Complete action of open superstring field theory

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1. Introduction

String field theory is one approach to nonperturbative formulations of string theory which plays a role complementary to other approaches such as the AdS/CFT correspondence, matrix models, and so on.

Since the bosonic string contains tachyons both in the open-string and closed-string channels, we need to formulate superstring field theory if we are interested in quantum aspects.

However, construction of an action including the Ramond sector has not been successful for about thirty years. This August we succeeded in constructing a gauge-invariant action of open superstring field theory including both the Neveu-Schwarz sector and the Ramond sector. This is the first construction of a complete formulation of superstring field theory.

Let us begin with explaining the difficulty in constructing an action for the Ramond sector.

2. Kinetic terms

Unintegrated vertex operators of the open bosonic string: cV^{matter}
$\rightarrow$ the open bosonic string field Ψ: ghost number 1

In string field theory, the physical state condition and the equivalence relation

$$Q\Psi = 0\,, \qquad \Psi \sim \Psi + Q\Lambda\,,$$

where Q is the BRST operator, are implemented as the equation of motion and a gauge symmetry. The action for the free theory

$$S = -\frac{1}{2}\langle\, \Psi, Q\Psi \,\rangle\,,$$

where $\langle\, A, B \,\rangle$ is the BPZ inner product of A and B, is invariant under the following gauge transformation:

$$\delta\Psi = Q\Lambda\,.$$

The gauge invariance follows from

$$\langle\, B, A \,\rangle = (-1)^{AB}\,\langle\, A, B \,\rangle\,, \quad Q^2 = 0\,,$$
$$\langle\, QA, B \,\rangle = -\,(-1)^{A}\,\langle\, A, QB \,\rangle\,.$$

The BPZ inner product in the open string is defined by a correlation function of the boundary CFT on a disk, and the total ghost number has to be 3 for the BPZ inner product to be nonvanishing.

$$S = -\frac{1}{2}\,\langle\, \Psi, Q\Psi \,\rangle \quad \rightarrow \quad 1 + 1 + 1 = 3 \quad \text{OK!}$$

For the open superstring, there are infinitely many vertex operators labeled by the picture number to describe each physical state. In formulating string field theory, we need to choose the picture number of the open superstring field.

The choice of the picture number is related to the choice of the vacuum of the superconformal ghost sector with the commutation relation

$$[\gamma_n, \beta_m] = \delta_{n+m,0}.$$

A natural choice in the Neveu-Schwarz sector is -1 picture.

annihilation operators: $\gamma_{1/2},\ \beta_{1/2},\ \gamma_{3/2},\ \beta_{3/2},\ \ldots$

creation operators: $\gamma_{-1/2},\ \beta_{-1/2},\ \gamma_{-3/2},\ \beta_{-3/2},\ \ldots$

the open superstring field Ψ: ghost number 1, picture number -1

$$S = -\frac{1}{2}\langle\,\Psi, Q\Psi\,\rangle$$

$$\rightarrow \begin{cases} \text{the ghost number:} \quad 1+1+1=3 \quad \text{OK!} \\ \text{the picture number:} \quad (-1)+0+(-1)=-2 \quad \text{OK!} \end{cases}$$

Natural choices in the Ramond sector would be
$-1/2$ picture:

$$\text{annihilation operators:} \quad \beta_0, \ \gamma_1, \ \beta_1, \ \gamma_2, \ \beta_2, \ \ldots$$
$$\text{creation operators:} \quad \gamma_0, \ \gamma_{-1}, \ \beta_{-1}, \ \gamma_{-2}, \ \beta_{-2}, \ \ldots$$

and $-3/2$ picture:

$$\text{annihilation operators:} \quad \gamma_0, \ \gamma_1, \ \beta_1, \ \gamma_2, \ \beta_2, \ \ldots$$
$$\text{creation operators:} \quad \beta_0, \ \gamma_{-1}, \ \beta_{-1}, \ \gamma_{-2}, \ \beta_{-2}, \ \ldots$$

Suppose that we choose the string field Ψ in the $-1/2$ picture.

$$S = -\frac{1}{2} \langle \Psi, Q\Psi \rangle$$

$$\rightarrow \begin{cases} \text{the ghost number:} \quad 1+1+1 = 3 \quad \text{OK!} \\ \text{the picture number:} \quad (-1/2) + 0 + (-1/2) \neq -2 \end{cases}$$

The picture number does not work out for any choice in the Ramond sector.

Actually, there is a similar problem in the closed bosonic string.

Unintegrated vertex operators of the closed bosonic string: $c\tilde{c}V^{\mathrm{matter}}$
$\rightarrow$ the closed bosonic string field Ψ: ghost number 2

$$S = -\frac{1}{2}\langle\,\Psi, Q\Psi\,\rangle \qquad \rightarrow \qquad \text{the ghost number:} \quad 2+1+2 \neq 6$$

Let us explain the solution to this problem by viewing surfaces for string propagators as Riemann surfaces.

The propagator strip in the open bosonic string can be generated by L_0 as e^{-tL_0}, where t is the modulus corresponding to the length of the strip.

In open bosonic string field theory, the integration over this modulus is implemented by the propagator in Siegel gauge as

$$\frac{b_0}{L_0} = \int_0^\infty dt\, b_0\, e^{-tL_0}\,.$$

The propagator surface in the closed bosonic string can be generated by $L_0 + \widetilde{L}_0$ and $i\,(L_0 - \widetilde{L}_0)$ as $e^{-t(L_0+\widetilde{L}_0)+i\theta(L_0-\widetilde{L}_0)}$, where t and θ are moduli.

In closed bosonic string field theory, the integration over t is implemented by the propagator in Siegel gauge as in the open bosonic string:

$$\frac{b_0^+}{L_0^+} = \int_0^\infty dt\, b_0^+\, e^{-t\,L_0^+}\,,$$

where

$$L_0^+ = L_0 + \widetilde{L}_0\,, \qquad b_0^+ = b_0 + \widetilde{b}_0\,.$$

On the other hand, the integration over θ is implemented as a constraint on the space of string fields. The integration over θ yields the operator given by

$$B = b_0^- \int_0^{2\pi} \frac{d\theta}{2\pi} \, e^{i\theta L_0^-} \,,$$

where

$$L_0^- = L_0 - \widetilde{L}_0 \,, \qquad b_0^- = b_0 - \widetilde{b}_0 \,.$$

Schematically, $B \sim \delta(b_0^-) \, \delta(L_0^-)$.

The closed bosonic string field Ψ is constrained to satisfy

$$b_0^- \, \Psi = 0 \,, \qquad L_0^- \, \Psi = 0 \,,$$

and the BRST cohomology on this restricted space is known to give the correct spectrum of the closed bosonic string.

The appropriate inner product of Ψ_1 and Ψ_2 in the restricted space can be written as

$$\langle \, \Psi_1, c_0^- \Psi_2 \, \rangle \, ,$$

where

$$c_0^- = \frac{1}{2} \left(\, c_0 - \widetilde{c}_0 \, \right) .$$

The kinetic term of closed bosonic string field theory is then given by

$$S = -\frac{1}{2} \langle \, \Psi, c_0^- Q \Psi \, \rangle \, .$$

$\rightarrow$ the ghost number: $2 + 1 + 1 + 2 = 6$ OK!

The operator B can also be written as

$$B = -i \int_0^{2\pi} \frac{d\theta}{2\pi} \int d\tilde{\theta} \, e^{i\theta L_0^- + i\tilde{\theta} b_0^-}$$

using a Grassmann-odd variable $\tilde{\theta}$.
(Note that the extended BRST transformation introduced by Witten in arXiv:1209.5461 maps θ to $\tilde{\theta}$.)

Since $B \, c_0^- \, B = B$, the operator $B \, c_0^-$ is a projector, and the closed bosonic string field Ψ in the restricted space can be characterized as

$$B \, c_0^- \, \Psi = \Psi \, .$$

The propagator strip for the Ramond sector of the open superstring has a fermionic modulus in addition to the bosonic modulus corresponding to the length of the strip.

The fermionic direction of the moduli space can be parameterized as $e^{\zeta G_0}$, where G_0 is the zero mode of the supercurrent and ζ is the fermionic modulus. The integration over ζ with the associated ghost insertion yields the operator X given by

$$X = \int d\zeta \int d\tilde{\zeta}\, e^{\zeta G_0 - \tilde{\zeta}\beta_0}\,,$$

where $\tilde{\zeta}$ is a Grassmann-even variable.
(The extended BRST transformation maps ζ to $\tilde{\zeta}$.)

If we perform the integration over ζ, we obtain

$$X = -\delta(\beta_0)\,G_0 + \delta'(\beta_0)\,b_0\,.$$

Now the open superstring field Ψ of picture $-1/2$ in the Ramond sector can be expanded as

$$\Psi = \sum_{n=0}^{\infty} (\gamma_0)^n \left(\phi_n + c_0\, \psi_n \right),$$

where

$$b_0\, \phi_n = 0 \,, \qquad \beta_0\, \phi_n = 0 \,, \qquad b_0\, \psi_n = 0 \,, \qquad \beta_0\, \psi_n = 0 \,.$$

It is known that the correct spectrum of the open superstring can be reproduced by the BRST cohomology with Ψ restricted to the following form:

$$\Psi = \phi - \left(\gamma_0 + c_0\, G \right) \psi \,,$$

where $G = G_0 + 2\, b_0 \gamma_0$ and

$$b_0\, \phi = 0 \,, \qquad \beta_0\, \phi = 0 \,, \qquad b_0\, \psi = 0 \,, \qquad \beta_0\, \psi = 0 \,.$$

The appropriate inner product of Ψ_1 and Ψ_2 in the restricted space can be written as
$$\langle\, \Psi_1, Y\Psi_2 \,\rangle$$
with
$$Y = -\, c_0\, \delta'(\gamma_0)\,,$$
and the kinetic term of open superstring field theory for the Ramond sector is given by
$$S = -\frac{1}{2}\,\langle\, \Psi, YQ\Psi \,\rangle\,.$$

$\rightarrow \begin{cases} \text{the ghost number:} & 1+0+1+1 = 3 \quad \text{OK!} \\ \text{the picture number:} & (-1/2)+(-1)+0+(-1/2) = -2 \quad \text{OK!} \end{cases}$

The important point is that the open superstring field Ψ in the restricted space can be characterized using the operator X as

Kugo and Terao, Phys. Lett. B 208 (1988) 416

$$XY\Psi = \Psi \, .$$

Since $XYX = X$, the operator XY is a projector to the restricted space. This is analogous to

$$B \, c_0^- \, \Psi = \Psi$$

for the closed bosonic string field, and we regard this characterization of the string field in the Ramond sector as fundamental.

If we impose an arbitrary constraint on the string field, there will be no hope of constructing a gauge-invariant action. However, this constraint in the Ramond sector has an interpretation in the context of the super-moduli space. *Can we can introduce interactions which are consistent with this constraint?*

3. The Neveu-Schwarz sector

For open bosonic string field theory, Witten constructed a gauge-invariant action for the interacting theory by introducing a product of string fields A and B called the star product $A*B$. The star product has the following properties:

$$Q\,(\,A*B\,) = QA*B + (-1)^A A*QB\,, \quad (\,A*B\,)*C = A*(\,B*C\,)\,,$$
$$\langle\,A, B*C\,\rangle = \langle\,A*B, C\,\rangle\,.$$

The Chern-Simons-like action

$$S = -\frac{1}{2}\,\langle\,\Psi, Q\Psi\,\rangle - \frac{g}{3}\,\langle\,\Psi, \Psi*\Psi\,\rangle\,,$$

where g is the coupling constant, is invariant under the gauge transformation given by

$$\delta\Psi = Q\Lambda + g\,(\,\Psi*\Lambda - \Lambda*\Psi\,)\,.$$

Consider the cubic interaction of open superstring field theory in the Neveu-Schwarz sector of the following form:

$$-\frac{g}{3}\left\langle \Psi, \Psi * \Psi \right\rangle.$$

$$\rightarrow \begin{cases} \text{the ghost number:} \quad 1+1+1=3 \quad \text{OK!} \\ \text{the picture number:} \quad (-1)+(-1)+(-1) \neq -2 \end{cases}$$

An insertion of a local picture-changing operator at the open-string midpoint *almost* works, but it turned out that the gauge symmetry is singular because of the collision of picture-changing operators.

Berkovits invented a clever way to avoid this problem using the description of the superconformal ghost sector in term of $\xi(z)$, $\eta(z)$, and $\phi(z)$.

The Hilbert space we usually use for the $\beta\gamma$ system is called the small Hilbert space, and the Hilbert space for $\xi(z)$, $\eta(z)$, and $\phi(z)$ called the large Hilbert space is indeed larger, but it can be compensated by an additional gauge symmetry generated by the zero mode of $\eta(z)$ which we denote by η.

The action of the free theory is given by

$$S = -\frac{1}{2} \langle \Phi, Q\eta\Phi \rangle,$$

where Φ is the string field in the large Hilbert space, and it is invariant under the following gauge transformations:

$$\delta\Phi = Q\Lambda + \eta\Omega.$$

These gauge transformations can be nonlinearly extended without using the picture-changing operator in the action. The action and the gauge transformations up to $O(g^2)$ are

$$S = -\frac{1}{2}\langle\, \Phi, Q\eta\Phi\,\rangle - \frac{g}{6}\langle\, \Phi, Q\,[\,\Phi, \eta\Phi\,]\,\rangle$$

$$-\frac{g^2}{24}\langle\, \Phi, Q\,[\,\Phi, [\,\Phi, \eta\Phi\,]\,]\,\rangle + O(g^3)\,,$$

$$\delta\Phi = Q\Lambda + \eta\Omega - \frac{g}{2}\,[\,\Phi, Q\Lambda\,] + \frac{g}{2}\,[\,\Phi, \eta\Omega\,]$$

$$+\frac{g^2}{12}\,[\,\Phi, [\,\Phi, Q\Lambda\,]\,] + \frac{g^2}{12}\,[\,\Phi, [\,\Phi, \eta\Omega\,]\,] + O(g^3)\,.$$

(All products of string fields are defined by the star product and here and in what follows we suppress the star symbol.)

The action to all orders in the coupling constant takes the Wess-Zumino-Witten-like (WZW-like) form: Berkovits, hep-th/9503099

$$S = \frac{1}{2} \langle e^{-\Phi} Q e^{\Phi}, e^{-\Phi} \eta e^{\Phi} \rangle$$
$$- \frac{1}{2} \int_0^1 dt \, \langle e^{-\Phi(t)} \partial_t e^{\Phi(t)}, \{ e^{-\Phi(t)} Q e^{\Phi(t)}, e^{-\Phi(t)} \eta e^{\Phi(t)} \} \rangle,$$

where we set $g = 1$ and Φ is the value of $\Phi(t)$ at $t = 1$. The action can also be written as

$$S = - \int_0^1 dt \, \langle A_t(t), Q A_\eta(t) \rangle$$

with

$$A_\eta(t) = (\eta e^{\Phi(t)}) e^{-\Phi(t)}, \qquad A_t(t) = (\partial_t e^{\Phi(t)}) e^{-\Phi(t)}.$$

The t dependence is topological, and the action is a functional of Φ.

The action is invariant under the gauge transformations given by

$$A_\delta = Q\Lambda + D_\eta \Omega \,,$$

where

$$A_\delta = \left(\delta e^\Phi \right) e^{-\Phi} \,,$$

and D_η is the "covariant derivative" with respect to the gauge transformation generated by η:

$$D_\eta A = \eta A - A_\eta \, A + (-1)^A \, A \, A_\eta \quad \text{with} \quad A_\eta = \left(\eta e^\Phi \right) e^{-\Phi} \,.$$

4. The Ramond sector

Let us now construct a gauge-invariant action including the Ramond sector. We choose the action of the free theory to be

$$S^{(0)} = -\frac{1}{2} \langle\, \Phi, Q\eta\Phi \,\rangle - \frac{1}{2} \langle\!\langle\, \Psi, YQ\Psi \,\rangle\!\rangle \,,$$

where $\langle\, A, B \,\rangle$ is the BPZ inner product in the large Hilbert space and $\langle\!\langle\, A, B \,\rangle\!\rangle$ is the BPZ inner product in the small Hilbert space. The gauge transformations of the free theory are

$$\delta^{(0)}\Phi = Q\Lambda + \eta\Omega \,,$$
$$\delta^{(0)}\Psi = Q\lambda \,.$$

Consider the action with the following cubic interactions:

$$S = S^{(0)} + g\, S^{(1)} + O(g^2) \,,$$

where

$$S^{(1)} = -\frac{1}{6} \langle\, \Phi, Q\,[\, \Phi, \eta\Phi \,] \,\rangle - \langle\, \Phi, \Psi^2 \,\rangle \,.$$

The operator X can be written as $X = \{\,Q, \Xi\,\}$ in the large Hilbert space, and the action up to this order is invariant under the gauge transformation given by

$$\delta \Phi = \delta^{(0)}\Phi + g\,\delta^{(1)}\Phi + O(g^2)\,,$$

$$\delta \Psi = \delta^{(0)}\Psi + g\,\delta^{(1)}\Psi + O(g^2)\,,$$

where

$$\delta^{(0)}\Phi = Q\Lambda + \eta\Omega\,,$$

$$\delta^{(1)}\Phi = -\frac{1}{2}\,[\,\Phi, Q\Lambda\,] + \frac{1}{2}\,[\,\Phi, \eta\Omega\,] - \{\Psi, \Xi\lambda\,\}\,,$$

$$\delta^{(0)}\Psi = Q\lambda\,,$$

$$\delta^{(1)}\Psi = X\eta\,\{\,\Psi, \Lambda\,\} - X\eta\,\{\,\eta\Phi, \Xi\lambda\,\}\,.$$

Because $XYX = X$, $\delta\Psi$ is consistent with the projection:

$$XY\delta\Psi = \delta\Psi\,.$$

We expand the action and the gauge transformations up to $O(g^2)$ as

$$S = S^{(0)} + g\, S^{(1)} + g^2\, S^{(2)} + O(g^3)\,,$$

and

$$\delta\Phi = \delta^{(0)}\Phi + g\, \delta^{(1)}\Phi + g^2\, \delta^{(2)}\Phi + O(g^3)\,,$$
$$\delta\Psi = \delta^{(0)}\Psi + g\, \delta^{(1)}\Psi + g^2\, \delta^{(2)}\Psi + O(g^3)\,.$$

The action is given by

$$S^{(0)} = -\frac{1}{2}\,\langle\, \Phi, Q\eta\Phi\,\rangle - \frac{1}{2}\,\langle\!\langle\, \Psi, YQ\Psi\,\rangle\!\rangle\,,$$
$$S^{(1)} = -\frac{1}{6}\,\langle\, \Phi, Q\,[\,\Phi, \eta\Phi\,]\,\rangle - \langle\, \Phi, \Psi^2\,\rangle\,,$$
$$S^{(2)} = -\frac{1}{24}\,\langle\, \Phi, Q\,[\,\Phi, [\,\Phi, \eta\Phi\,]\,]\,\rangle - \frac{1}{2}\,\langle\, \Phi, \{\Psi, \Xi\,\{\eta\Phi, \Psi\}\}\,\rangle\,.$$

The gauge transformations are

$$\delta^{(0)}\Phi = Q\Lambda + \eta\Omega\,,$$

$$\delta^{(1)}\Phi = -\frac{1}{2}\left[\Phi, Q\Lambda\right] + \frac{1}{2}\left[\Phi, \eta\Omega\right] - \left\{\Psi, \Xi\lambda\right\}\,,$$

$$\delta^{(2)}\Phi = \frac{1}{12}\left[\Phi, \left[\Phi, Q\Lambda\right]\right] + \left\{\Psi, \Xi\left\{\Psi, \Lambda\right\}\right\} + \frac{1}{12}\left[\Phi, \left[\Phi, \eta\Omega\right]\right]$$

$$- \left\{\Psi, \Xi\left\{\eta\Phi, \Xi\lambda\right\}\right\} - \left\{\Xi\left\{\eta\Phi, \Psi\right\}, \Xi\lambda\right\} + \frac{1}{2}\left[\Phi, \left\{\Psi, \Xi\lambda\right\}\right]\,,$$

$$\delta^{(0)}\Psi = Q\lambda\,,$$

$$\delta^{(1)}\Psi = X\eta\left\{\Psi, \Lambda\right\} - X\eta\left\{\eta\Phi, \Xi\lambda\right\}\,,$$

$$\delta^{(2)}\Psi = X\eta\left\{\Xi\left\{\eta\Phi, \Psi\right\}, \Lambda\right\} + X\eta\left\{\eta\Phi, \Xi\left\{\Psi, \Lambda\right\}\right\}$$

$$- X\eta\left\{\eta\Phi, \Xi\left\{\eta\Phi, \Xi\lambda\right\}\right\} - \frac{1}{2}X\eta\left\{\left[\Phi, \eta\Phi\right], \Xi\lambda\right\}\,.$$

The complete action S is given by

$$S = -\frac{1}{2} \langle\!\langle \Psi, YQ\Psi \rangle\!\rangle - \int_0^1 dt \, \langle A_t(t), QA_\eta(t) + (F(t)\Psi)^2 \rangle,$$

where

$$F(t)\Psi = \Psi + \Xi\{A_\eta(t), \Psi\} + \Xi\{A_\eta(t), \Xi\{A_\eta(t), \Psi\}\} + \dots$$

$$= \sum_{n=0}^{\infty} \underbrace{\Xi\{A_\eta(t), \Xi\{A_\eta(t), \dots, \Xi\{A_\eta(t), \Psi\} \dots\}\}}_{n}.$$

It is invariant under the gauge transformations given by

$$A_\delta = Q\Lambda + D_\eta\Omega + \{F\Psi, F\Xi(\{F\Psi, \Lambda\} - \lambda)\},$$
$$\delta\Psi = Q\lambda + X\eta\, F\Xi\, D_\eta(\{F\Psi, \Lambda\} - \lambda).$$

The action of F is defined by

$$FA = A + \Xi\,[\,A_\eta, A\,] + \Xi\,[\,A_\eta, \Xi\,[\,A_\eta, A\,]\,] + \dots$$

$$= \sum_{n=0}^{\infty} \underbrace{\Xi\,[\,A_\eta, \Xi\,[\,A_\eta, \dots, \Xi\,[\,A_\eta}_{n}, A\,]\dots]\,]$$

when A is a Grassmann-even state and

$$FA = A + \Xi\,\{\,A_\eta, A\,\} + \Xi\,\{\,A_\eta, \Xi\,\{\,A_\eta, A\,\}\,\} + \dots$$

$$= \sum_{n=0}^{\infty} \underbrace{\Xi\,\{\,A_\eta, \Xi\,\{\,A_\eta, \dots, \Xi\,\{\,A_\eta}_{n}, A\,\}\dots\}\,\}$$

when A is a Grassmann-odd state.

5. Future directions

We constructed a gauge-invariant action of open superstring field theory including both the Neveu-Schwarz sector and the Ramond sector. This is the first construction of a complete action of superstring field theory in a covariant form.

- Quantization of open superstring field theory

- Generalization to closed superstring field theory

- Integration of the large Hilbert space and the supermoduli space

- The relation to the approach by Sen in arXiv:1508.05387

We hope that these exciting developments in superstring field theory will help us unveil the nature of the nonperturbative theory underlying the perturbative superstring theory.