

$\mathbb{R}P^2$ index and its applications

AKINORI TANAKA

(RIKEN)

Based on

A. T, H. Mori, and T. Morita, [Phys. Rev. D91 \(2015\) 105023](#), [arXiv:1408.3371 \[hep-th\]](#).

A. T, H. Mori, and T. Morita, [JHEP 09 \(2015\) 154](#), [arXiv:1505.07539 \[hep-th\]](#).

H. Mori, A.T., [arXiv:151x.xxxxx \[hep-th\]](#).

(2+1)d Index

$$I = \text{Tr}_{\mathcal{H}} (-1)^F q^{\frac{1}{2}(H+j_3)}$$

- Exactly calculable (Localization)

Y. Imamura & S. Yokoyama(2011)

$\mathcal{H} \otimes$ 4-supercharges

A.T, H. Mori & T. Morita(2014-2015)

$\mathcal{H} \otimes$ 4-supercharges

$/\mathbb{Z}_2$

\mathcal{P}
or
 CP

Applications $/\mathbb{Z}_2$

- Explicit checks of 3d duality

Krattenthaler, Spiridonov, Vartanov (2011), Kapustin, Willett (2011)

$$\left(\begin{array}{c} \text{3d mirror symmetry} \\ \text{(1-flavor)} \end{array} \right) = \left(\begin{array}{c} \text{q-binomial Formula} \\ + \\ \text{Ramanujan's Sum} \end{array} \right)$$

- 3d-3d correspondence

Dimofte, Gaiotto, Gukov (2011), Gang, Koh, Lee, Park (2013)

$$\left(\begin{array}{c} \text{3d mirror symmetry} \\ \text{(1-flavor)} \end{array} \right) = \left(\begin{array}{c} \text{3d polyhedron} \leftrightarrow \text{3d polyhedra} \end{array} \right)$$

Today's talk

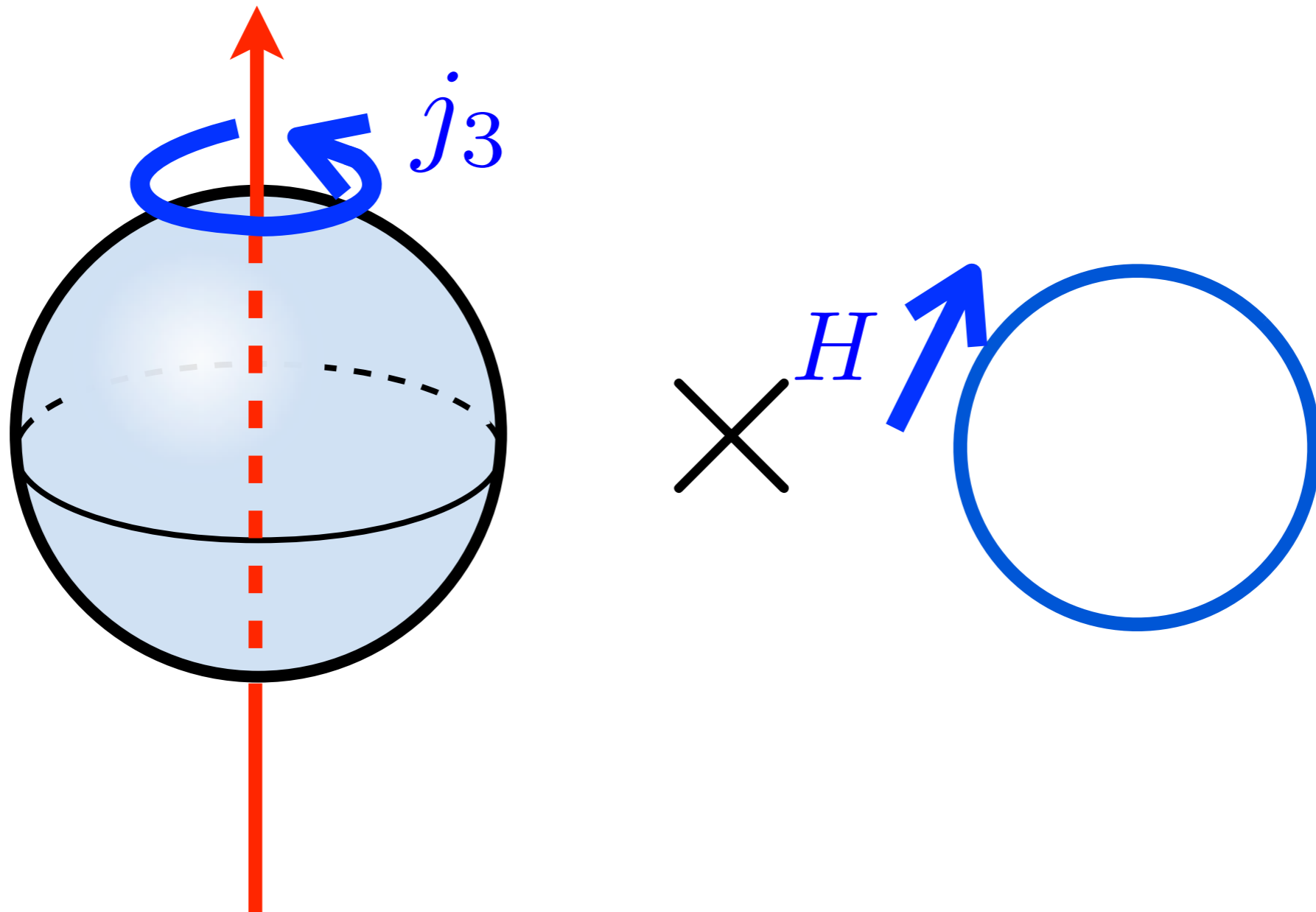
- Index formula (review)
- $\mathbb{R}P^2$ index formula
- Role of \mathbb{Z}_2 in 3d duality

- Index formula (review)

Definition

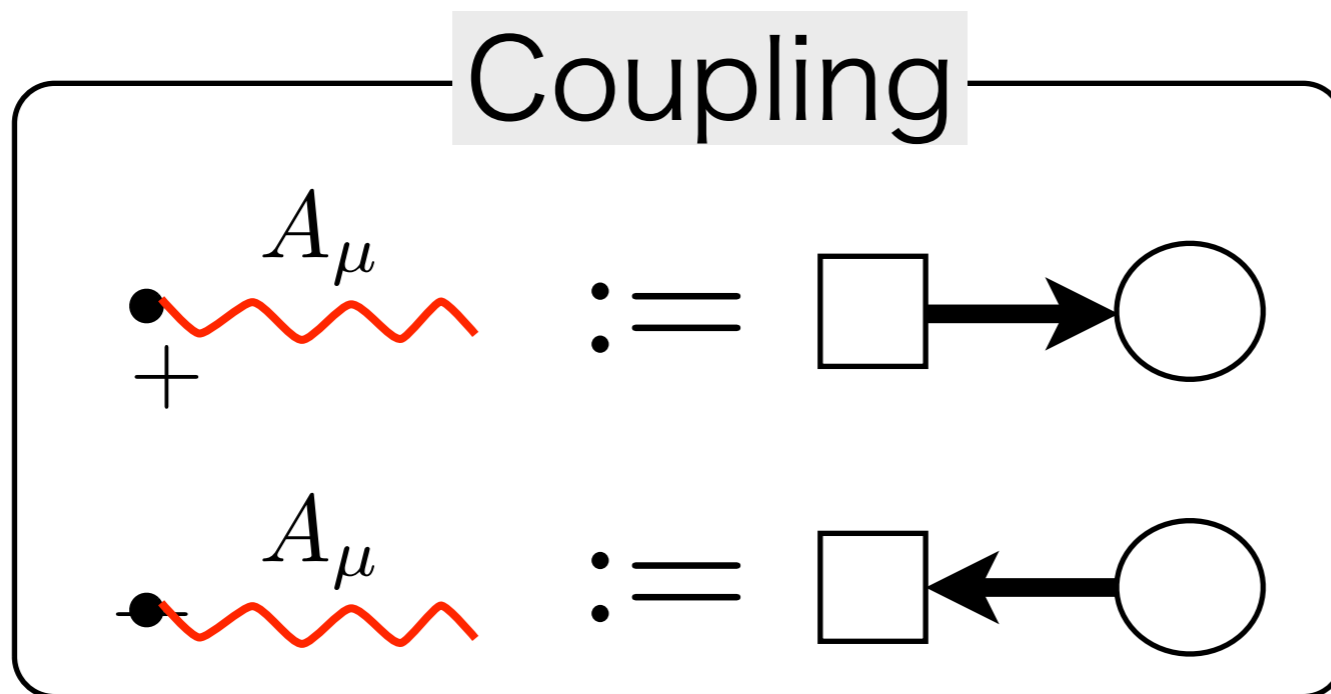
$$I = \text{Tr}_{\mathcal{H}} (-1)^F q^{\frac{1}{2}(H+j_3)}$$

SUSY QFT on



- Index formula (review)

Our notation for quiver diagram



$+$ $-$
 \bullet A_μ (red wavy line) \circ

$\square \rightarrow \bigcirc \rightarrow \square$

$$-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \dots$$

$$+ \bar{\psi}_1 (\not{\partial} - i\not{A}) \psi_1 + \dots$$

$$+ \bar{\psi}_2 (\not{\partial} + i\not{A}) \psi_2 + \dots$$

- Index formula (review)

Localization formula (2-steps)

Y. Imamura & S. Yokoyama(2011)

$$I = \text{Tr}_{\mathcal{H}} (-1)^F q^{\frac{1}{2}} (H + j_3)$$

STEP 1

$$\bigcirc = \sum_{2s=-\infty}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} d\theta \bigcirc_{s, \theta}$$

STEP 2

$$\square \dashrightarrow \bigcirc_{s, \theta} = \left(q^{\frac{1-\Delta}{2}} \right)^s \frac{(e^{-iQ\theta} q^{|Qs| + \frac{2-\Delta}{2}}; q)_{\infty}}{(e^{+iQ\theta} q^{|Qs| + \frac{\Delta}{2}}; q)_{\infty}}$$

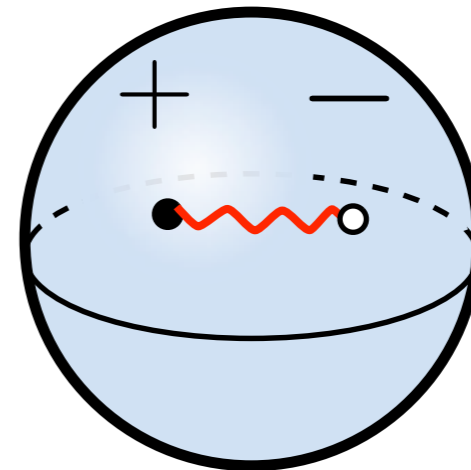
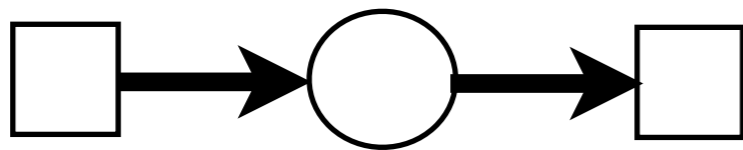
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Y. Imamura & S. Yokoyama(2011)

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Example



$$\text{Circle} = \sum_{2s=-\infty}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} d\theta \text{ (dashed circle with } s, \theta \text{)}$$

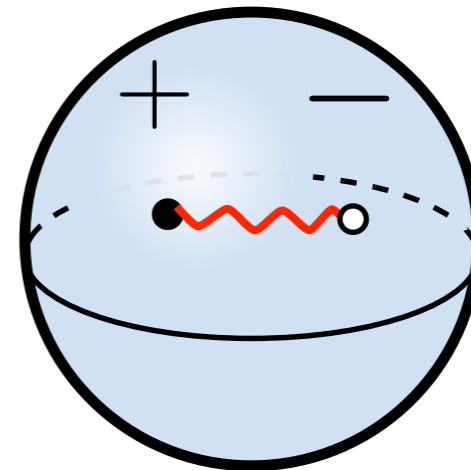
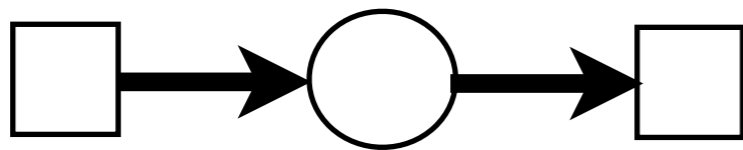
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Localization formula (2-steps)

Y. Imamura & S. Yokoyama(2011)

$$I = \text{Tr}_{\mathcal{H}} (-1)^F q^{\frac{1}{2}} (H + j_3)$$

Example



$$= \sum_{2s=-\infty}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} d\theta \left(\square \dashrightarrow s, \theta \dashrightarrow \square \right)$$

$$\square \dashrightarrow s, \theta = \left(q^{\frac{1-\Delta}{2}} \right)^s \frac{(e^{-iQ\theta} q^{|Qs| + \frac{2-\Delta}{2}}; q)_{\infty}}{(e^{+iQ\theta} q^{|Qs| + \frac{\Delta}{2}}; q)_{\infty}}$$

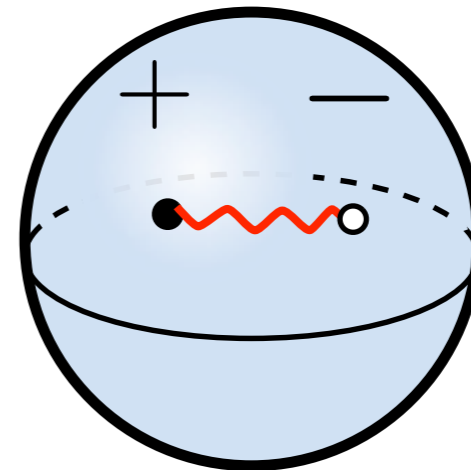
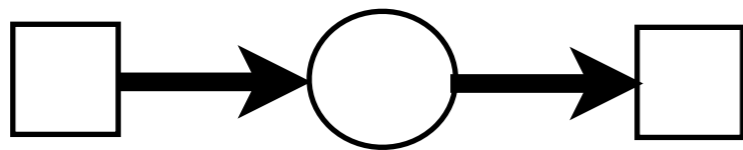
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$$I = \text{Tr}_{\mathcal{H}} (-1)^F q^{\frac{1}{2}} (H + j_3)$$

Example



$$= \sum_{2s=-\infty}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} d\theta \left(\square \dashrightarrow (s, \theta) \dashrightarrow \square \right) \quad z = e^{i\theta}$$

$$= \sum_{2s=-\infty}^{\infty} \oint \frac{dz}{2\pi i z} \left(q^{\frac{1-\Delta}{2}} \right)^{|2s|} \frac{(z^{-1} q^{|s| + \frac{2-\Delta}{2}}; q)_{\infty}}{(z q^{|s| + \frac{0+\Delta}{2}}; q)_{\infty}} \times \frac{(z q^{|s| + \frac{2-\Delta}{2}}; q)_{\infty}}{(z^{-1} q^{|s| + \frac{0+\Delta}{2}}; q)_{\infty}}$$

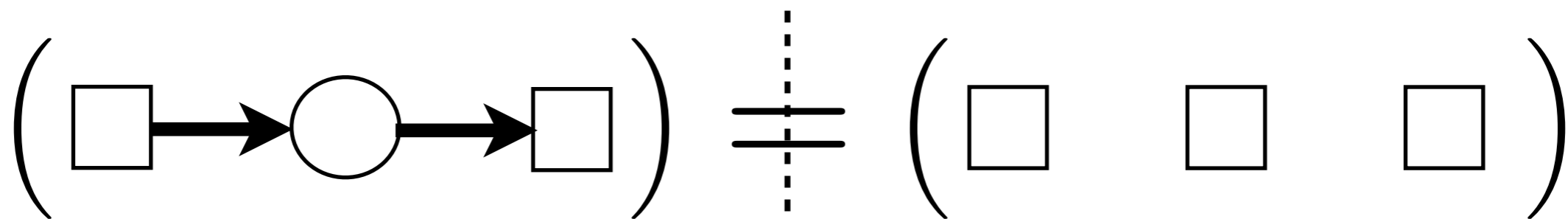
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$$I = \text{Tr}_{\mathcal{H}}(-1)^F q^{\frac{1}{2}} (H + j_3)$$

Check: 3d mirror symmetry



Series [

$$\text{Sum} \left[q^{\text{Abs}[k]/4} \frac{(\text{QPochhammer}[q^{(\text{Abs}[k]+j+1)}, q])}{(\text{QPochhammer}[q^{-1/2+(\text{Abs}[k]+j+1)}, q])} \right. \\ \left. \frac{(\text{QPochhammer}[q^{(1-2j)/2}, q])}{(\text{QPochhammer}[q, q])} \frac{q^{j(j+1)/2} (-1)^j}{(\text{QPochhammer}[q, q, j])} \right] \\ \{j, 0, 20, 1\}, \{k, -50, 50, 1\}, \{q, 0, 3\} \mid$$

$$1 + 2q^{1/4} + 3\sqrt{q} + 2q^{3/4} + q + 2q^{5/4} + 4q^{3/2} + \\ 4q^{7/4} - 2q^{9/4} + 3q^{5/2} + 6q^{11/4} + 2q^3 + O[q]^{13/4}$$

$$\text{Series} \left[\frac{(\text{QPochhammer}[q^{3/4}, q])^2}{(\text{QPochhammer}[q^{1/4}, q])^2}, \{q, 0, 3\} \right]$$

$$1 + 2q^{1/4} + 3\sqrt{q} + 2q^{3/4} + q + 2q^{5/4} + 4q^{3/2} + \\ 4q^{7/4} - 2q^{9/4} + 3q^{5/2} + 8q^{11/4} + 5q^3 + O[q]^{13/4}$$

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Localization formula (2-steps)

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$$I = \text{Tr}_{\mathcal{H}} (-1)^F q^{\frac{1}{2}} (H + j_3)$$

Check: 3d mirror symmetry

$$\left(\square \longrightarrow \bigcirc \longrightarrow \square \right) = \left(\square \quad \square \quad \square \right)$$

q-binomial theorem
+
Ramanujan's sum formula

Krattenthaler, Spiridonov, Vartanov (2011),
Kapustin, Willett (2011)

Today's talk



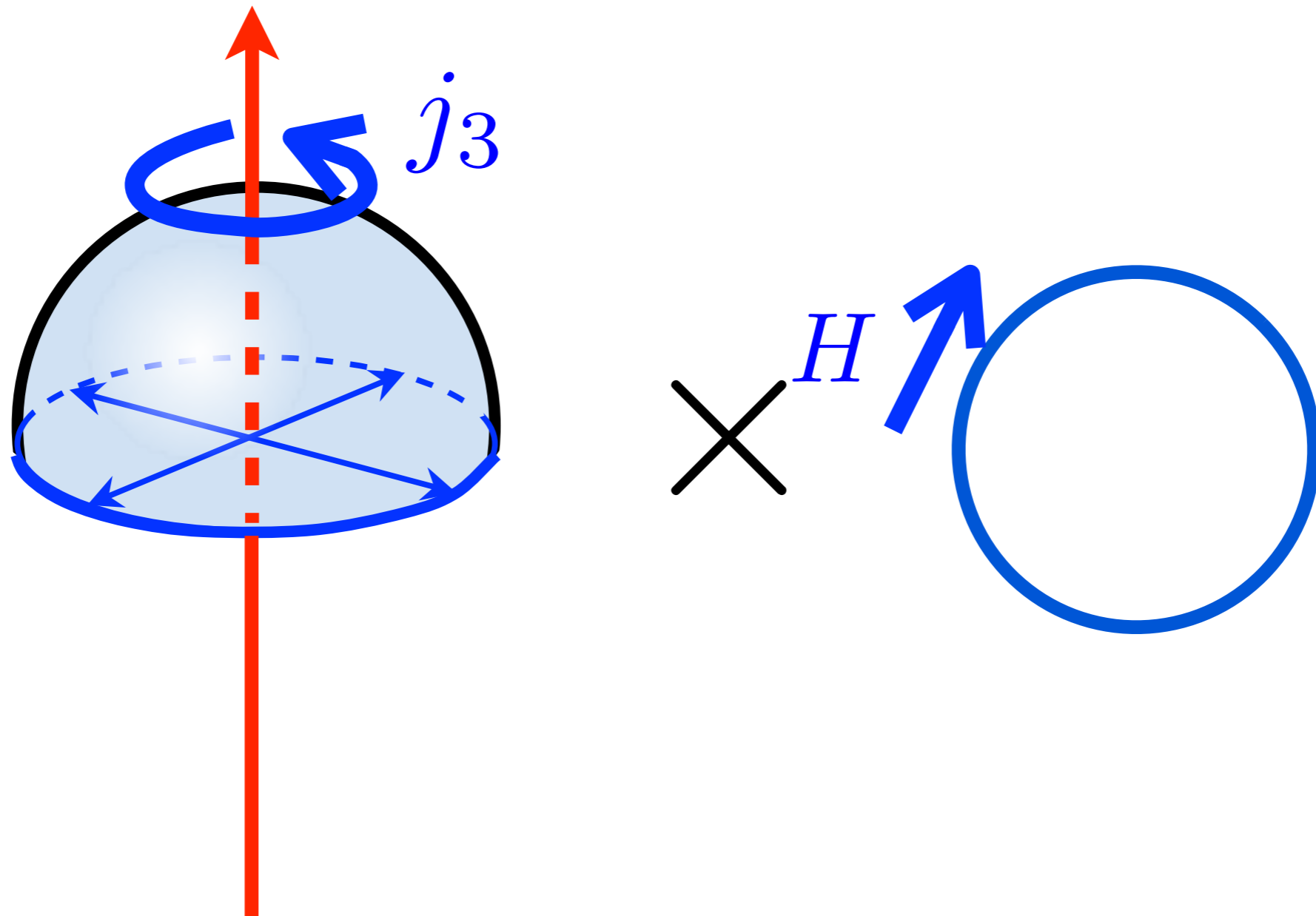
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- \mathbb{RP}^2 index formula

Definition

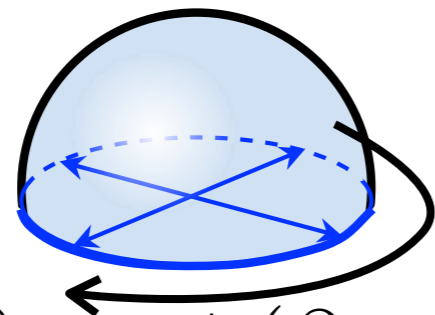
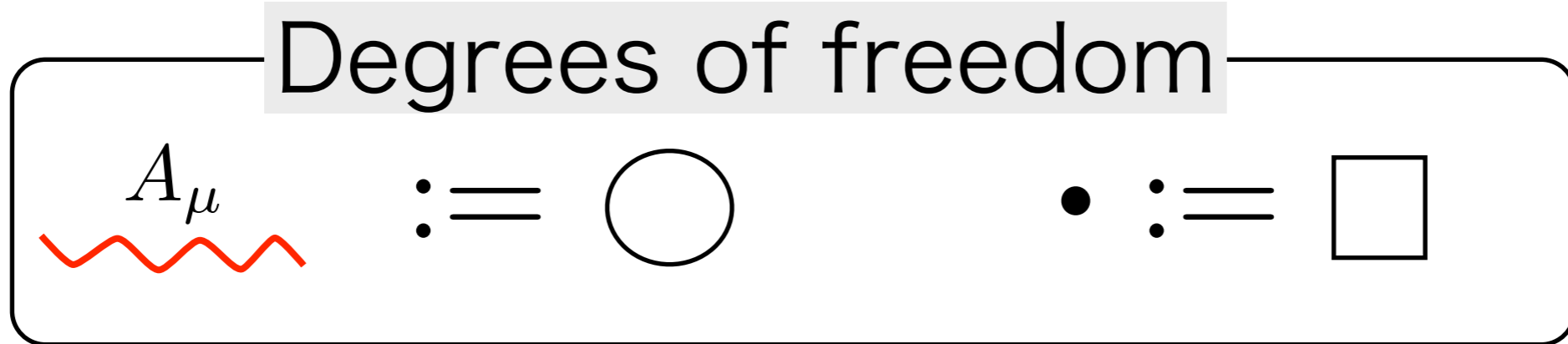
$$I = \text{Tr}_{\mathcal{H}/\mathbb{Z}_2} (-1)^F q^{\frac{1}{2}(H+j_3)}$$

SUSY QFT on



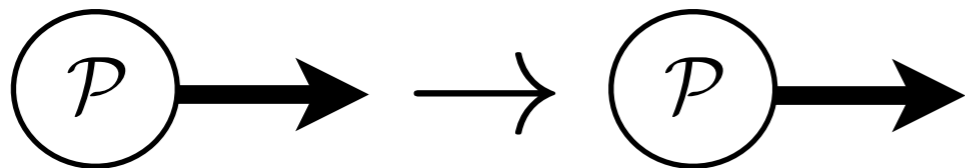
- \mathbb{RP}^2 index formula

Our notation for quiver diagram

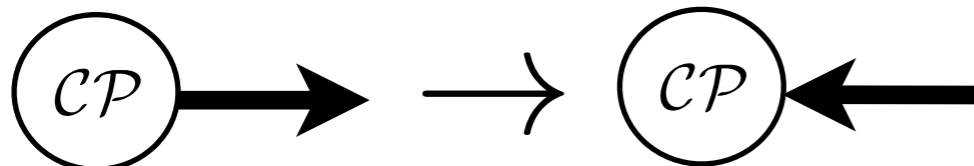


\Rightarrow SUSY parity conds

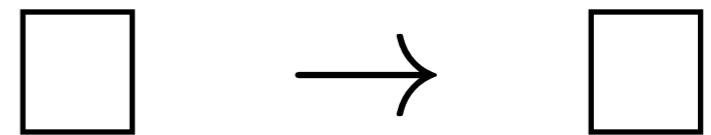
- $(\partial_\mu - iA_\mu) \rightarrow \pm(\partial_\mu - iA_\mu)$



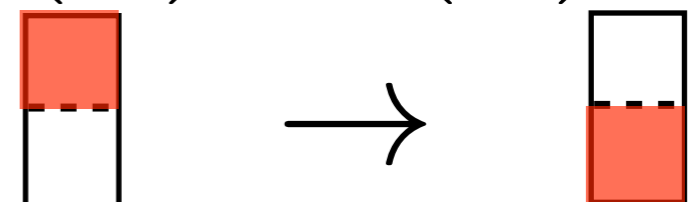
- $(\partial_\mu - iA_\mu) \rightarrow \pm(\partial_\mu + iA_\mu)$



- $\phi \rightarrow \phi$

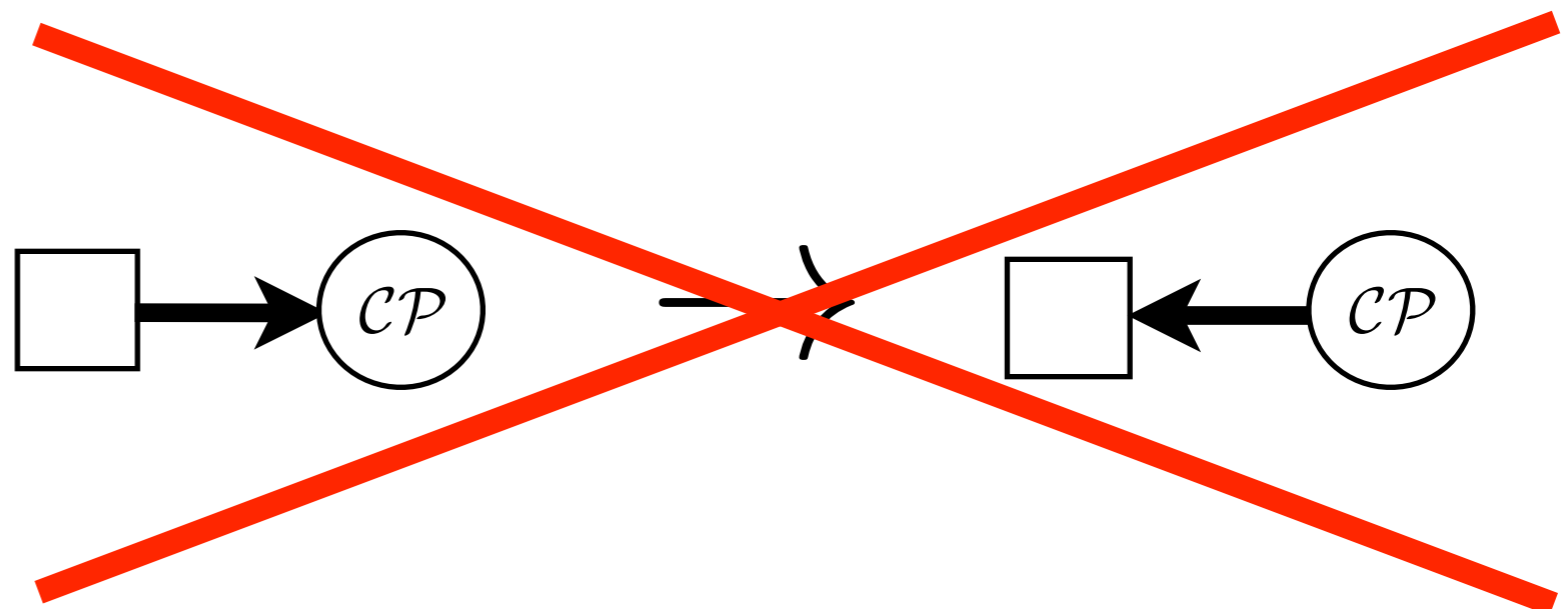
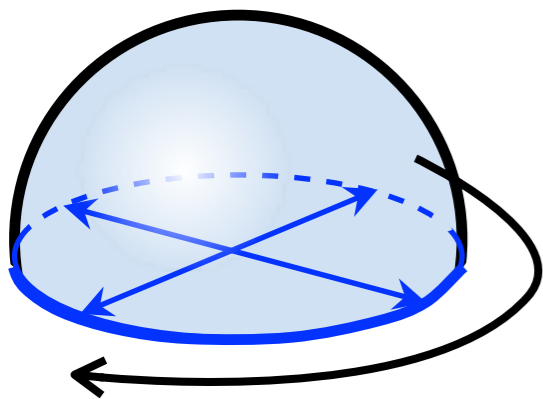
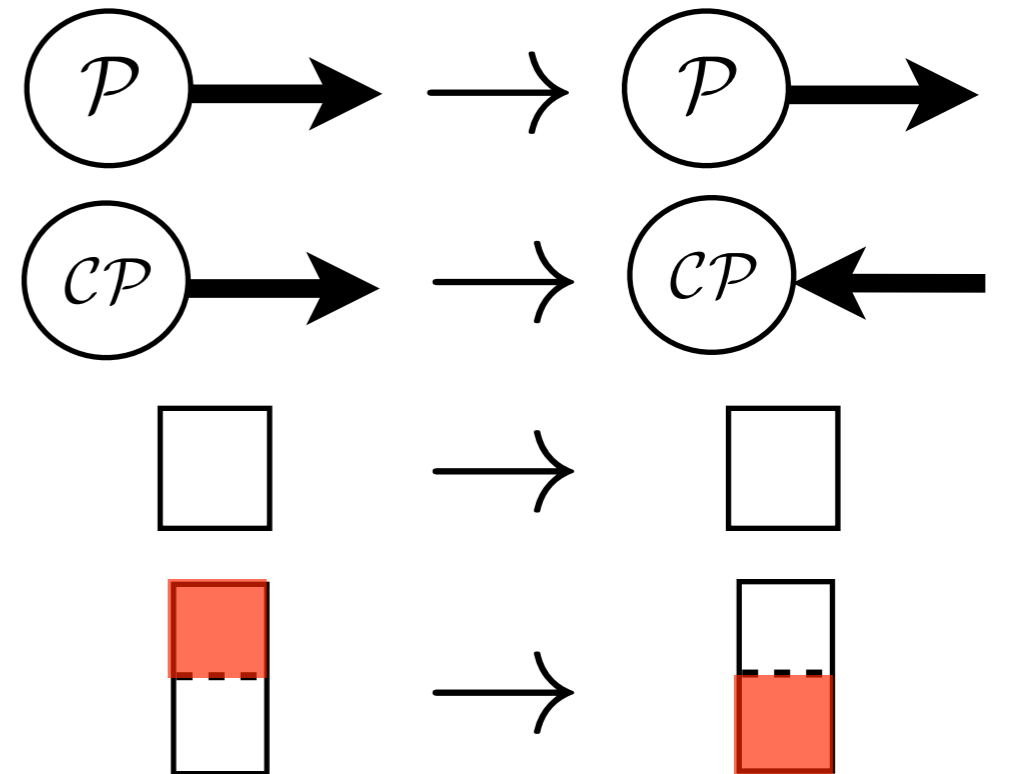
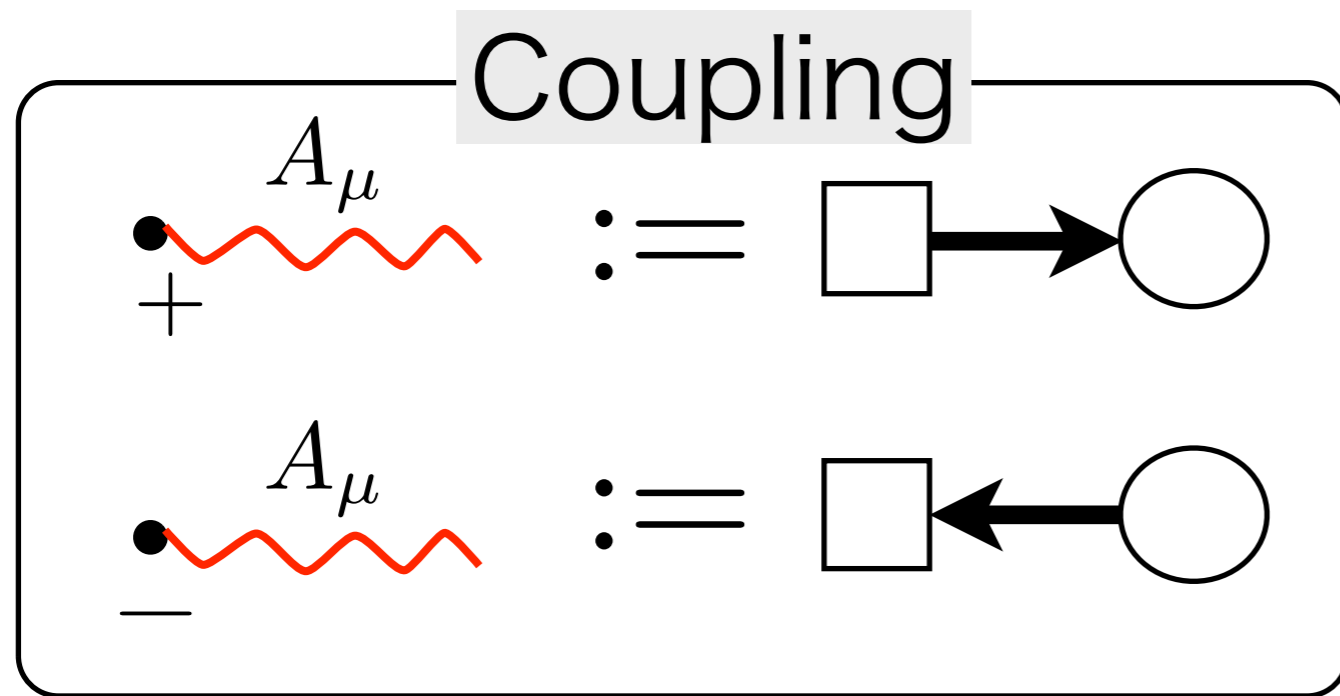


- $\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \rightarrow \begin{pmatrix} \phi_2 \\ \phi_1 \end{pmatrix}$



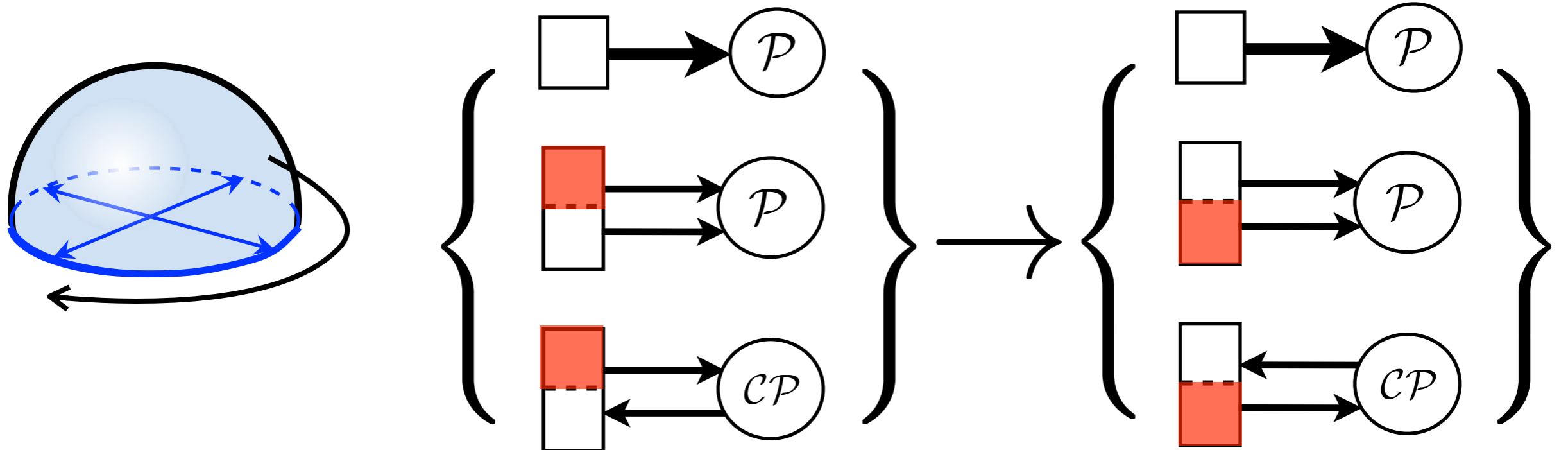
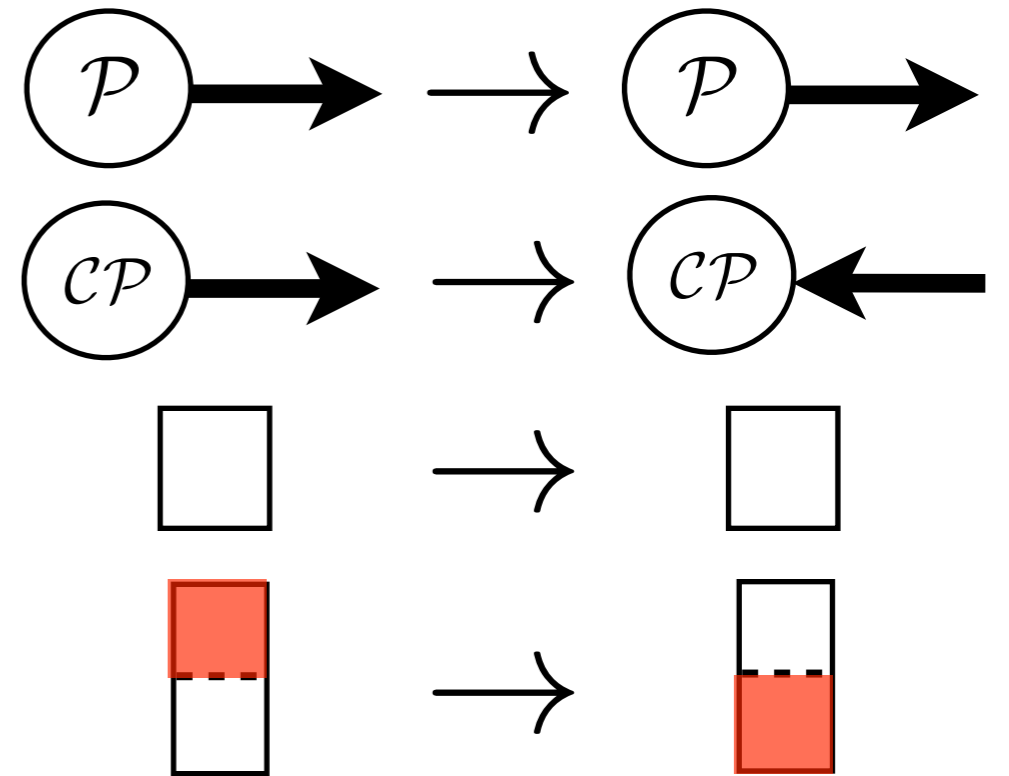
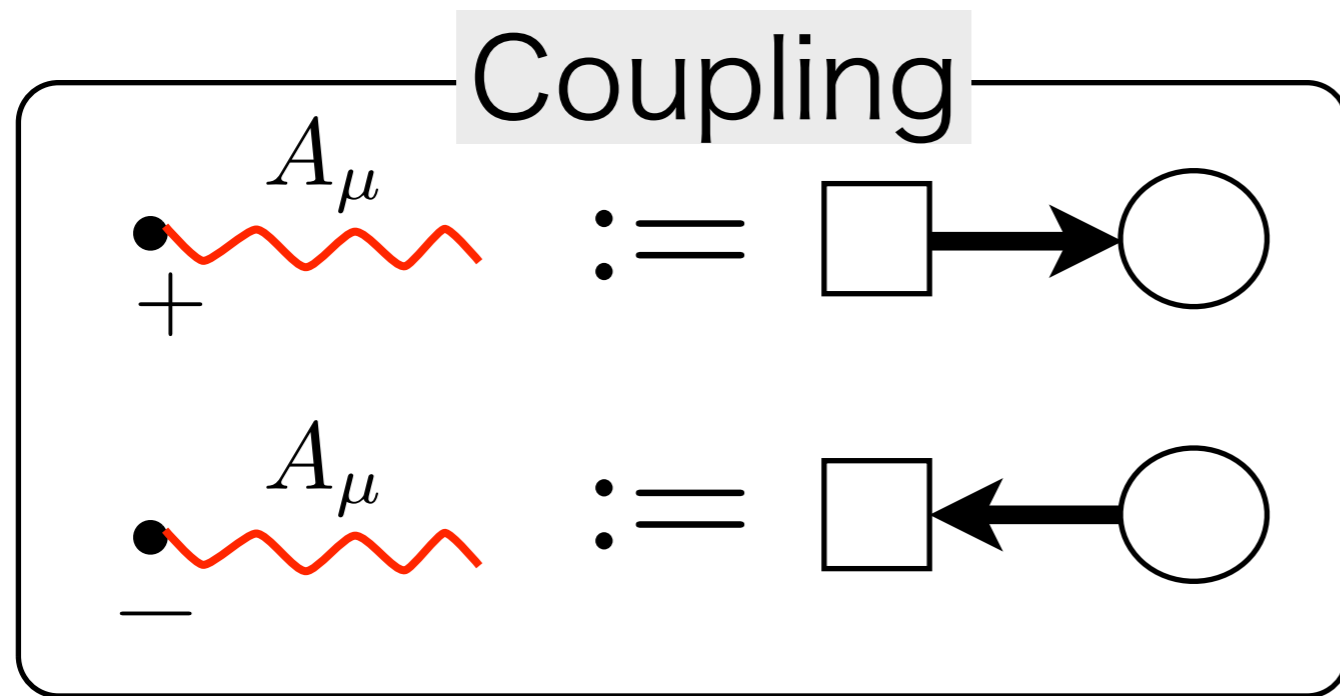
- $\mathbb{R}P^2$ index formula

Our notation for quiver diagram



- $\mathbb{R}P^2$ index formula

Our notation for quiver diagram

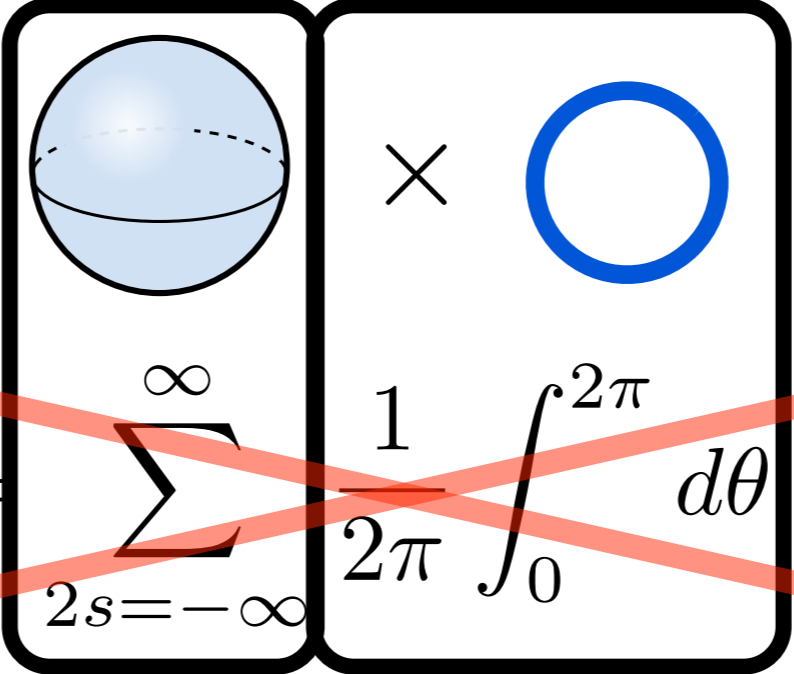


- $\mathbb{R}P^2$ index formula

Localization formula (2-steps)

$$I = \text{Tr}_{\mathcal{H}/\mathbb{Z}_2} (-1)^F q^{\frac{1}{2}(H+j_3)}$$

STEP 1



$$\bigcirc = \sum_{2s=-\infty}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} d\theta \quad (s, \theta)$$

The diagram above shows a sphere and a circle with a multiplication sign between them, enclosed in a box. Below this, the equation $\bigcirc = \sum_{2s=-\infty}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} d\theta \quad (s, \theta)$ is shown. A large red 'X' is drawn over the entire diagram and equation.

$$\bigcirc_{\mathcal{P}} = q^{+\frac{1}{8}} \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}} \sum_{s^{\pm}=0,1} \frac{1}{2\pi} \int_0^{2\pi} d\theta \quad (s^{\pm}, \theta)$$

$$\bigcirc_{CP} = q^{-\frac{1}{8}} \frac{(q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \sum_{s=-\infty}^{\infty} \frac{1}{2} \sum_{\theta_{\pm}=0,\pi} \quad (s, \theta_{\pm})$$

- \mathbb{RP}^2 index formula

Localization formula (2-steps)

$$I = \text{Tr}_{\mathcal{H}/\mathbb{Z}_2} (-1)^F q^{\frac{1}{2}} (H + j_3)$$

STEP2

$$\square \dashrightarrow s^\pm, \theta = \begin{cases} \left(q^{\frac{\Delta-1}{8}} e^{\frac{iQ\theta}{4}} \right)^{+1} \frac{(e^{-iQ\theta} q^{\frac{2-\Delta}{2}}; q^2)_\infty}{(e^{+iQ\theta} q^{\frac{0+\Delta}{2}}; q^2)_\infty} & \text{for } e^{i\pi Qs^\pm} = +1, \\ \left(q^{\frac{\Delta-1}{8}} e^{\frac{iQ\theta}{4}} \right)^{-1} \frac{(e^{-iQ\theta} q^{\frac{4-\Delta}{2}}; q^2)_\infty}{(e^{+iQ\theta} q^{\frac{2+\Delta}{2}}; q^2)_\infty} & \text{for } e^{i\pi Qs^\pm} = -1, \end{cases}$$

$$\begin{array}{|c} \square \\ \hline \square \end{array} \dashrightarrow s, \theta_\pm = \left(q^{\frac{1-\Delta}{2}} \right)^{|Qs|} \frac{(e^{-iQ\theta_\pm} q^{|Qs| + \frac{(2-\Delta)}{2}}; q)_\infty}{(e^{+iQ\theta_\pm} q^{|Qs| + \frac{(0+\Delta)}{2}}; q)_\infty}$$

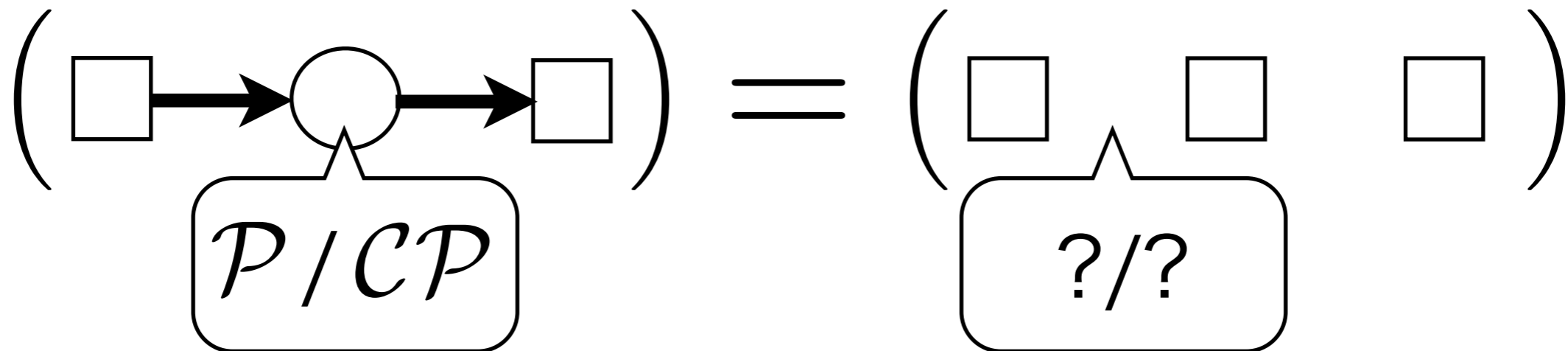
etc.

- \mathbb{RP}^2 index formula

Localization formula (2-steps)

$$I = \text{Tr}_{\mathcal{H}/\mathbb{Z}_2} (-1)^F q^{\frac{1}{2}(H+j_3)}$$

Check: 3d mirror symmetry



- \mathbb{RP}^2 index formula

Localization formula (2-steps)

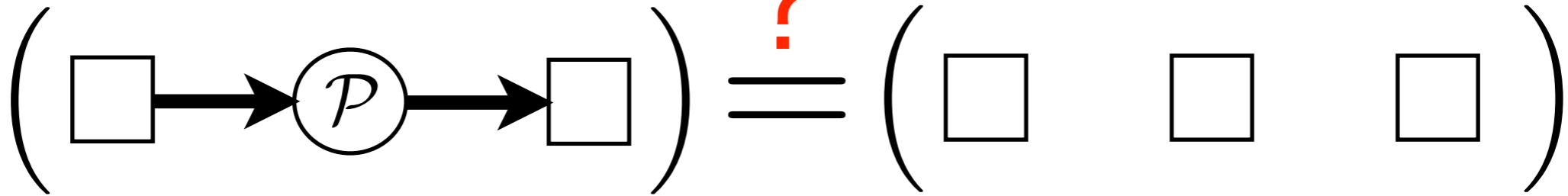
C

$$\begin{aligned} & \text{Series} \left[q^{1/8} \frac{(\text{QPochhammer}[q^2, q^2])}{(\text{QPochhammer}[q, q^2])} \right. \\ & \left(q^{-1/8} \frac{(\text{QPochhammer}[q^{1/2}, q^2])}{(\text{QPochhammer}[q^{1/2}, q^2])} \frac{(\text{QPochhammer}[q, q^2])}{(\text{QPochhammer}[q^2, q^2])} \right. \\ & \quad \text{QHypergeometricPFQ}[\{q^{3/2}, q^{1/2}\}, \{q\}, q^2, q^{1/2}] + \\ & \left. q^{+1/8} \frac{(\text{QPochhammer}[q^{1/2}, q^2])}{(\text{QPochhammer}[q^{5/2}, q^2])} \frac{(\text{QPochhammer}[q^3, q^2])}{(\text{QPochhammer}[q^2, q^2])} \right. \\ & \quad \left. \text{QHypergeometricPFQ}[\{q^{3/2}, q^{5/2}\}, \{q^3\}, q^2, q^{1/2}] \right), \\ & \{q, 0, 3\} \\ & 1 + q^{1/4} + \sqrt{q} + q^{5/4} + q^{3/2} - q^2 + 2q^{5/2} + q^{11/4} - q^3 + O[q]^{25/8} \end{aligned}$$

$$(-1)^F q^{\frac{1}{2}} (H + j_3)$$

metry

$$\begin{aligned} & \text{Series} \left[q^{(2-1/2)/8} \frac{(\text{QPochhammer}[q^{3/4}, q^2])^2}{(\text{QPochhammer}[q^{1/4}, q^2])^2}, \{q, \dots\} \right. \\ & q^{3/16} + 2q^{7/16} + 3q^{11/16} + 2q^{15/16} + \\ & \left. q^{19/16} + 2q^{39/16} + 4q^{43/16} + 4q^{47/16} + O[q]^{51/16} \right] \end{aligned}$$



- $\mathbb{R}P^2$ index formula

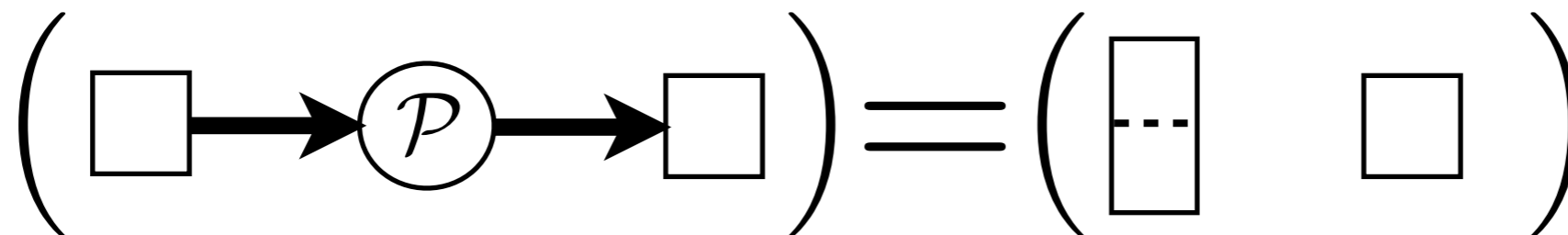
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$$-1)^F q^{\frac{1}{2}} (H + j_3)$$

$$\begin{aligned} & \text{Simplify} \left[\right. \\ & \text{Series} \left[\frac{\text{QPochhammer}[q^{(1+1/2)/2}, q]}{\text{QPochhammer}[q^{(1-1/2)/2}, q]} \right. \\ & \left. \left(q^{\frac{2/2-1}{8}} \frac{\text{QPochhammer}[q^{(2-2/2)/2}, q^2]}{\text{QPochhammer}[q^{1/2}, q^2]} \right), \{q, 0, 3\} \right] \\ & 1 + q^{1/4} + \sqrt{q} + q^{5/4} + q^{3/2} - q^2 + 2q^{5/2} + q^{11/4} - q^3 + O[q]^{13/4} \end{aligned}$$

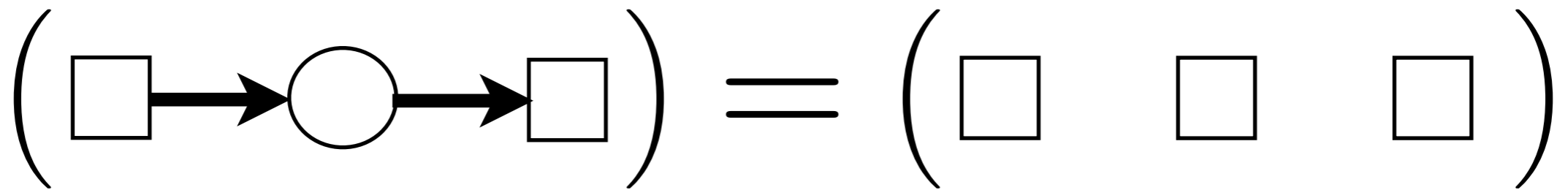


- $\mathbb{R}P^2$ index formula

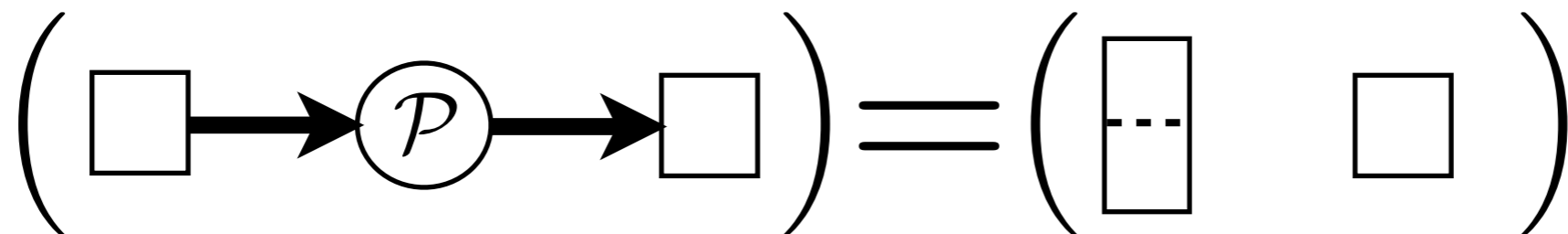
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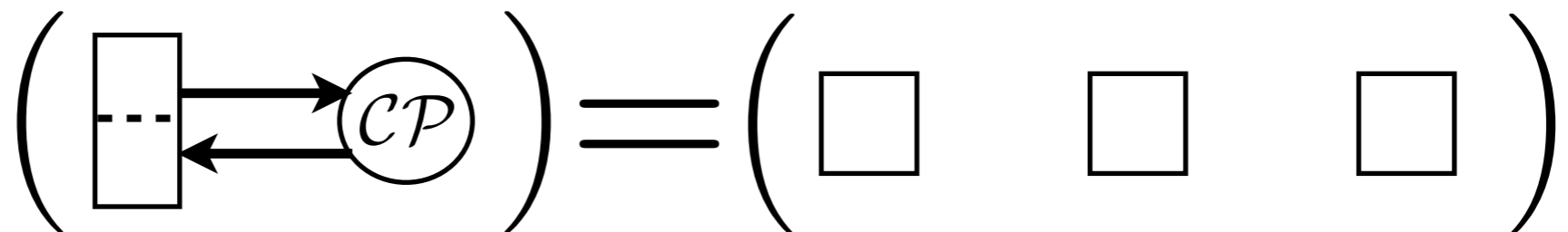
Check: 3d mirror symmetry



A.T, H. Mori, T. Morita (2014)



A.T, H. Mori, T. Morita (2015)



- $\mathbb{R}P^2$ index formula

Mirror symmetry eq decomposition

$$\left(\square \longrightarrow \bigcirc \longrightarrow \square \right) = \left(\square \quad \square \quad \square \right)$$

q-binomial theorem
+
Ramanujan's sum formula

A.T, H. Mori, T. Morita (2014)

$$\left(\square \longrightarrow \bigcirc \mathcal{P} \longrightarrow \square \right) = \left(\begin{array}{c} \square \\ \vdots \\ \square \end{array} \quad \square \right) \quad \text{q-binomial theorem}$$

A.T, H. Mori, T. Morita (2015)

$$\left(\begin{array}{c} \square \\ \vdots \\ \square \end{array} \longleftarrow \bigcirc \mathcal{CP} \right) = \left(\square \quad \square \quad \square \right) \quad \text{Ramanujan's sum formula}$$

Today's talk



- Index formula (review) $\bigcirc = \sum_s \int d\theta$



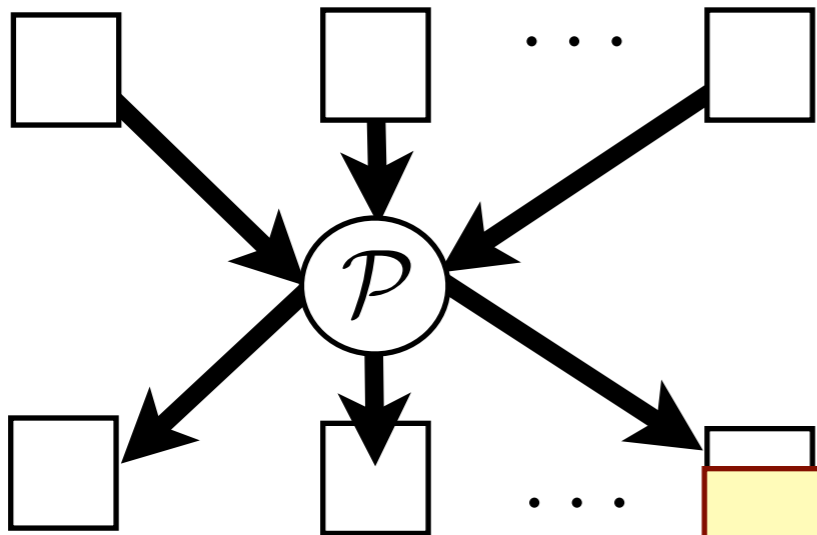
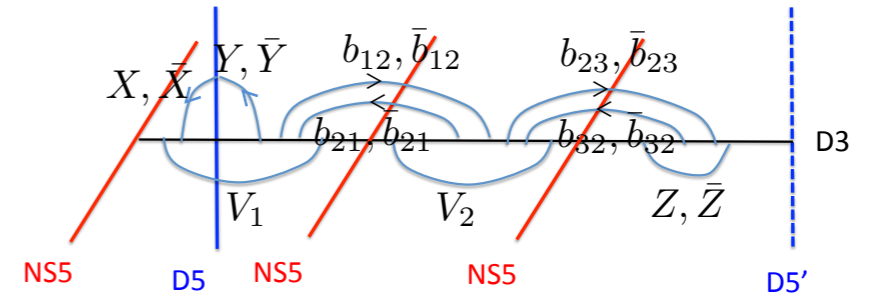
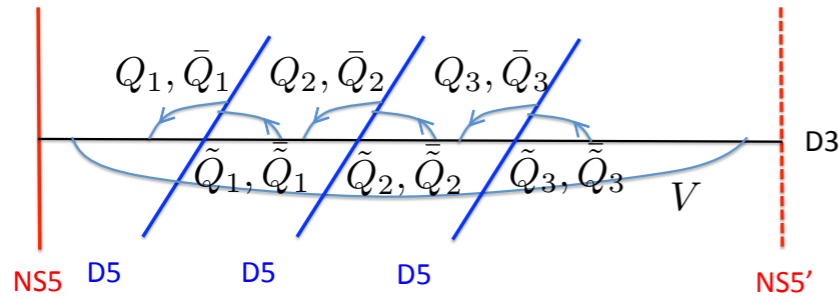
- \mathbb{RP}^2 index formula $\left\{ \begin{array}{l} \bigcirc_{\mathcal{P}} = \sum_{s \in \mathbb{Z}_2} \int d\theta \\ \bigcirc_{\mathcal{CP}} = \sum_s \sum_{\theta \in \mathbb{Z}_2} \end{array} \right.$

- Role of \mathbb{Z}_2 in 3d duality

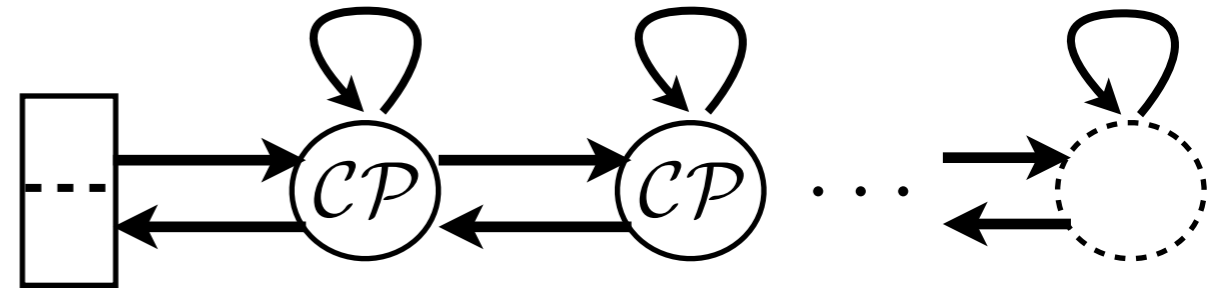
- Role of \mathbb{Z}_2 in 3d duality

3d abelian mirror symmetry

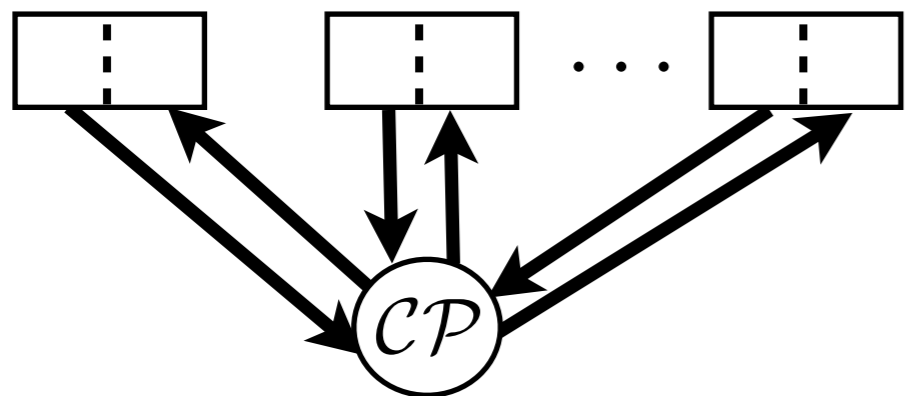
H. Mori, A.T (To appear)



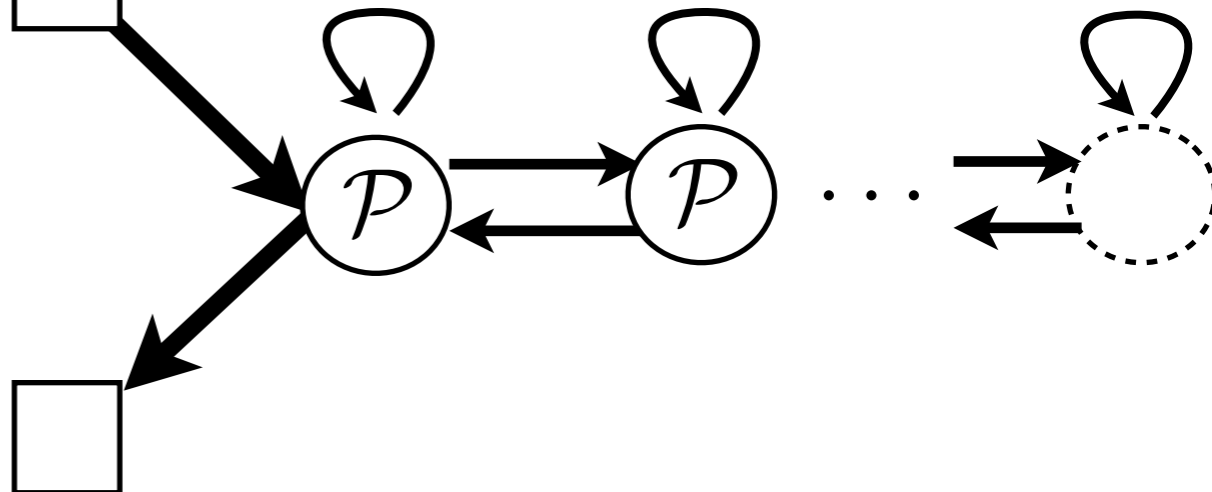
=



$\mathcal{P} \Leftrightarrow CP$



=



- Role of \mathbb{Z}_2 in 3d duality

Duality between loop operators

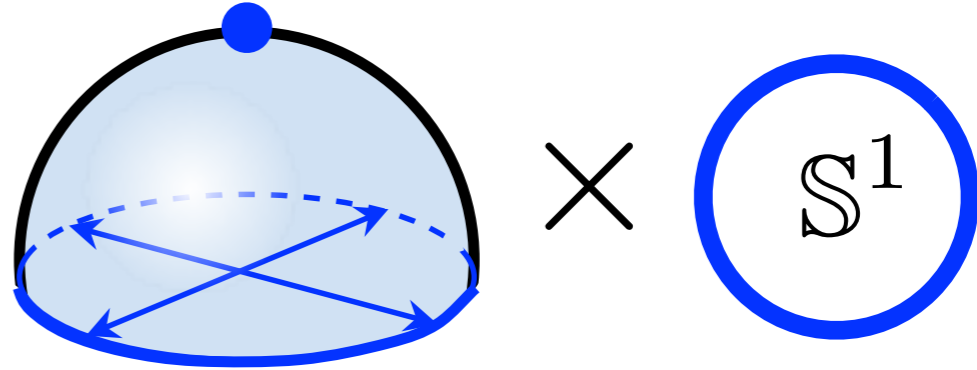
H. Mori, A.T (To appear)

Wilson loop :

$$\left. \begin{array}{l} \textcircled{\mathcal{P}} \quad e^{ie \int dt A_t} \\ \textcircled{\mathcal{CP}} \quad e^{ie \int dt \sigma} \end{array} \right\} \text{ on } \left(\text{Dome with red dot} \right) \times \left(\text{Red } S^1 \right)$$


The diagram shows a light blue dome with a black outline and a red dot at its top. Two blue arrows cross on the base of the dome. To the right of the dome is a red circle containing the text S^1 . A large 'X' symbol is placed between the dome and the circle.

Vortex loop :

$$\left. \begin{array}{l} \textcircled{\mathcal{P}} \quad S^{-1} e^{ie \int dt A_t} S \\ \textcircled{\mathcal{CP}} \quad S^{-1} e^{ie \int dt \sigma} S \end{array} \right\} \text{ on } \left(\text{Dome with blue dot} \right) \times \left(\text{Blue } S^1 \right)$$


The diagram shows a light blue dome with a black outline and a blue dot at its top. Two blue arrows cross on the base of the dome. To the right of the dome is a blue circle containing the text S^1 . A large 'X' symbol is placed between the dome and the circle.

- Role of \mathbb{Z}_2 in 3d duality

Duality between loop operators

H. Mori, A.T (To appear)

$$\mathcal{P} \Leftrightarrow \mathcal{CP}$$

- Gauge $\begin{Bmatrix} \mathcal{P} \\ \mathcal{CP} \end{Bmatrix} \mathbf{W} = \text{Topological} \begin{Bmatrix} \mathcal{CP} \\ \mathcal{P} \end{Bmatrix} \mathbf{V}$
- Gauge $\begin{Bmatrix} \mathcal{P} \\ \mathcal{CP} \end{Bmatrix} \mathbf{V} = \text{Topological} \begin{Bmatrix} \mathcal{CP} \\ \mathcal{P} \end{Bmatrix} \mathbf{W}$

Today's talk



- Index formula (review) $\bigcirc = \sum_s \int d\theta$



- \mathbb{RP}^2 index formula $\left\{ \begin{array}{l} \bigcirc_{\mathcal{P}} = \sum_{s \in \mathbb{Z}_2} \int d\theta \\ \bigcirc_{\mathcal{CP}} = \sum_s \sum_{\theta \in \mathbb{Z}_2} \end{array} \right.$



- Role of \mathbb{Z}_2 in 3d duality

$$\mathcal{P} \Leftrightarrow \mathcal{CP}$$

- Comments

Thank you !

● CS term?

→ Generically, no...

But (AdA - BdB) type is OK.

● Parity anomaly = Gauge anomaly?

$$\square \dashrightarrow s^\pm, \theta = \begin{cases} \left(q^{\frac{\Delta-1}{8}} e^{\frac{iQ\theta}{4}} \right)^{+1} \frac{(e^{-iQ\theta} q^{\frac{2-\Delta}{2}}; q^2)_\infty}{(e^{+iQ\theta} q^{\frac{0+\Delta}{2}}; q^2)_\infty} & \text{for } e^{i\pi Q s^\pm} = +1, \\ \left(q^{\frac{\Delta-1}{8}} e^{\frac{iQ\theta}{4}} \right)^{-1} \frac{(e^{-iQ\theta} q^{\frac{4-\Delta}{2}}; q^2)_\infty}{(e^{+iQ\theta} q^{\frac{2+\Delta}{2}}; q^2)_\infty} & \text{for } e^{i\pi Q s^\pm} = -1, \end{cases}$$

Unless $Q \in 4\mathbb{Z}$, they are anomalous.