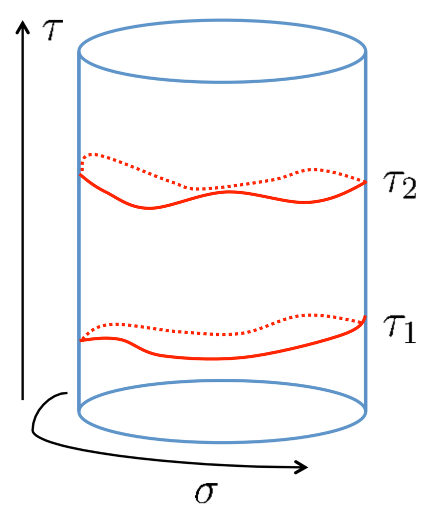
5. Monodormy relation

Motivation

Classical monodormy matrix plays essential role at strong coupling.

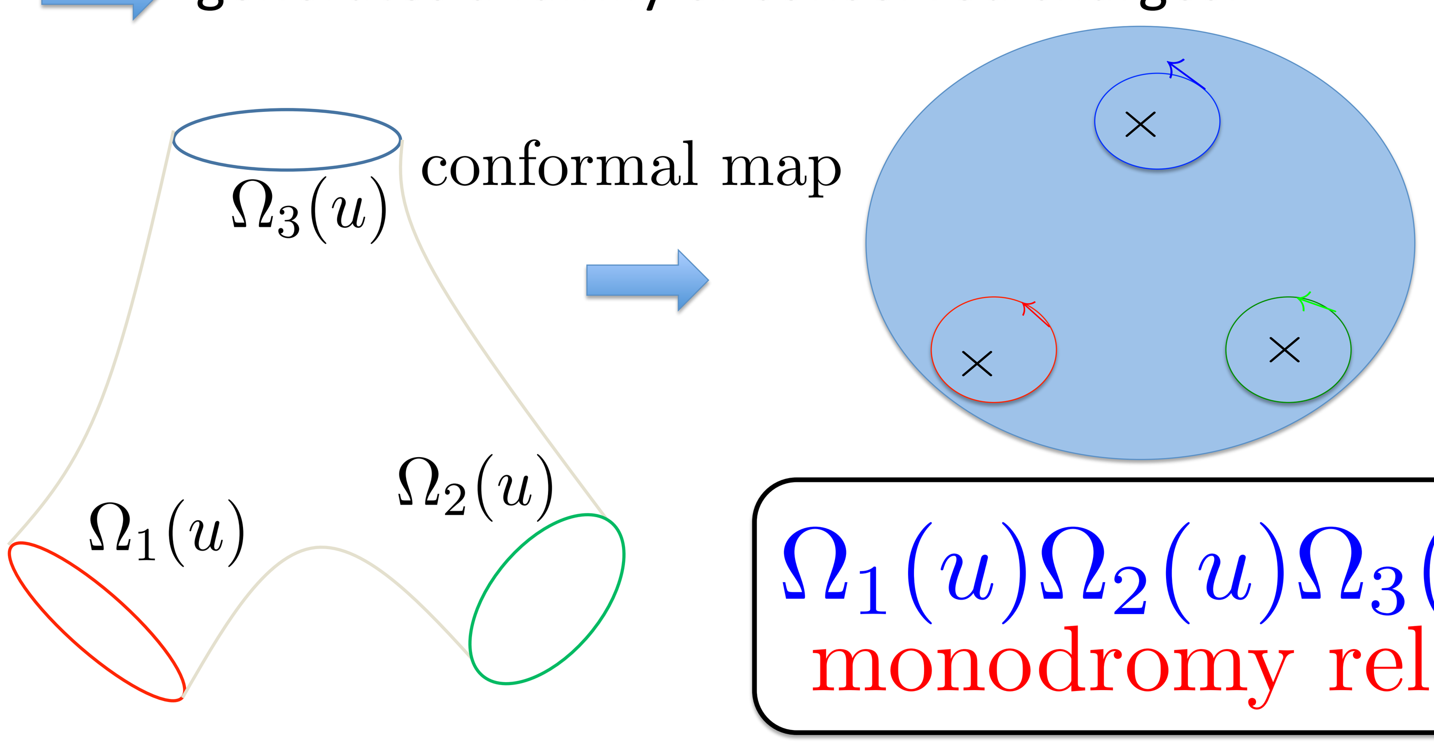
$$\text{E.O.M} \Leftrightarrow (d+A(u))^2=0 \quad u$$: spectral parameter

$$\Omega_i(u)=\mathrm{P}\exp\left[\oint_{C_i} A(u)\right]$$



Due to the flatness condition, it does not depend on the world sheet.

➡ generates a family of conserved charges.



$$\Omega_1(u)\Omega_2(u)\Omega_3(u)=1$$

monodromy relation

Combined with the analyticity, the monodromy relation determines semiclassical 3-point function uniquely! [Janik,Wereszczyński'11] [Kazama,Komatsu'11,'12,'13]

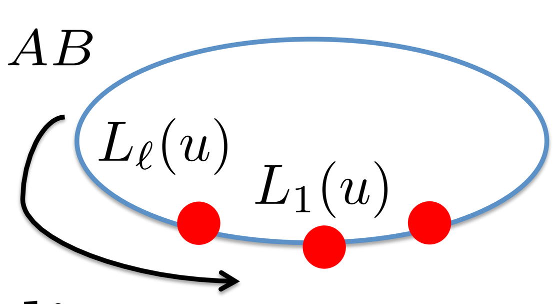
Q: *What is a weak coupling analogue of this relation?*

Definition of monodormy

$$(L(u))^A{}_B=u\delta^A{}_B+\eta(-1)^{|B|}J^A{}_B \quad \text{Lax operator}$$

$$=\left(\begin{array}{cc|c} u+\eta Y_\alpha{}^\beta & i\eta P_{\alpha\dot\beta} & -\eta Q^b_\alpha \\ i\eta K^{\dot\alpha\beta} & u+\eta Y^{\dot\alpha}{}_{\dot\beta} & -i\eta \bar S^{\dot\alpha b} \\ \hline \eta S^\beta_a & i\eta\bar Q_{\dot\beta a} & u-\eta W^b_a\end{array}\right)_{AB}$$

$$\Omega(u)=L_1(u)\cdots L_\ell(u)$$



Monodormy relation@ weak copling

Using the property of singlet and definition of the Lax operator, we find

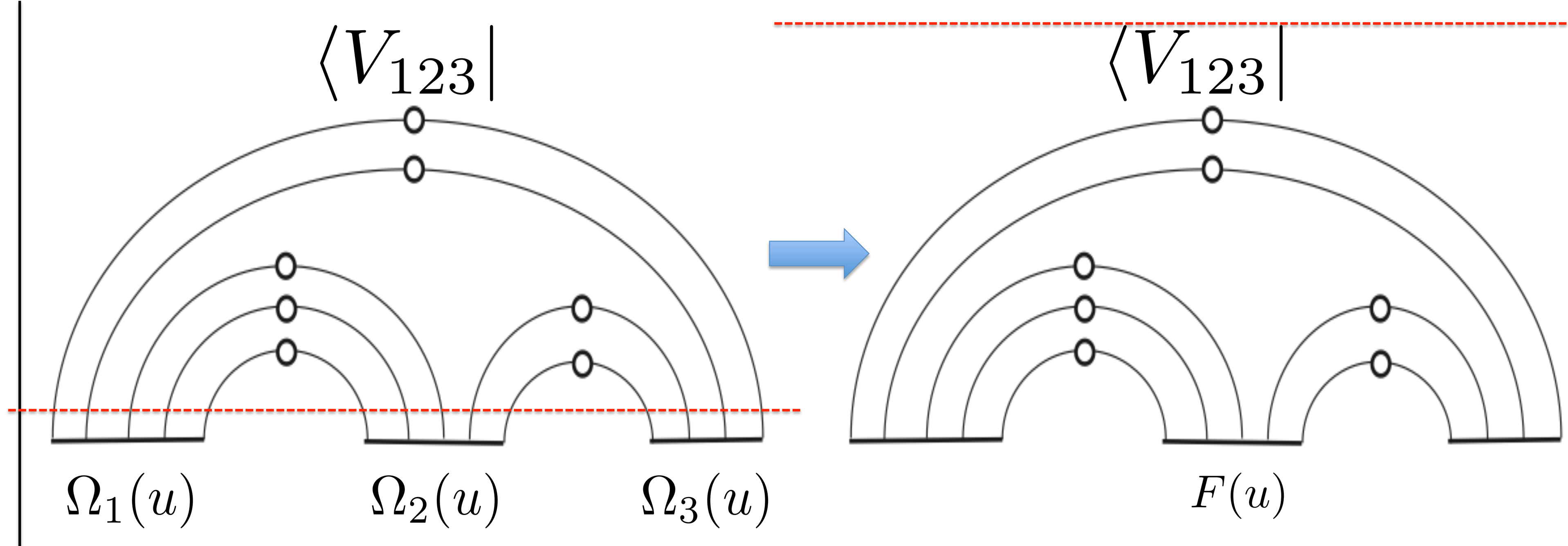
Crossing $\langle \mathbf{1}_{12}|L^{(1)}(u)=-\langle \mathbf{1}_{12}|L^{(2)}(-u+\eta)$

Inversion $L^{(i)}(u)^{(i)}L(\eta-u)=u(\eta-u)\mathbf{1}$

With these relations, we can derive the following monodormy relation.

[Jiang,Kostov,Petrovskii,Serban'14] [Kazama,Komatsu,T.N.'14,'15]

$$\langle V_{123}|\Omega_1(u)\Omega_2(u)\Omega_3(u)=F(u)\langle V_{123}|$$
$$F(u)=(u(u-\eta))^{\ell_1+\ell_2+\ell_3}$$



- Expanding the relation in power of 1/u around u=∞, we find the Ward identity at the leading order.
- Higher order terms of 1/u expansion give non-trivial identities for various 3-point functions. ➡ *``Ward identities'' of Yangians.*

Harmonic R-matrix

We can also derive another variant of the monodormy relation using so-called the harmonic R-matrix. [Kazam,Komatsu,T.N.'15]

$$\langle V_{123}|\Omega^{(1)}(u)\Omega^{(2)}(u)\Omega^{(3)}(u)=\langle V_{123}|$$

$$\Omega^{(i)}(u):=\mathbf{R}^{(i)}_{a1}(u)\cdots\mathbf{R}^{(i)}_{a\ell_i}(u)$$

$$\mathbf{R}_{12}(u)=\sum_{k,l,m,n}\mathcal{A}^{(\mathbf{N})}_{k,l,m,n}(u)\mathbf{Hop}^{(12)}_{k,l,m,n}$$

$$\mathbf{Hop}^{(12)}_{k,l,m,n}=:\frac{(\bar\alpha_2\alpha^1)^k}{k!}\frac{(\bar\beta^2\beta_1)^l}{l!}\frac{(\bar\alpha_1\alpha^2)^m}{m!}\frac{(\bar\beta^1\beta_2)^n}{n!}:$$

$$\bar\alpha^{\mathsf{A}}=\begin{pmatrix}\bar a_\alpha\\ \bar c_i\end{pmatrix}\quad \alpha_{\mathsf{A}}=\begin{pmatrix}a^\alpha\\ c^i\end{pmatrix}\quad \bar\beta^{\dot{\mathsf{A}}}=\begin{pmatrix}\bar b_{\dot\alpha}\\ \bar d_i\end{pmatrix}\quad \beta_{\dot{\mathsf{A}}}=\begin{pmatrix}b^{\dot\alpha}\\ d^i\end{pmatrix}$$

$$\mathcal{A}^{(\mathbf{N})}_{k,l,m,n}(u)=\frac{(-1)^{I+\frac{\mathbf{N}}{2}}\Gamma(u+1)\Gamma(1-u)\Gamma(I+1)}{\Gamma(I+1-u-\frac{\mathbf{N}}{2})\Gamma(u+1+\frac{\mathbf{N}}{2})}(-1)^{(k+l)(m+n)}\delta_{k+n,m+l}$$

$$\mathbf{N}=\mathbf{N}^{(1)}+\mathbf{N}^{(2)}\qquad \mathbf{N}^{(i)}=\mathbf{N}^{(i)}_\alpha+\mathbf{N}^{(i)}_\beta=\bar\alpha^{\mathsf{A}}_i\alpha^i_{\mathsf{A}}+\bar\beta^i_{\dot{\mathsf{A}}}\beta^{\dot{\mathsf{A}}}_i$$

- It is important to note that the 1-loop dilatation operator (Hamiltonian) is closely related to the harmonic R-matrix: [Beisert,Staudacher'04]

$$\mathbf{H}_{12}=\frac{d}{du}\ln\mathbf{R}_{12}(u)|_{u=0}$$

- The harmonic R-matrix is used to construct building blocks for the scattering amplitude as Yangian invariant. [Chicherin,Kirschner'13] [Ferro,Lukowski,Meneghelli,Plefka,Staudacher,'13] [Broedel,de Leeuw,Rosso'14]

6. Outlooks

1. Use of monodormy relation.

- Semi-classical three-point functions from Landau-Lifshitz model. [Kazama,Komatsu,T.N. to appear]
- Application to Chern-Simons vector models. [Kiryu,Komatsu,T.N. in progress]

2. 1-loop correction. [Komatsu,T.N. in progress]

It would be nice to detrmine the 1-loop correction using symmetry:

$$(\langle V^{(0)}_{123}|+g\langle V^{(1)}_{123}|+\cdots)\sum_{i=1}^{3}(J^{(0)}_i+gJ^{(1)}_i+\cdots)=0$$

$$\Rightarrow\sum_{i=1}^{3}\langle V^{(0)}_{123}|J^{(1)}_i+\sum_{i=1}^{3}\langle V^{(1)}_{123}|J^{(0)}_i=0$$

Hamiltonian insertion? Relation to scattering amplitude?

Integrable deformation? [Ferro,Lukowski,Meneghelli,Plefka,Staudacher,'13] [Bargheer,Huang,Loebbert,Yamazaki'14]

3. Relation to recent non-perturbative approach.

- SFT vertex (form factor) [Bajnok,Janik'15]
- Hexagon form factor [Basso,Komatsu,Vieira'15]

Theme of Komatsu's talk!