$\mathbb{RP}^2$ index and its applications

AKINORI TANAKA

(RIKEN)

Based on

A. T, H. Mori, and T. Morita, Phys. Rev. D91 (2015) 105023, arXiv:1408.3371 [hep-th].

A. T, H. Mori, and T. Morita, JHEP 09 (2015) 154, arXiv:1505.07539 [hep-th].

H. Mori, A.T., arXiv:151x.xxxxx [hep-th].

(2+1)d Index

$$I = \mathrm{Tr}_{\mathcal{H}}(-1)^F q^{\frac{1}{2}(H+j_3)}$$

- Exactly calculable (Localization)

Y. Imamura & S. Yokoyama(2011)

$$\mathcal{H} \circlearrowleft^{\text{4-supercharges}}$$

A.T, H. Mori & T. Morita(2014-2015)

$$\mathcal{H} \circlearrowleft^{\text{4-supercharges}} \quad /\mathbb{Z}_2$$

$\mathcal{P}$ or $\mathcal{CP}$

Applications $/\mathbb{Z}_2$

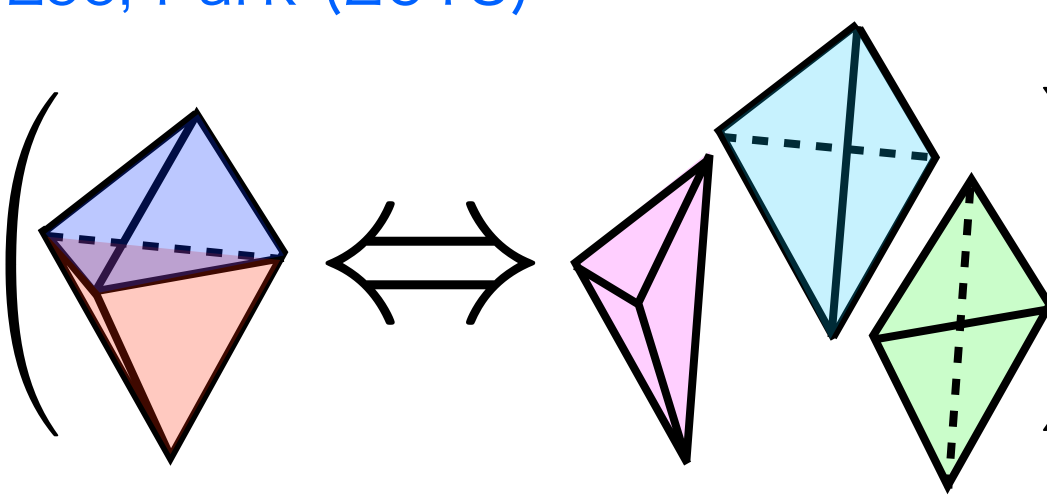
- Explicit checks of 3d duality

Krattenthaler, Spiridonov, Vartanov (2011), Kapustin, Willett (2011)

(3d mirror symmetry (1-flavor)) = (q-binomial Formula + Ramanujan's Sum)

- 3d-3d correspondence

Dimofte, Gaiotto, Gukov (2011), Gang, Koh, Lee, Park (2013)

(3d mirror symmetry (1-flavor)) = ()

Today's talk

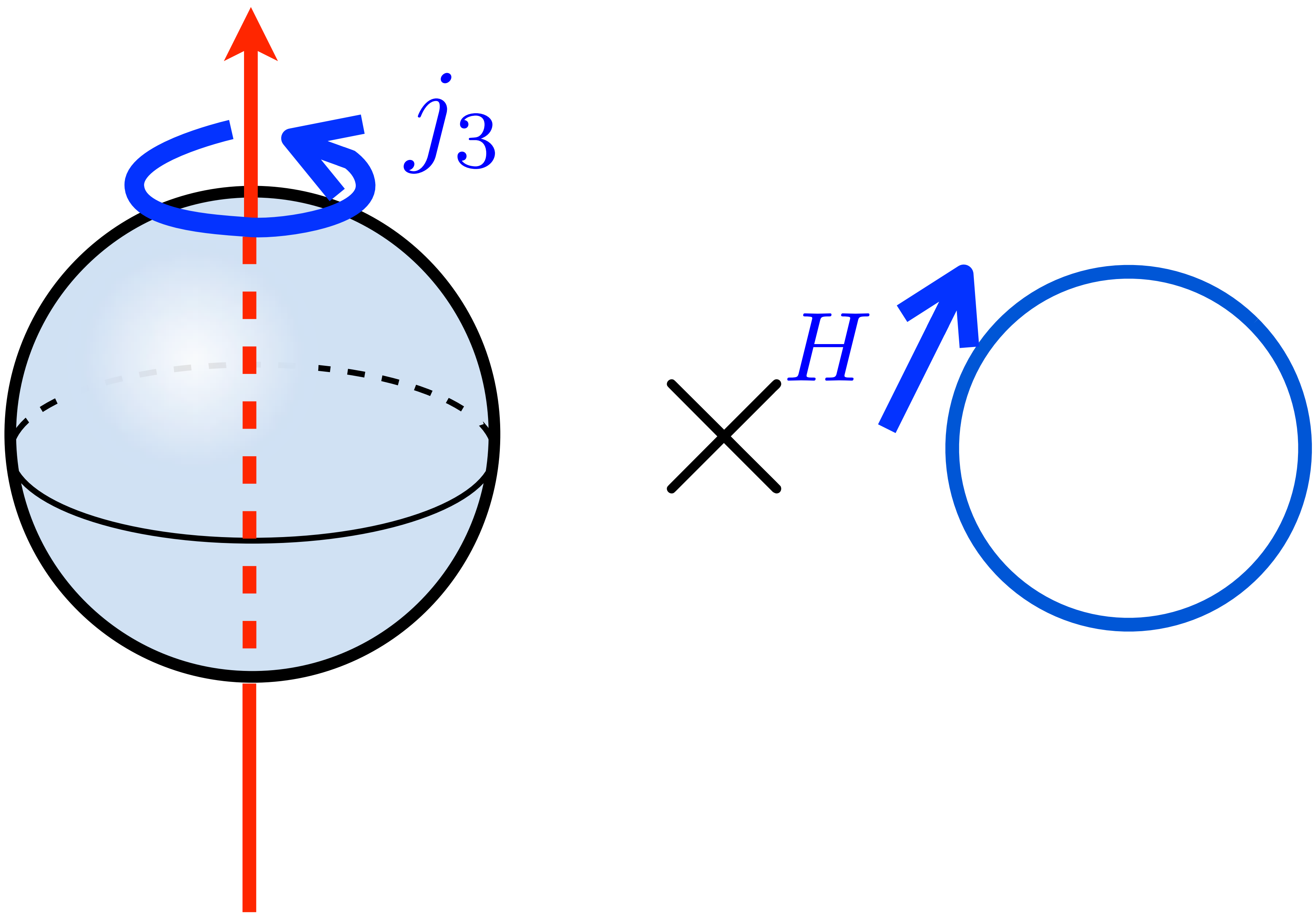
- Index formula (review)
- $\mathbb{RP}^2$ index formula
- Role of $\mathbb{Z}_2$ in 3d duality

· Index formula (review)

Definition

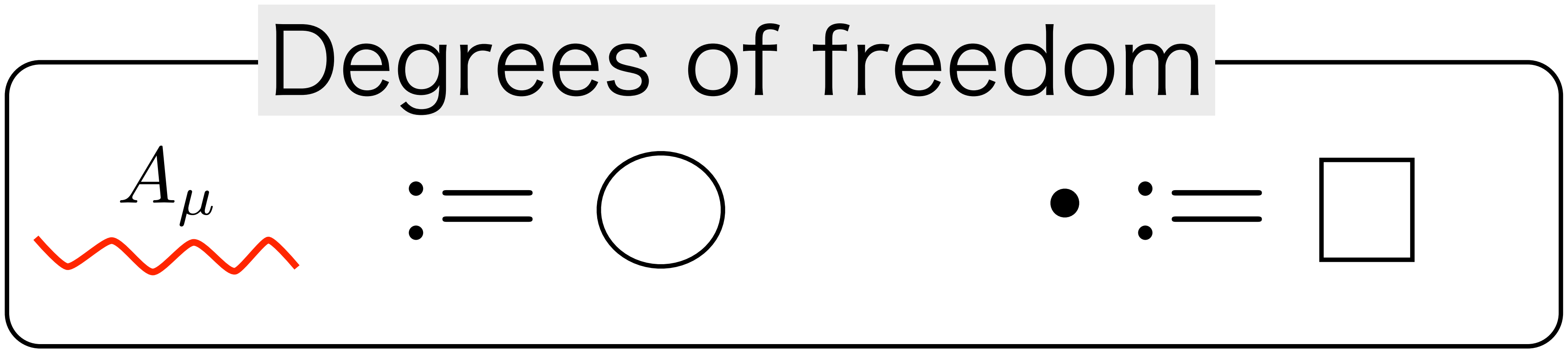
$$I = \mathrm{Tr}_{\mathcal{H}}(-1)^F q^{\frac{1}{2}(H+j_3)}$$

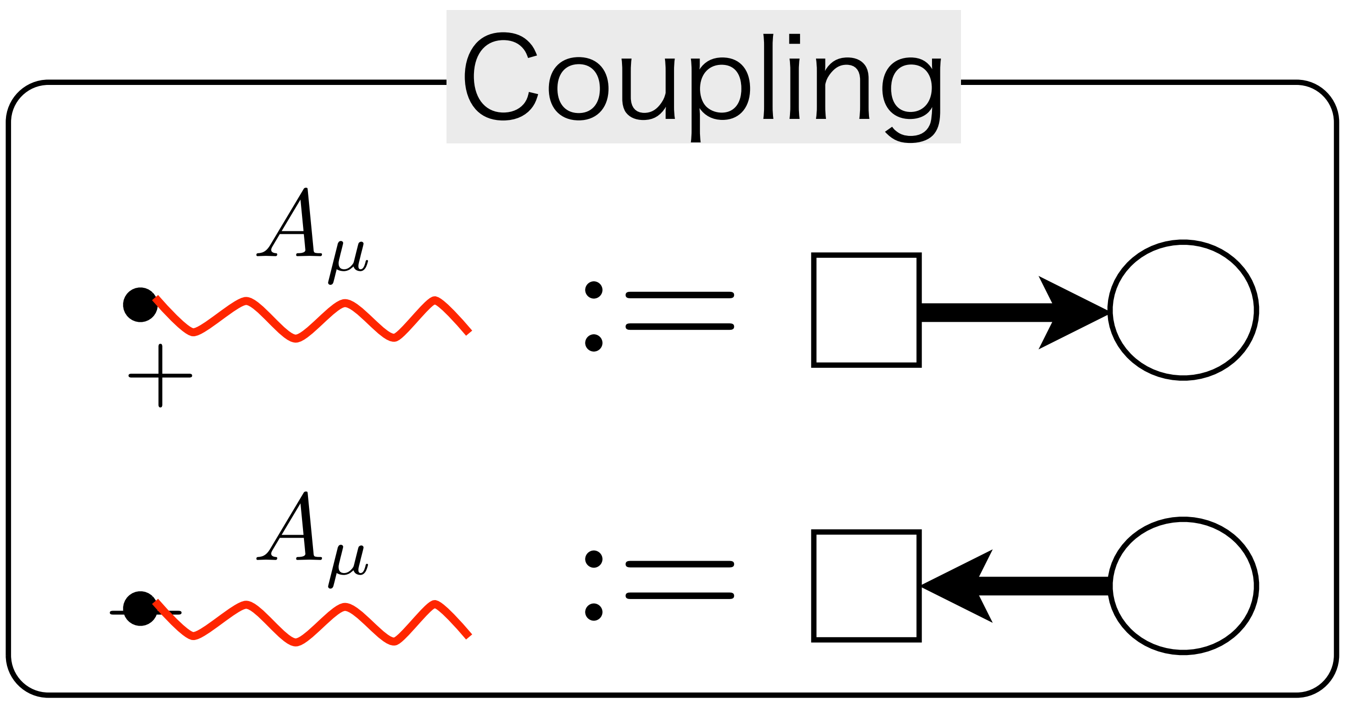
SUSY QFT on

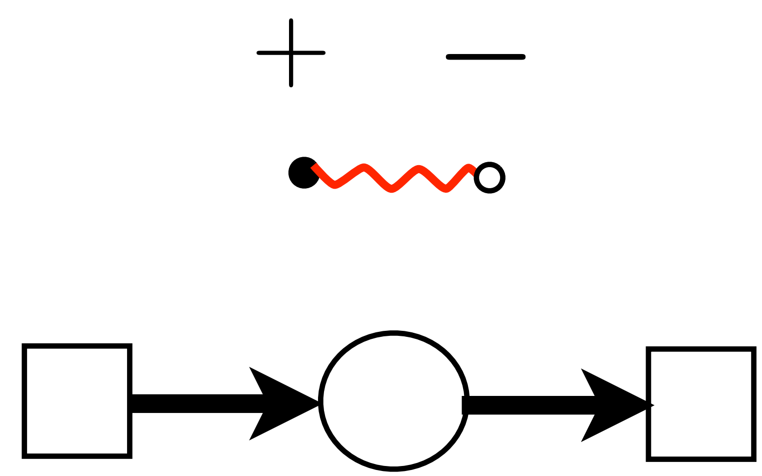


・ Index formula (review)

Our notation for quiver diagram







$$
\begin{aligned}
&-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\dots\\
&+\bar{\psi}_1(\partial\!\!\!/-iA\!\!\!/)\psi_1+\dots\\
&+\bar{\psi}_2(\partial\!\!\!/+iA\!\!\!/)\psi_2+\dots
\end{aligned}
$$

・ Index formula (review)

Localization formula (2-steps)

Y. Imamura & S. Yokoyama(2011)

$$I = \mathrm{Tr}_{\mathcal{H}}(-1)^F q^{\frac{1}{2}(H+j_3)}$$

STEP1

$$\bigcirc = \sum_{2s=-\infty}^{\infty} \frac{1}{2\pi}\int_0^{2\pi} d\theta \ \textcircled{s,\theta}$$

STEP2

$$\square \dashrightarrow \textcircled{s,\theta} = \left(q^{\frac{1-\Delta}{2}}\right)^s \frac{(e^{-i\boldsymbol{Q}\theta} q^{|\boldsymbol{Q}s| + \frac{2-\Delta}{2}}; q)_\infty}{(e^{+i\boldsymbol{Q}\theta} q^{|\boldsymbol{Q}s| + \frac{\Delta}{2}}; q)_\infty}$$

・ Index formula (review)

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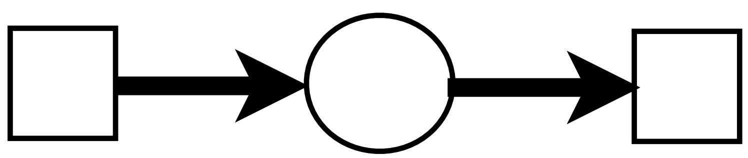
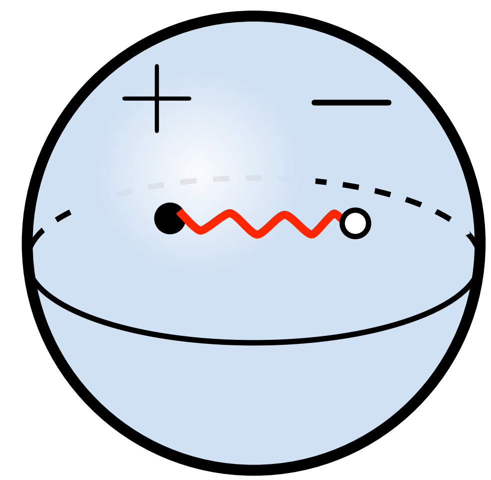
Example





$$\bigcirc = \sum_{2s=-\infty}^{\infty} \frac{1}{2\pi}\int_0^{2\pi} d\theta \; (s,\theta)$$

・ Index formula (review)

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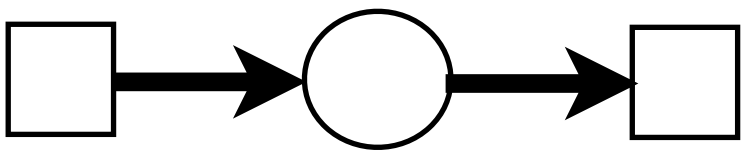
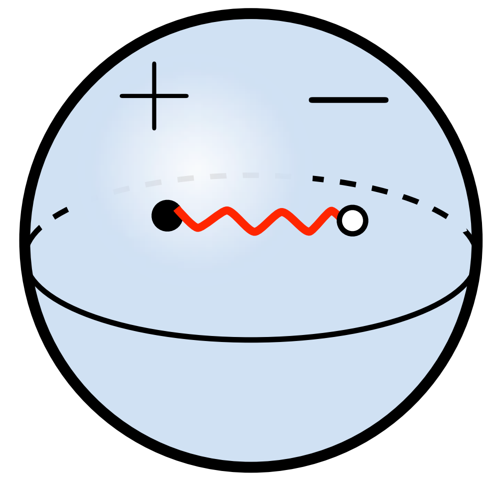
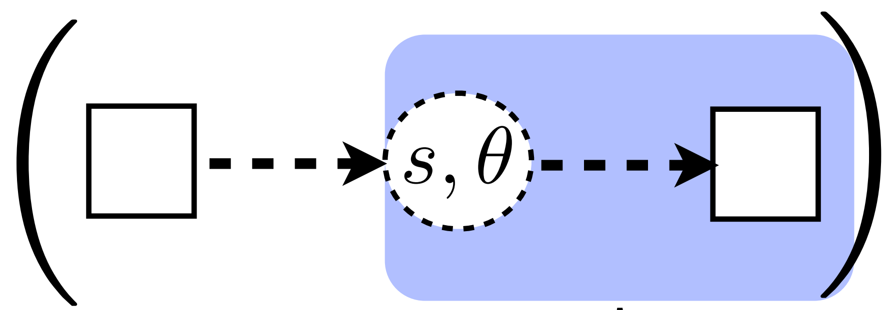
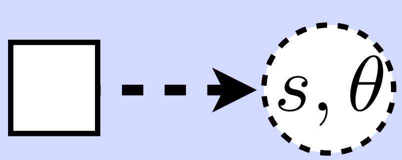




$$= \sum_{2s=-\infty}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} d\theta$$





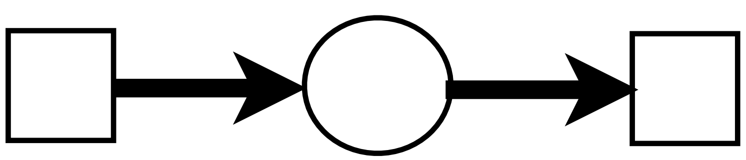
$$= \left(q^{\frac{1-\Delta}{2}}\right)^s \frac{(e^{-i\boldsymbol{Q}\theta} q^{|\boldsymbol{Q}s|+\frac{2-\Delta}{2}}; q)_\infty}{(e^{+i\boldsymbol{Q}\theta} q^{|\boldsymbol{Q}s|+\frac{\Delta}{2}}; q)_\infty}$$

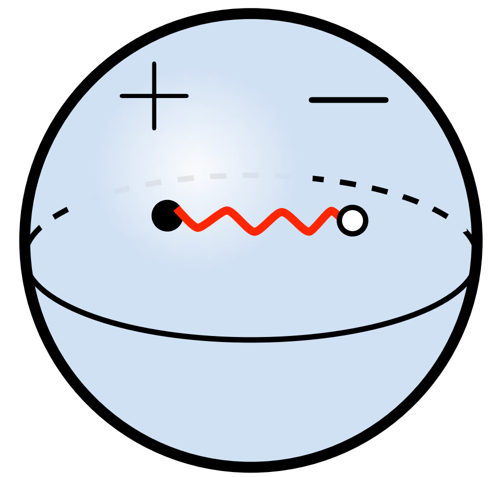
・ Index formula (review)

Localization formula (2-steps) Y. Imamura & S. Yokoyama(2011)

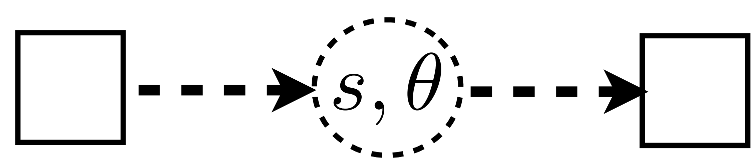
$$I = \mathrm{Tr}_{\mathcal{H}}(-1)^F q^{\frac{1}{2}(H+j_3)}$$

Example





$$= \sum_{2s=-\infty}^{\infty} \frac{1}{2\pi}\int_0^{2\pi} d\theta \Bigg(\quad \Bigg)$$



$$z = e^{i\theta}$$

$$= \sum_{2s=-\infty}^{\infty} \oint \frac{dz}{2\pi i z}\left(q^{\frac{1-\Delta}{2}}\right)^{|2s|} \frac{(z^{-1}q^{|s|+\frac{2-\Delta}{2}};q)_\infty}{(zq^{|s|+\frac{0+\Delta}{2}};q)_\infty} \times \frac{(zq^{|s|+\frac{2-\Delta}{2}};q)_\infty}{(z^{-1}q^{|s|+\frac{0+\Delta}{2}};q)_\infty}$$

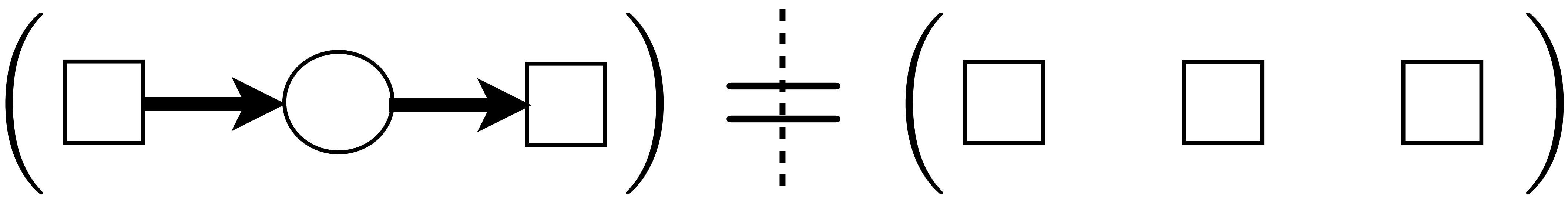
• Index formula (review)

Localization formula (2-steps)

Y. Imamura & S. Yokoyama(2011)

$$I = \mathrm{Tr}_{\mathcal{H}}(-1)^F q^{\frac{1}{2}(H+j_3)}$$

Check: 3d mirror symmetry



$$\mathrm{Series}\left[\mathrm{Sum}\left[q^{\mathrm{Abs}[k]/4}\frac{(\mathrm{QPochhammer}[q^{(\mathrm{Abs}[k]+j+1)}, q])}{(\mathrm{QPochhammer}[q^{-1/2+(\mathrm{Abs}[k]+j+1)}, q])}\frac{(\mathrm{QPochhammer}[q^{(1-2j)/2}, q])}{(\mathrm{QPochhammer}[q, q])}\frac{q^{j(j+1)/2}(-1)^j}{(\mathrm{QPochhammer}[q, q, j])}, \{j, 0, 20, 1\}, \{k, -50, 50, 1\}\right], \{q, 0, 3\}\right]$$

$$1 + 2q^{1/4} + 3\sqrt{q} + 2q^{3/4} + q + 2q^{5/4} + 4q^{3/2} + 4q^{7/4} - 2q^{9/4} + 3q^{5/2} + 6q^{11/4} + 2q^3 + O[q]^{13/4}$$

$$\mathrm{Series}\left[\frac{(\mathrm{QPochhammer}[q^{3/4}, q])^2}{(\mathrm{QPochhammer}[q^{1/4}, q])^2}, \{q, 0, 3\}\right]$$

$$1 + 2q^{1/4} + 3\sqrt{q} + 2q^{3/4} + q + 2q^{5/4} + 4q^{3/2} + 4q^{7/4} - 2q^{9/4} + 3q^{5/2} + 8q^{11/4} + 5q^3 + O[q]^{13/4}$$

・ Index formula (review)

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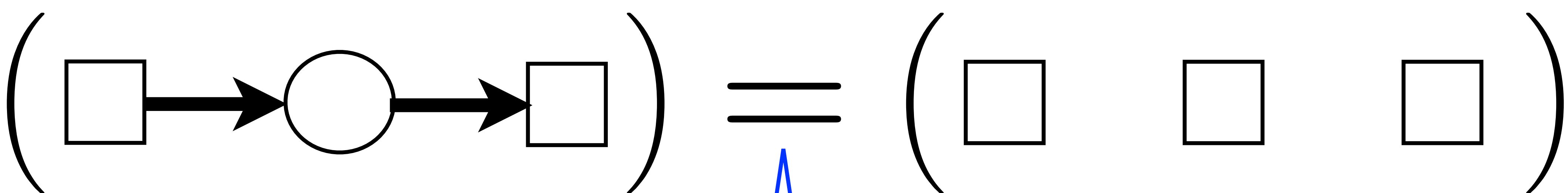


q-binomial theorem

+

Ramanujan's sum formula

Krattenthaler, Spiridonov, Vartanov (2011),
Kapustin, Willett (2011)

Today's talk

✔ • Index formula (review) $\bigcirc = \sum_s \int d\theta$

• $\mathbb{RP}^2$ index formula

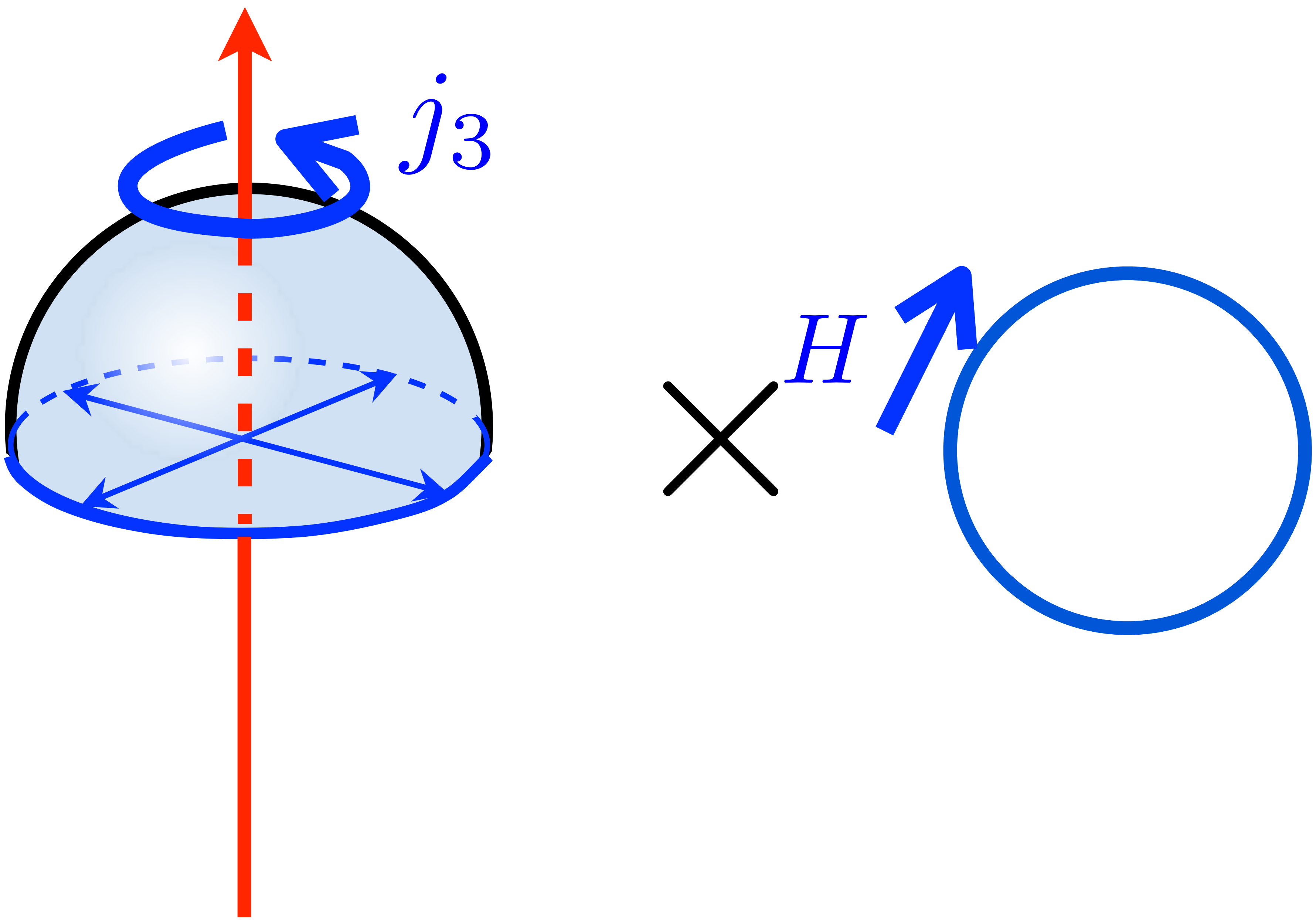
• Role of $\mathbb{Z}_2$ in 3d duality

· $\mathbb{RP}^2$ index formula

Definition

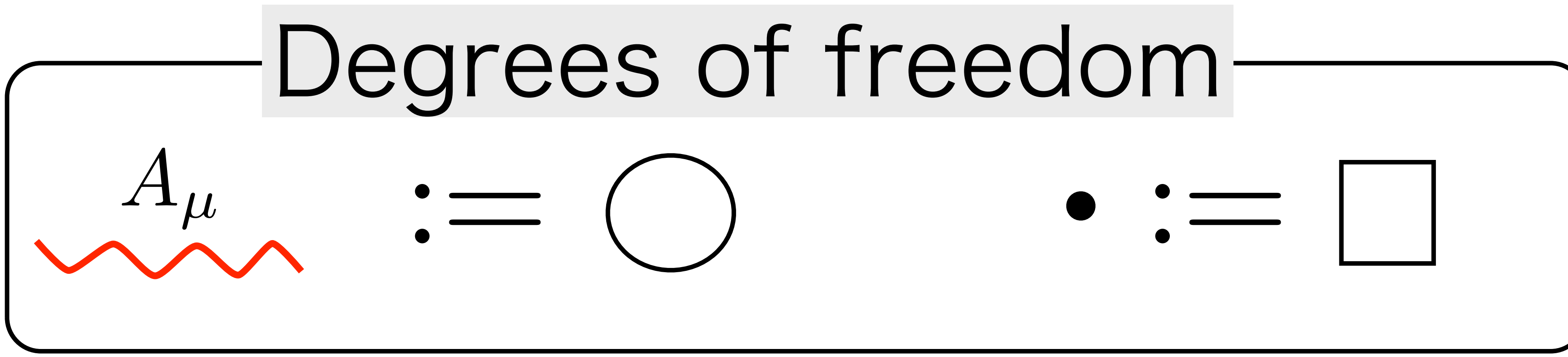
$$I = \mathrm{Tr}_{\mathcal{H}/\mathbb{Z}_2}(-1)^F q^{\frac{1}{2}(H+j_3)}$$

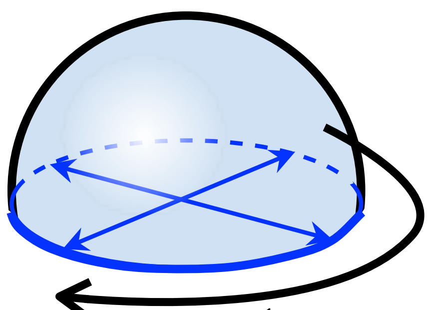
SUSY QFT on



· $\mathbb{RP}^2$ index formula

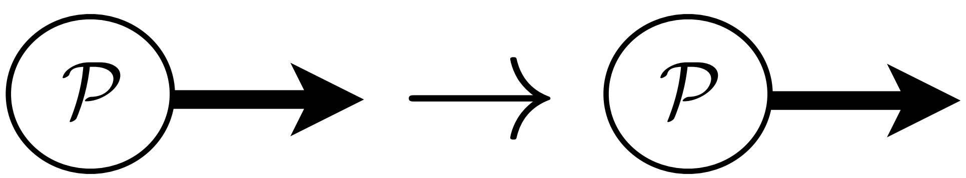
Our notation for quiver diagram



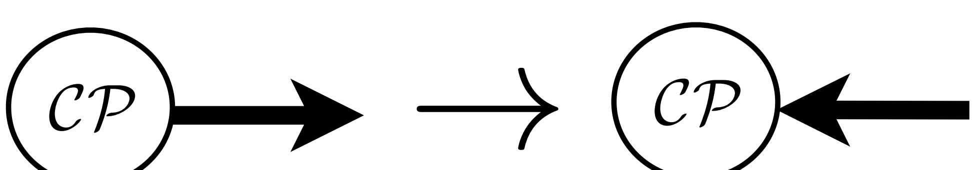


$\Rightarrow$ SUSY parity conds

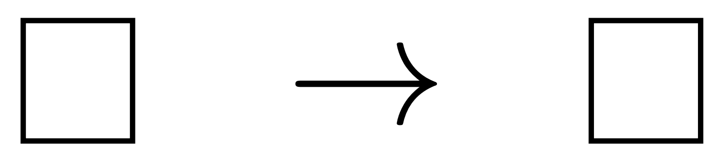
- $(\partial_\mu - iA_\mu) \to \pm(\partial_\mu - iA_\mu)$



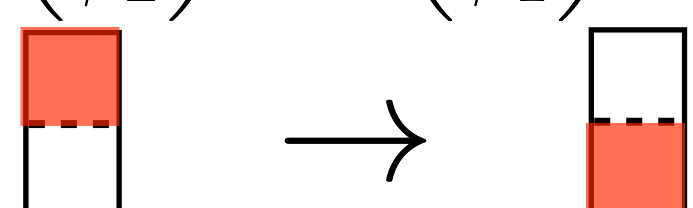
- $(\partial_\mu - iA_\mu) \to \pm(\partial_\mu + iA_\mu)$



- $\phi \to \phi$

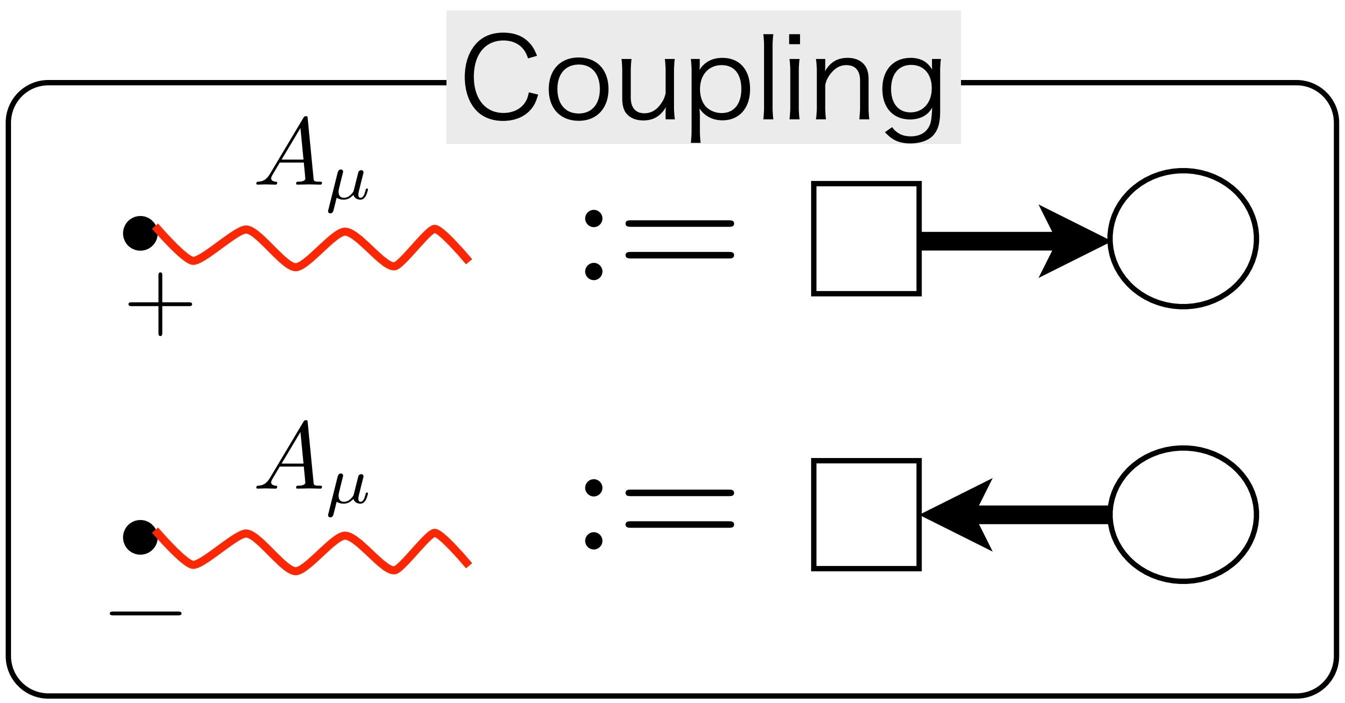


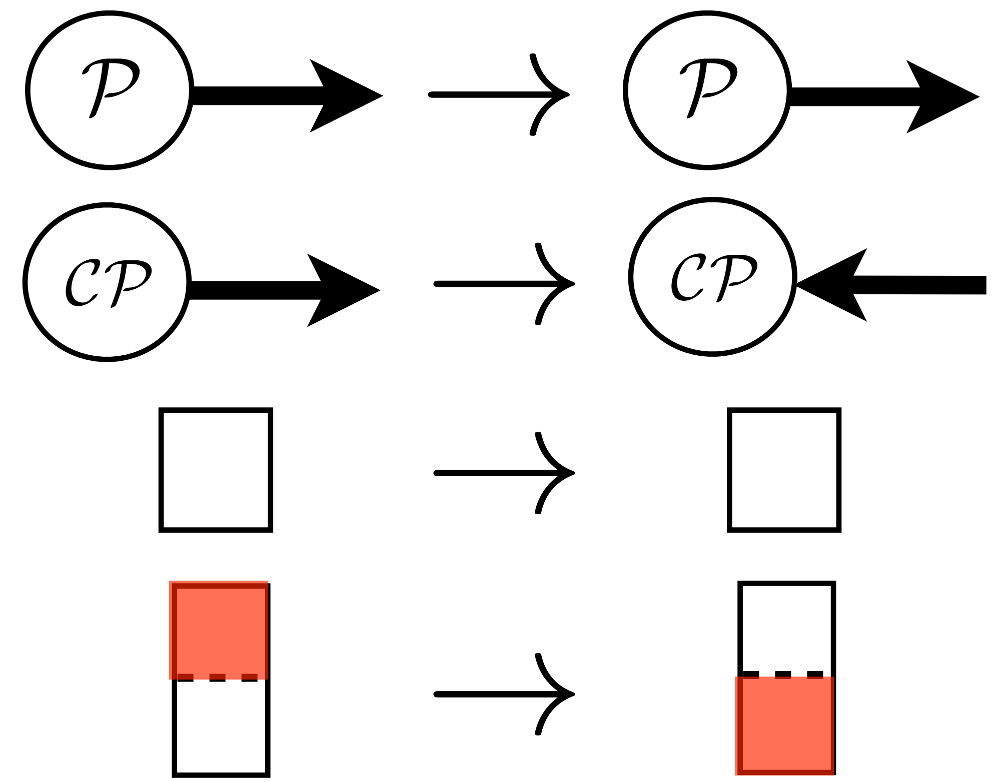
- $\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \to \begin{pmatrix} \phi_2 \\ \phi_1 \end{pmatrix}$

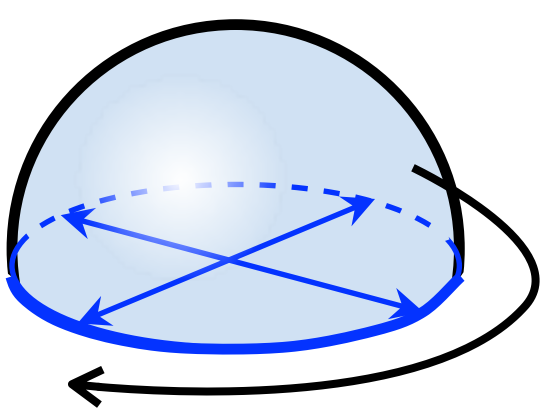


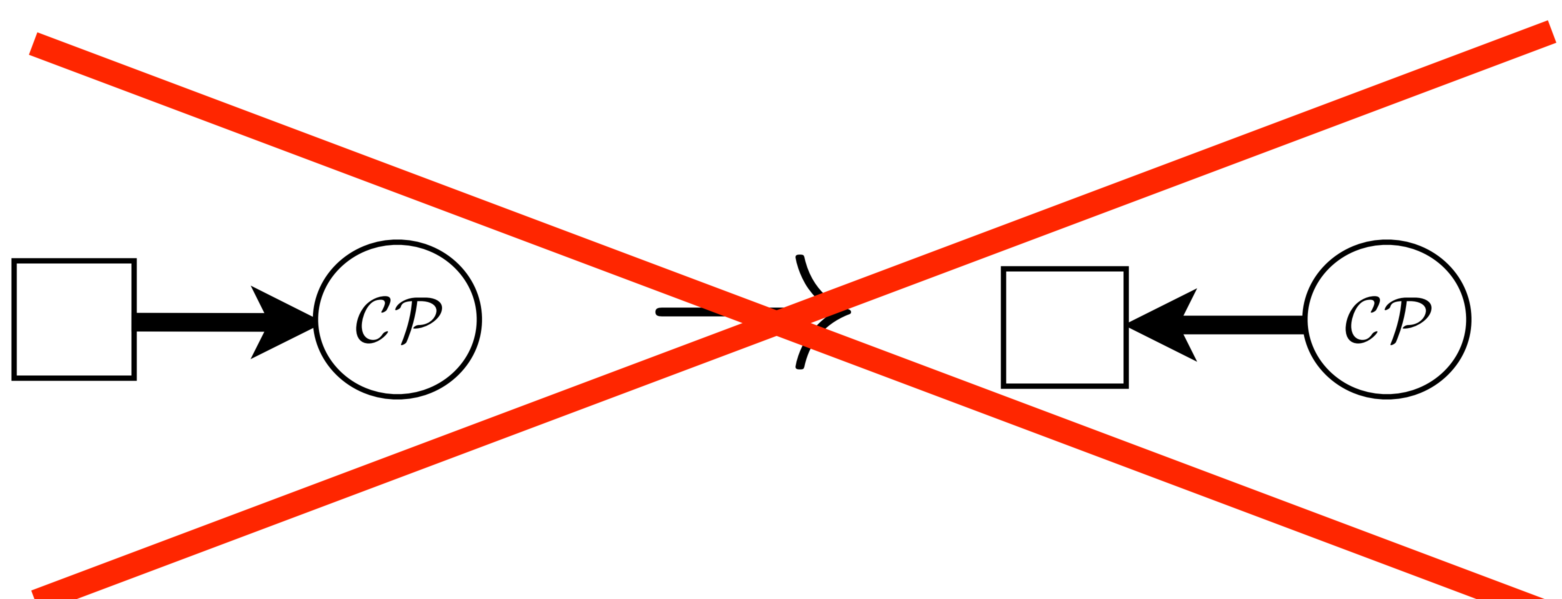
· $\mathbb{RP}^2$ index formula

Our notation for quiver diagram



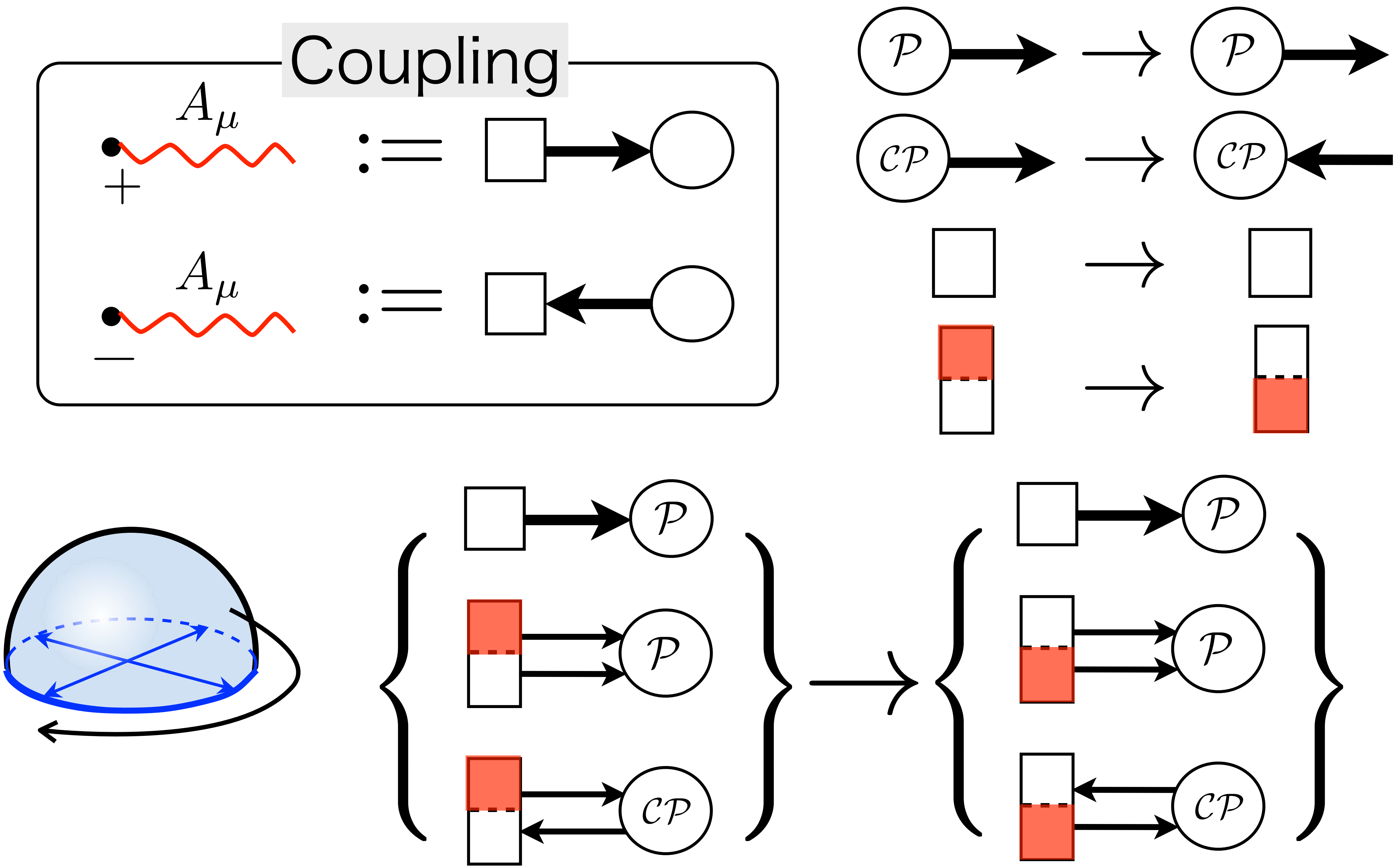






· $\mathbb{RP}^2$ index formula

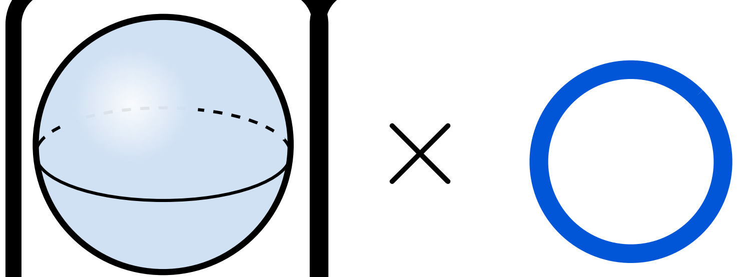
Our notation for quiver diagram



· $\mathbb{RP}^2$ index formula

Localization formula (2-steps)

$$I = \mathrm{Tr}_{\mathcal{H}/\mathbb{Z}_2}(-1)^F q^{\frac{1}{2}(H+j_3)}$$



STEP1

$$\bigcirc = \sum_{2s=-\infty}^{\infty} \frac{1}{2\pi}\int_0^{2\pi} d\theta \; \left(s,\theta\right)$$

$$\left(\mathcal{P}\right) = q^{+\frac{1}{8}} \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \sum_{s^\pm=0,1} \frac{1}{2\pi}\int_0^{2\pi} d\theta \; \left(s^\pm,\theta\right)$$

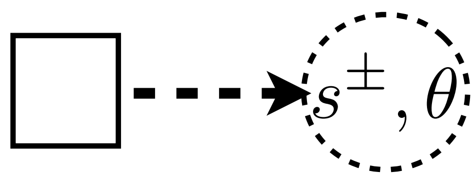
$$\left(\mathcal{CP}\right) = q^{-\frac{1}{8}} \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{s=-\infty}^{\infty} \frac{1}{2} \sum_{\theta_\pm=0,\pi} \left(s,\theta_\pm\right)$$

· $\mathbb{RP}^2$ index formula

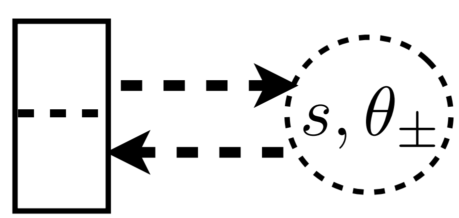
Localization formula (2-steps)

$$I = \mathrm{Tr}_{\mathcal{H}/\mathbb{Z}_2}(-1)^F q^{\frac{1}{2}(H+j_3)}$$

STEP2



$$= \begin{cases} \left(q^{\frac{\Delta-1}{8}} e^{\frac{iQ\theta}{4}}\right)^{+1} \frac{(e^{-iQ\theta} q^{\frac{2-\Delta}{2}}; q^2)_\infty}{(e^{+iQ\theta} q^{\frac{0+\Delta}{2}}; q^2)_\infty} & \text{for } e^{i\pi Q s^{\pm}} = +1, \\ \left(q^{\frac{\Delta-1}{8}} e^{\frac{iQ\theta}{4}}\right)^{-1} \frac{(e^{-iQ\theta} q^{\frac{4-\Delta}{2}}; q^2)_\infty}{(e^{+iQ\theta} q^{\frac{2+\Delta}{2}}; q^2)_\infty} & \text{for } e^{i\pi Q s^{\pm}} = -1, \end{cases}$$



$$= \left(q^{\frac{1-\Delta}{2}}\right)^{|Qs|} \frac{\left(e^{-iQ\theta_{\pm}} q^{|Qs| + \frac{(2-\Delta)}{2}}; q\right)_\infty}{\left(e^{+iQ\theta_{\pm}} q^{|Qs| + \frac{(0+\Delta)}{2}}; q\right)_\infty}$$

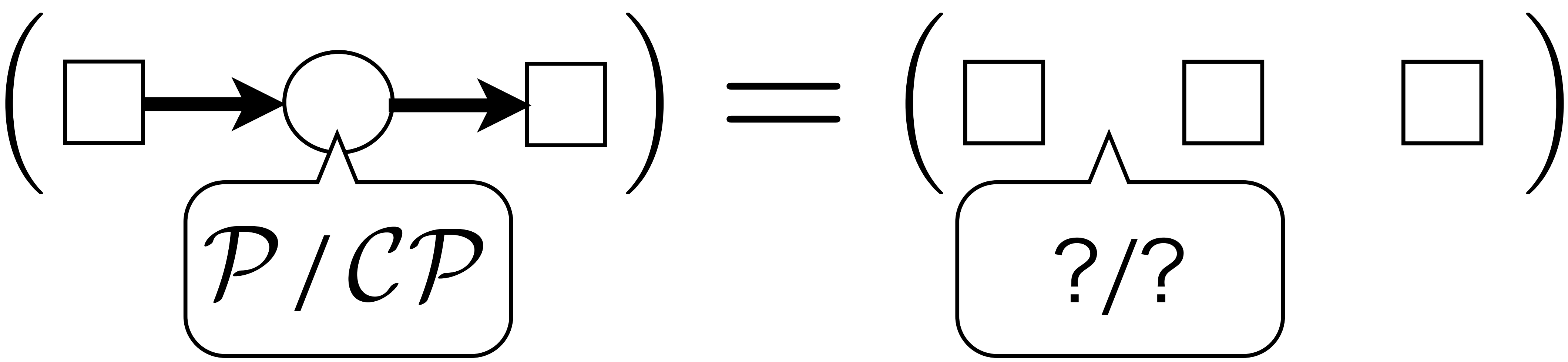
etc.

- $\mathbb{RP}^2$ index formula

Localization formula (2-steps)

$$I = \mathrm{Tr}_{\mathcal{H}/\mathbb{Z}_2}(-1)^F q^{\frac{1}{2}(H+j_3)}$$

Check: 3d mirror symmetry



· $\mathbb{RP}^2$ index formula

Localization formula (2-steps)

$-1)^F q^{\frac{1}{2}(H+j_3)}$

C … etry

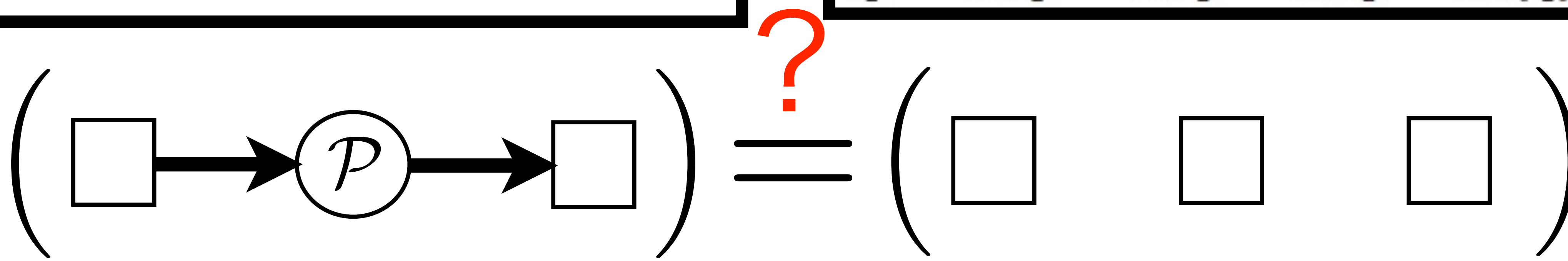
$$\texttt{Series}\Big[q^{1/8}\frac{(\texttt{QPochhammer}[q^2,q^2])}{(\texttt{QPochhammer}[q,q^2])}\Big(q^{-1/8}\frac{(\texttt{QPochhammer}[q^{1/2},q^2])}{(\texttt{QPochhammer}[q^{1/2},q^2])}\frac{(\texttt{QPochhammer}[q,q^2])}{(\texttt{QPochhammer}[q^2,q^2])}\texttt{QHypergeometricPFQ}[\{q^{3/2},q^{1/2}\},\{q\},q^2,q^{1/2}] + q^{+1/8}\frac{(\texttt{QPochhammer}[q^{1/2},q^2])}{(\texttt{QPochhammer}[q^{5/2},q^2])}\frac{(\texttt{QPochhammer}[q^3,q^2])}{(\texttt{QPochhammer}[q^2,q^2])}\texttt{QHypergeometricPFQ}[\{q^{3/2},q^{5/2}\},\{q^3\},q^2,q^{1/2}]\Big),\ \{q,0,3\}\Big]$$

$$1+q^{1/4}+\sqrt{q}+q^{5/4}+q^{3/2}-q^2+2q^{5/2}+q^{11/4}-q^3+O[q]^{25/8}$$

$=$

$$\texttt{Series}\Big[q^{(2-1/2)/8}\frac{(\texttt{QPochhammer}[q^{3/4},q^2])^2}{(\texttt{QPochhammer}[q^{1/4},q^2])^2},\ \{q,$$

$$q^{3/16}+2q^{7/16}+3q^{11/16}+2q^{15/16}+q^{19/16}+2q^{39/16}+4q^{43/16}+4q^{47/16}+O[q]^{51/16}$$



· $\mathbb{RP}^2$ index formula

Localization formula (2-steps)

$-1)^F q^{\frac{1}{2}(H+j_3)}$

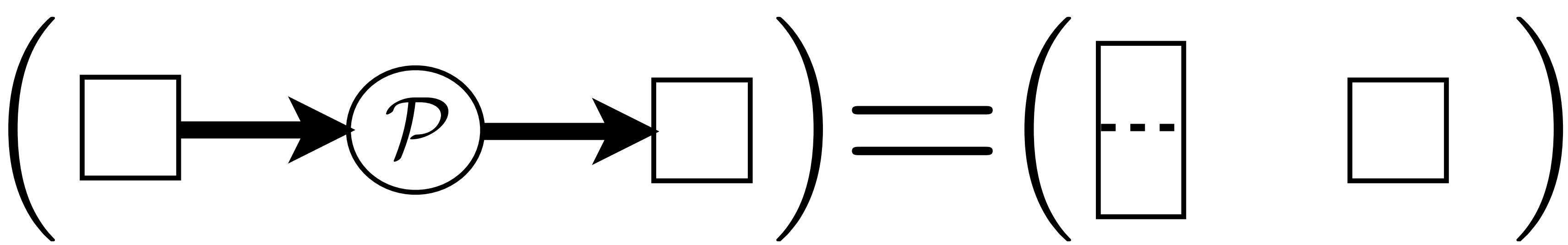
C … ne … =

```
Series[q^(1/8) (QPochhammer[q^2, q^2])/(QPochhammer[q, q^2])
 (q^(-1/8) (QPochhammer[q^(1/2), q^2])/(QPochhammer[q^(1/2), q^2]) (QPochhammer[q, q^2])/(QPochhammer[q^2, q^2])
   QHypergeometricPFQ[{q^(3/2), q^(1/2)}, {q}, q^2, q^(1/2)] +
  q^(+1/8) (QPochhammer[q^(1/2), q^2])/(QPochhammer[q^(5/2), q^2]) (QPochhammer[q^3, q^2])/(QPochhammer[q^2, q^2])
   QHypergeometricPFQ[{q^(3/2), q^(5/2)}, {q^3}, q^2, q^(1/2)]),
 {q, 0, 3}]
```

$1+q^{1/4}+\sqrt{q}+q^{5/4}+q^{3/2}-q^2+2q^{5/2}+q^{11/4}-q^3+O[q]^{25/8}$

```
Simplify[
 Series[ QPochhammer[q^((1+1/2)/2), q]/QPochhammer[q^((1-1/2)/2), q]
  (q^((2/2-1)/8) QPochhammer[q^((2-2/2)/2), q^2]/QPochhammer[q^(1/2), q^2]), {q, 0, 3}]]
```

$1+q^{1/4}+\sqrt{q}+q^{5/4}+q^{3/2}-q^2+2q^{5/2}+q^{11/4}-q^3+O[q]^{13/4}$



· $\mathbb{RP}^2$ index formula

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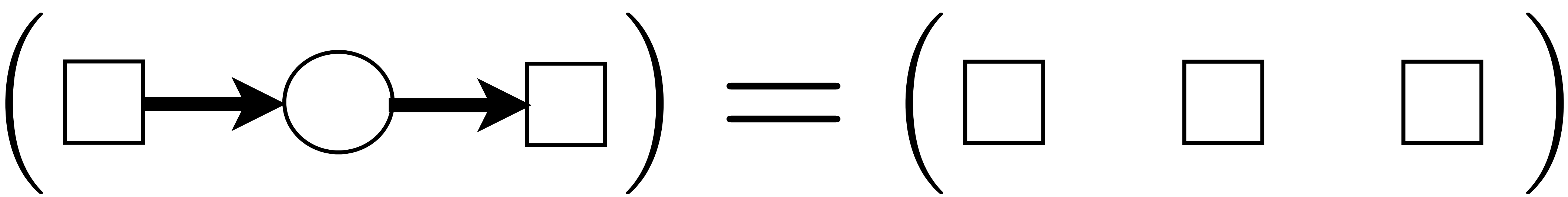
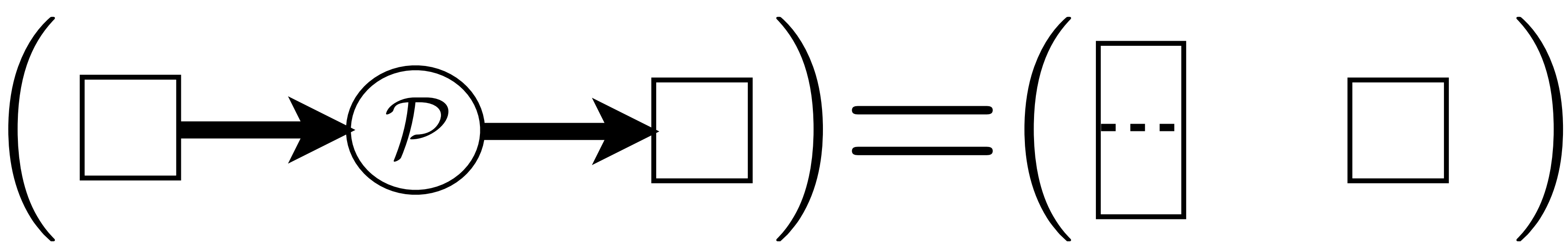
Check: 3d mirror symmetry



A.T, H. Mori, T. Morita (2014)



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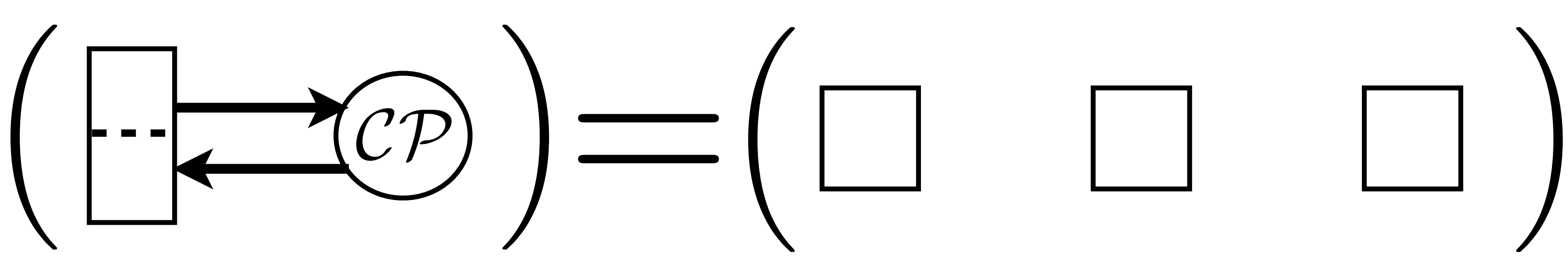
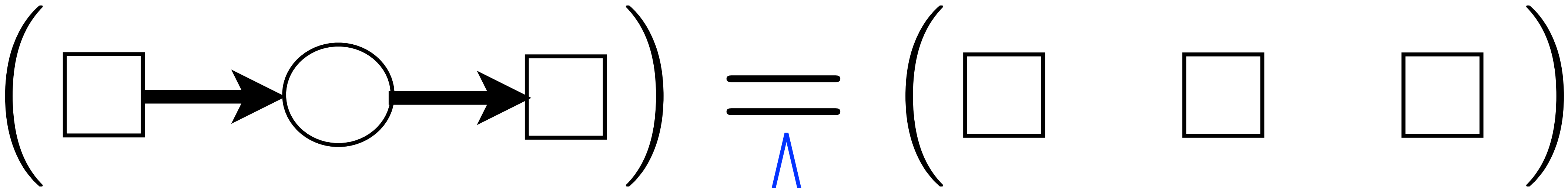


· $\mathbb{RP}^2$ index formula

Mirror symmetry eq decomposition



q-binomial theorem

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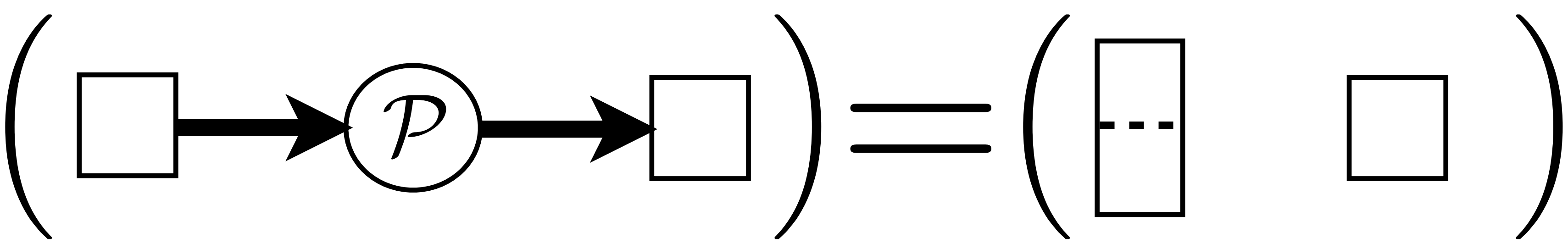
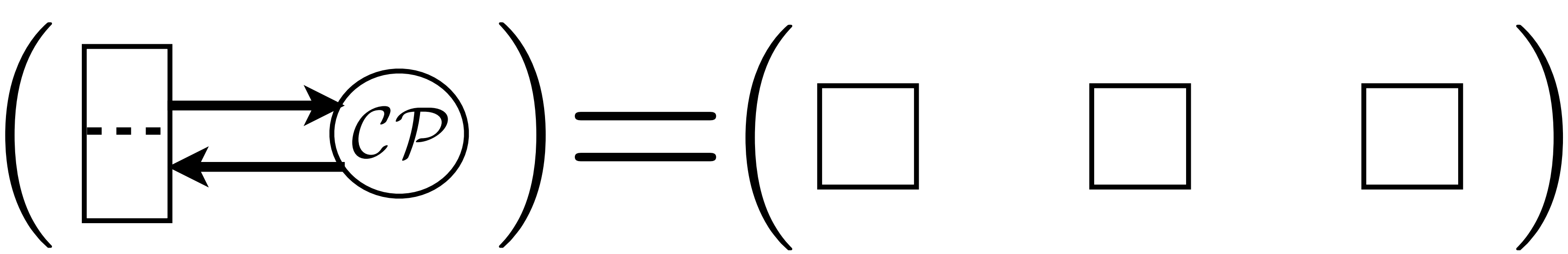
Ramanujan's sum formula

A.T, H. Mori, T. Morita (2014)



q-binomial theorem

A.T, H. Mori, T. Morita (2015)



Ramanujan's sum formula

Today's talk

- ✔ Index formula (review) $\bigcirc = \sum_{s} \int d\theta$
- ✔ $\mathbb{RP}^2$ index formula $\begin{cases} \textcircled{\mathcal{P}} = \sum_{s \in \mathbb{Z}_2} \int d\theta \\ \textcircled{\mathcal{CP}} = \sum_{s} \sum_{\theta \in \mathbb{Z}_2} \end{cases}$
- Role of $\mathbb{Z}_2$ in 3d duality

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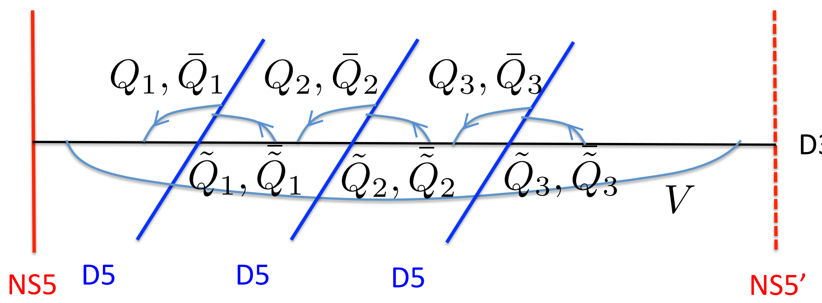
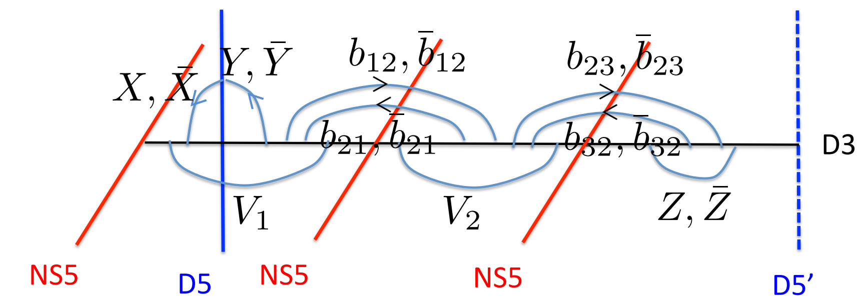
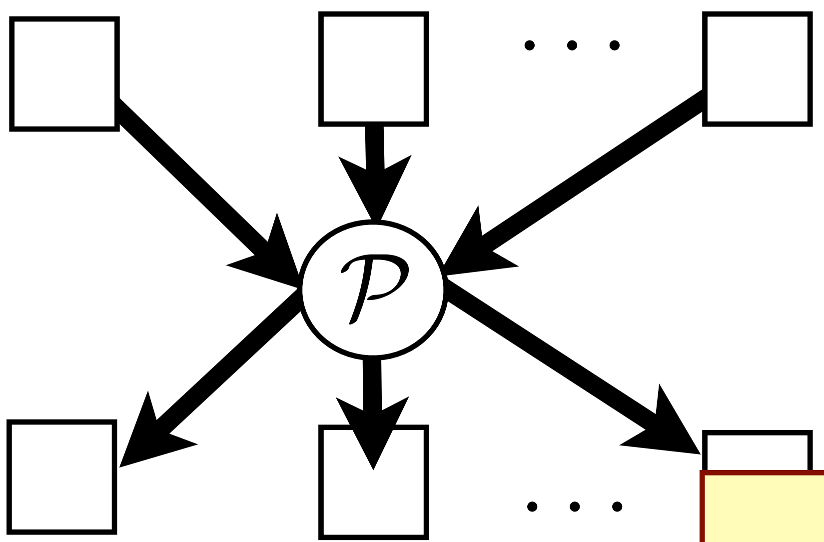
3d abelian mirror symmetry

H. Mori, A.T (To apper)







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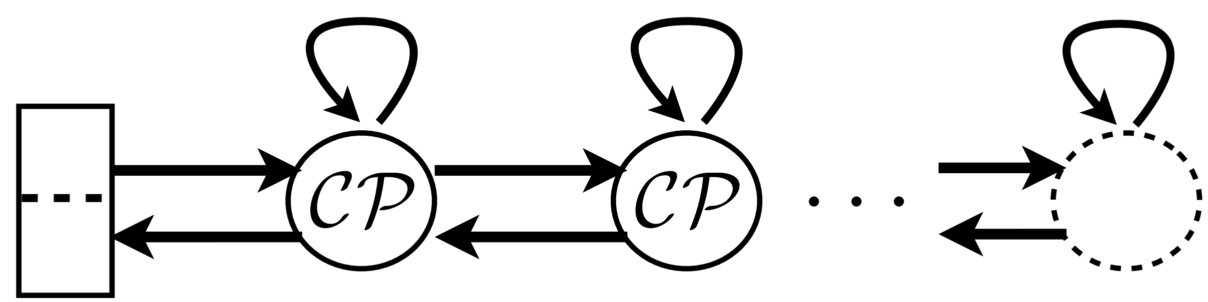
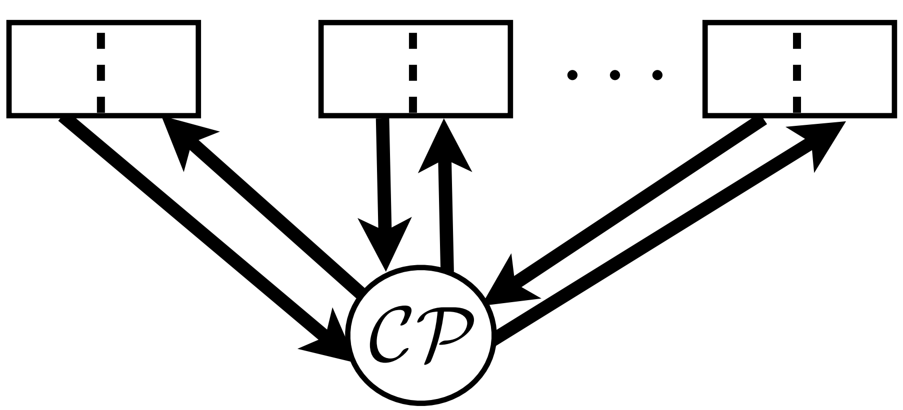


$$\mathcal{P} \Leftrightarrow \mathcal{CP}$$



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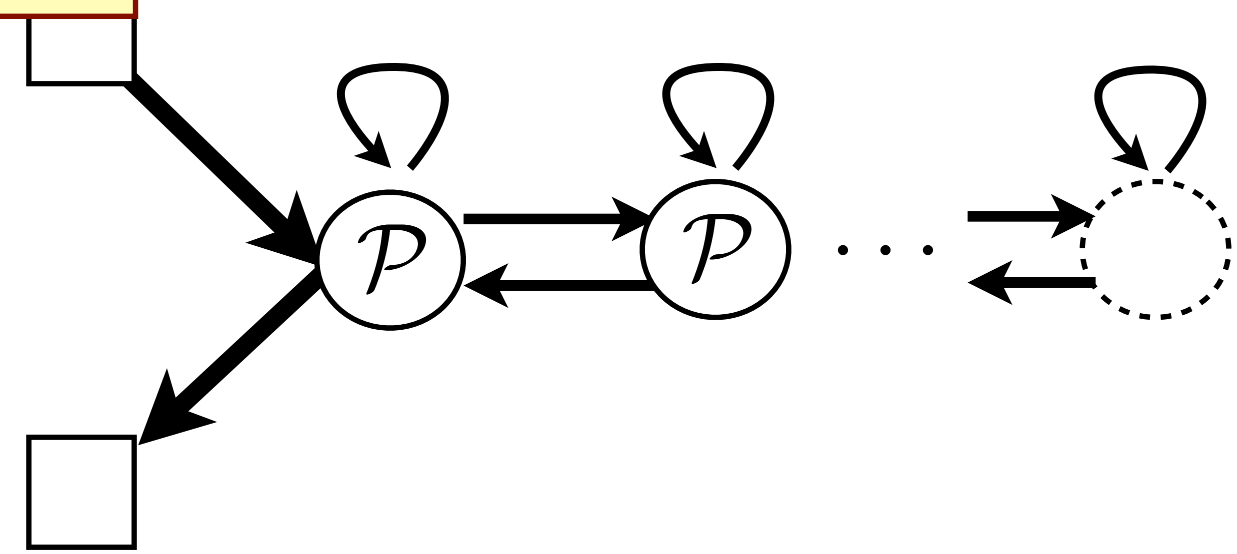


• Role of $\mathbb{Z}_2$ in 3d duality

Duality between loop operators

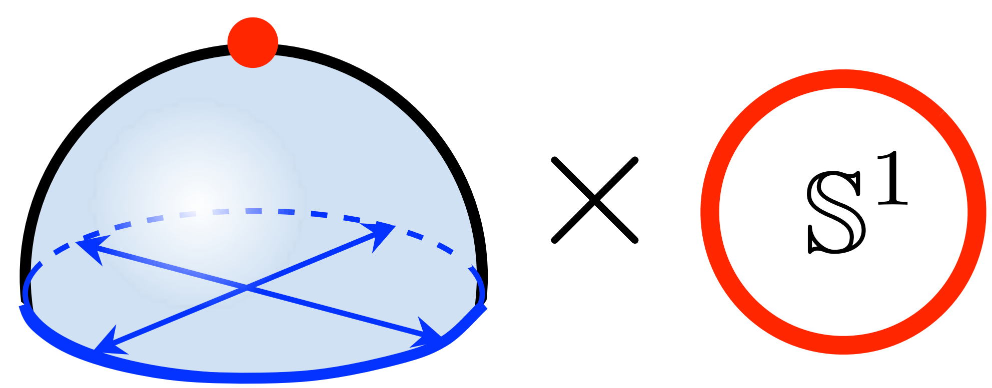
H. Mori, A.T (To apper)

Wilson loop :

($\mathcal{P}$) $e^{ie\int dt A_t}$

($\mathcal{CP}$) $e^{ie\int dt\sigma}$

} on

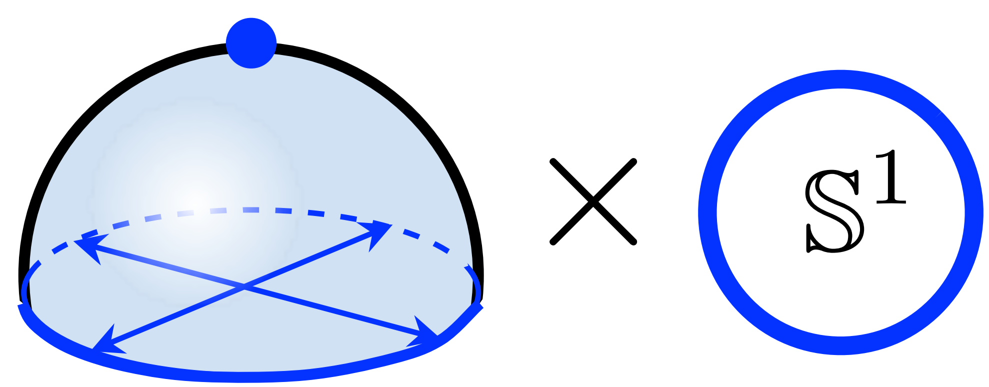


Vortex loop :

($\mathcal{P}$) $S^{-1} e^{ie\int dt A_t}\ S$

($\mathcal{CP}$) $S^{-1} e^{ie\int dt\sigma}\ S$

} on



・ Role of $\mathbb{Z}_2$ in 3d duality

Duality between loop operators

H. Mori, A.T (To apper)

$$\mathcal{P} \Leftrightarrow \mathcal{CP}$$

- Gauge $\left\{\begin{matrix}\mathcal{P}\\ \mathcal{CP}\end{matrix}\right\}$ **W** = Topological $\left\{\begin{matrix}\mathcal{CP}\\ \mathcal{P}\end{matrix}\right\}$ **V**

- Gauge $\left\{\begin{matrix}\mathcal{P}\\ \mathcal{CP}\end{matrix}\right\}$ **V** = Topological $\left\{\begin{matrix}\mathcal{CP}\\ \mathcal{P}\end{matrix}\right\}$ **W**

Today's talk

✔ · Index formula (review) $\bigcirc = \sum_s \int d\theta$

✔ · $\mathbb{RP}^2$ index formula $\begin{cases} \text{(}\mathcal{P}\text{)} = \sum_{s\in\mathbb{Z}_2} \int d\theta \\ \text{(}\mathcal{CP}\text{)} = \sum_s \sum_{\theta\in\mathbb{Z}_2} \end{cases}$

✔ · Role of $\mathbb{Z}_2$ in 3d duality $\mathcal{P} \Leftrightarrow \mathcal{CP}$

· Comments

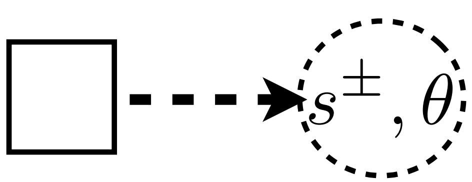
Thank you !

- **CS term?**

  →Generically, no...

  But (AdA - BdB) type is OK.

- **Parity anomaly = Gauge anomaly?**



$$
= \begin{cases} \left(q^{\frac{\Delta-1}{8}} e^{\frac{i\boldsymbol{Q}\theta}{4}} \right)^{+1} \frac{(e^{-i\boldsymbol{Q}\theta} q^{\frac{2-\Delta}{2}}; q^2)_\infty}{(e^{+i\boldsymbol{Q}\theta} q^{\frac{0+\Delta}{2}}; q^2)_\infty} & \text{for } e^{i\pi \boldsymbol{Q} s^{\pm}} = +1, \\ \left(q^{\frac{\Delta-1}{8}} e^{\frac{i\boldsymbol{Q}\theta}{4}} \right)^{-1} \frac{(e^{-i\boldsymbol{Q}\theta} q^{\frac{4-\Delta}{2}}; q^2)_\infty}{(e^{+i\boldsymbol{Q}\theta} q^{\frac{2+\Delta}{2}}; q^2)_\infty} & \text{for } e^{i\pi \boldsymbol{Q} s^{\pm}} = -1, \end{cases}
$$

Unless $\mathbf{Q} \in 4\mathbb{Z}$, they are anomalous.