Perturbative dynamics of fuzzy spheres at large N (hep-th/0410263)

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Contents

1 Introduction

Large-N reduced models $\Rightarrow$ promising candidates for the constructive definition of superstring theory.

The IIB matrix model

N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, hep-th/9612115

$$\boxed{S = -\frac{1}{g^2}\mathrm{tr}\left(\frac{1}{4}[A_\mu, A_\nu]^2 + \frac{1}{2}\bar{\psi}\Gamma^\mu[A_\mu, \psi]\right).}$$

Relation with the type IIB superstring theory:

- **Matrix regularization of the Green-Schwarz action of type IIB superstring theory.**
- **D-brane interaction.**
- **Derivation of the string field theory.**

Matrix models on a homogeneous space

Motivations of fuzzy manifold studies:

- **Relation between the non-commutative field theory and the superstring.**
- **Novel regularization scheme alternative to lattice regularization.**
- **Prototype of the curved-space background in the large-N reduced models.**

Matrix models on a homogeneous space G/H:

$G=$ (a Lie group), H = (a closed subgroup of G).

$\mathrm{S}^2 = \mathrm{SU}(2)/\mathrm{U}(1)$, $\mathrm{S}^2 \times \mathrm{S}^2$, $\mathrm{S}^4 = \mathrm{SO}(5)/\mathrm{U}(2)$, $\mathrm{CP}^2 = \mathrm{SU}(3)/\mathrm{U}(2)$, $\cdots$.

These fuzzy manifolds are compact, and thus realized by finite matrices.

The Chern-Simons term is added to accommodate the classical solution of the fuzzy manifolds.

2 The model and its classical solution

3d Yang-Mills-Chern-Simons (YMCS) model

$\Rightarrow$ **a toy model with fuzzy sphere solutions:**

S. Iso, Y. Kimura, K. Tanaka and K. Wakatsuki, hep-th/0101102.

$$\boxed{S[A] = N\mathrm{tr}\left(-\frac{1}{4}[A_\mu, A_\nu]^2 + \frac{2i\alpha}{3}\epsilon_{\mu\nu\rho}A_\mu A_\nu A_\rho\right).}$$

- Defined in the 3-dimensional Euclidean space $(\mu, \nu, \rho = 1, 2, 3)$.
- Convergence of the path integral P. Austing and J. F. Wheater, hep-th/0310170.
- Classical equation of motion: $[A_\nu, [A_\mu, A_\nu]] - i\alpha\epsilon_{\mu\nu\rho}[A_\nu, A_\rho] = 0$.
- fuzzy S^2 classical solutions: $A_\mu = X_\mu = \alpha L_\mu$, (where $[L_\mu, L_\nu] = i\epsilon_{\mu\nu\rho}L_\rho$).
  $L_\mu =$ ($N \times N$ representation of the SU(2) Lie algebra).
  Casimir operator: $Q = A_1^2 + A_2^2 + A_3^2 = R^2 1_N$.
  $R =$ (radius of the fuzzy sphere) $= \frac{\alpha}{2}\sqrt{N^2 - 1}$.

First-order phase transition

Monte Carlo simulation launched from fuzzy sphere classical solution:

Critical point at $\alpha_{\rm cr} \simeq \frac{2.1}{\sqrt{N}}$.

- **$\alpha < \alpha_{\rm cr}$: Yang-Mills phase**

  Strong quantum effects.

  behavior like the $\alpha = 0$ case.

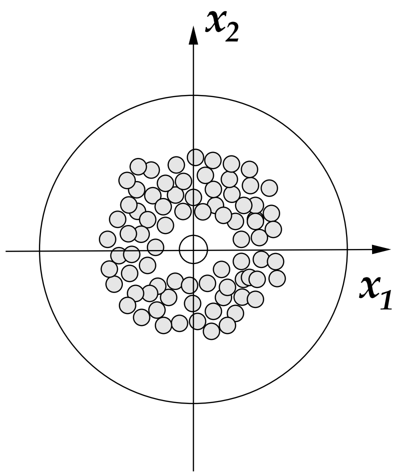
  T. Hotta, J. Nishimura and A. Tsuchiya, hep-th/9811220,

$$\langle \frac{S}{N^2} \rangle \simeq \mathrm{O}(1), \ \langle \frac{1}{N} \mathrm{tr}\, A_\mu^2 \rangle \simeq \mathrm{O}(1).$$

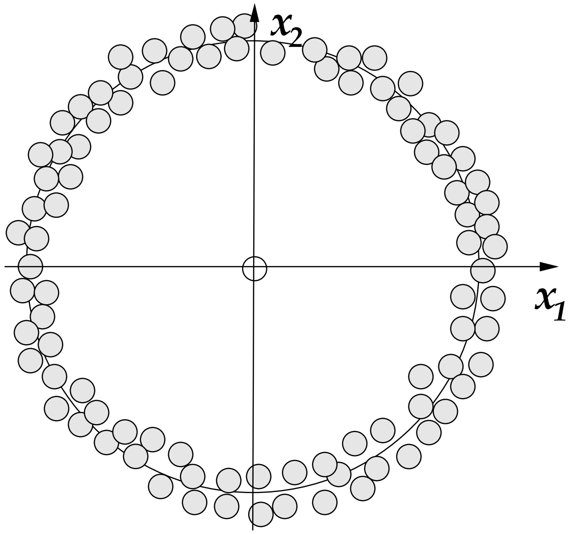
- **$\alpha > \alpha_{\rm cr}$: fuzzy sphere phase.**

  Fuzzy sphere configuration is stable.

Yang–Mills phase



Fuzzy sphere phase



Phase transition from the one-loop effective action

The effective action Γ is saturated at the one-loop level at large N.

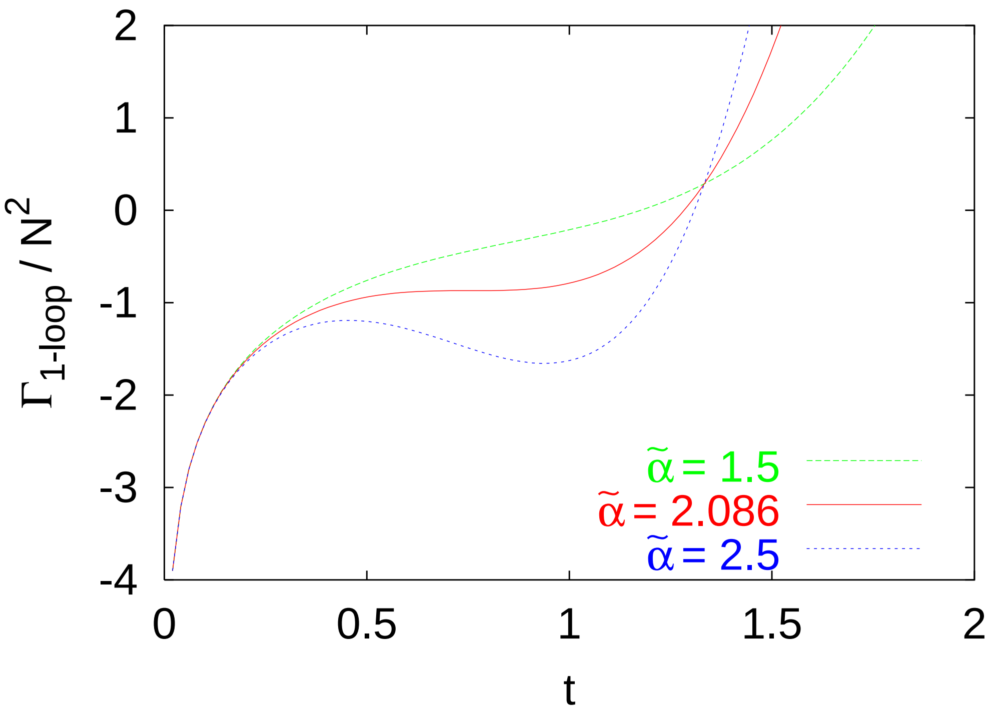
T. Imai, Y. Kitazawa, Y. Takayama and D. Tomino, hep-th/0307007.

Effective action at one-loop around $A_\mu = tX_\mu$ (where $\tilde{\alpha} = \alpha\sqrt{N}$).

$$\frac{\Gamma_{1-\text{loop}}}{N^2} \simeq \tilde{\alpha}^4 \left(\frac{t^4}{8} - \frac{t^3}{6}\right) + \log t.$$

The local minimum disappears at $\tilde{\alpha} < \tilde{\alpha}_{\text{cr}} = (\frac{8}{3})^{\frac{3}{4}} \simeq 2.086\cdots$.

Consistent with the Monte Carlo simulation.



3 Explicit two-loop calculation

Does the "one-loop dominance" hold true for generic observables?

Fluctuation from the fuzzy sphere configuration $A_\mu = \alpha L_\mu + \tilde{A}_\mu$.

$$
\begin{aligned}
S_{\text{g.f.}} &= -\frac{1}{2} N \,\text{tr}\,([X_\mu, A_\mu]^2), \qquad S_{\text{gh}} = -N\,\text{tr}\,([X_\mu, \bar{c}][A_\mu, c]), \\
S_{\text{total}} &= S + S_{\text{g.f.}} + S_{\text{gh}} = S[X] + S_{\text{kin}} + S_{\text{int}}, \ \text{where} \\
S_{\text{kin}} &= \frac{1}{2} N\,\text{tr}\,(\tilde{A}_\mu [X_\lambda, [X_\lambda, \tilde{A}_\mu]]) + N\,\text{tr}\,(\bar{c}[X_\lambda, [X_\lambda, c]])\ , \\
S_{\text{int}} &= -\frac{1}{4} N\,\text{tr}\,([\tilde{A}_\mu, \tilde{A}_\nu]^2) - N\,\text{tr}\,([\tilde{A}_\mu, \tilde{A}_\nu][X_\mu, \tilde{A}_\nu]) \\
&\quad + \frac{2}{3} i\,\alpha\, N\, \epsilon_{\mu\nu\rho}\,\text{tr}\,(\tilde{A}_\mu \tilde{A}_\nu \tilde{A}_\rho) - N\,\text{tr}\,([X_\mu, \bar{c}][\tilde{A}_\mu, c]).
\end{aligned}
$$

Definition of the free energy W:

$$W = -\log \int d\tilde{A}\, dc\, d\bar{c}\, e^{-S_{\text{total}}}.$$

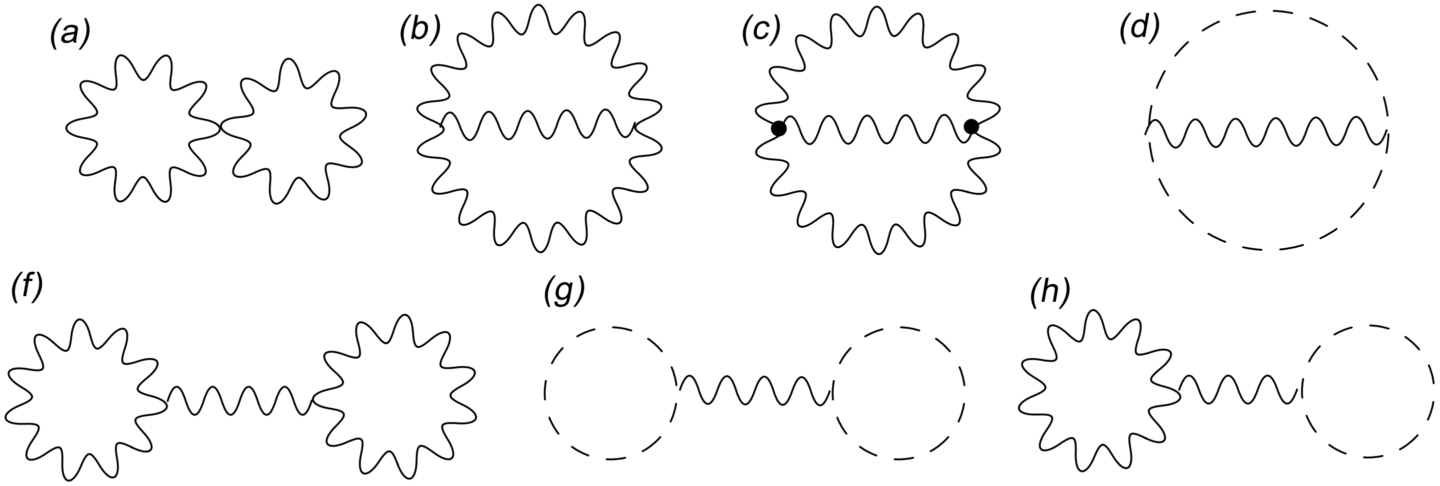
Its perturbative expansion $W = \sum_{j=0}^{\infty} W_j$ (W_j=(j-loop contribution)).

$$
\begin{aligned}
W_0 &= S[X] = -\frac{\tilde{\alpha}^4}{24}(N^2 - 1), \\
W_1 &= \frac{1}{2} \sum_{l=1}^{N-1} (2l + 1) \log(\tilde{\alpha}^2 l(l + 1)).
\end{aligned}
$$

Vacuum expectation value of the action:

$$\frac{1}{N^2}\langle S\rangle = \frac{3}{4}\left(1-\frac{1}{N^2}\right)+\frac{\tilde{\alpha}}{4N^2}\frac{\partial W}{\partial\tilde{\alpha}}, \Rightarrow \frac{1}{N^2}\langle S\rangle_{\text{1-loop}} = \left(-\frac{\tilde{\alpha}^4}{24}+1\right)\left(1-\frac{1}{N^2}\right).$$

Evaluation of W_2 $\Rightarrow$ we explicitly calculate 2-loop diagrams:



$$W_j = -N^2 w_j(N)\tilde{\alpha}^{4(1-j)}, \quad \Rightarrow \frac{1}{N^2}\langle S\rangle = \frac{1}{N^2}\langle S\rangle_{\text{1-loop}} + \sum_{j=2}^{\infty}(j-1)w_j(N)\tilde{\alpha}^{4(1-j)}.$$

- **1PI diagrams ((a) $\sim$ (d)):** $w_2^{(\mathrm{1PI})}(N) = \mathcal{O}\left(\frac{(\log N)^2}{N^2}\right)$ **vanishes at large N.**

  T. Imai, Y. Kitazawa, Y. Takayama and D. Tomino, hep-th/0303120

- **1PR diagrams ((e)(f)(g)):** $w_2^{(\mathrm{1PR})}(N) \simeq 1-\frac{1}{N^2}$ **survives large-N limit.**

4 All order calculation from one-loop effective action

The effective action Γ is saturated at one loop at large N

Y. Kitazawa, Y. Takayama and D. Tomino, hep-th/0403242

The free energy W can be obtained by the extremum of the effective action.

Expansion around $A_\mu = \beta L_\mu$: ($\tilde{\beta} = \beta\sqrt{N}$)

$$\lim_{N\to\infty} \frac{1}{N^2}\Gamma(\tilde{\beta}) = \left(\frac{\tilde{\beta}^4}{8} - \frac{1}{6}\tilde{\alpha}\tilde{\beta}^3\right) + \log\tilde{\beta}.$$

Local minimum for $\tilde{\alpha} > \tilde{\alpha}_{\rm cr} = \sqrt[4]{\frac{512}{27}}$:

$$\begin{aligned}
\tilde{\beta} &= f(\tilde{\alpha}) = \frac{\tilde{\alpha}}{4}\left(1 + \sqrt{1+\delta} + \sqrt{2 - \delta + \frac{2}{\sqrt{1+\delta}}}\right) \\
&= \tilde{\alpha}\left(1 - \frac{2}{\tilde{\alpha}^4} - \frac{12}{\tilde{\alpha}^8} - \frac{120}{\tilde{\alpha}^{12}} - \frac{1456}{\tilde{\alpha}^{16}} - \cdots\right), \text{ where} \\
\delta &= 4\tilde{\alpha}^{-\frac{4}{3}}\left[\left(1 + \sqrt{1 - \frac{512}{27\tilde{\alpha}^4}}\right)^{\frac{1}{3}} + \left(1 - \sqrt{1 - \frac{512}{27\tilde{\alpha}^4}}\right)^{\frac{1}{3}}\right].
\end{aligned}$$

Free energy and observables:

$$\begin{aligned}
\lim_{N\to\infty}\frac{1}{N^2}W &= \left(\frac{1}{8}f(\tilde{\alpha})^4 - \frac{1}{6}\tilde{\alpha}f(\tilde{\alpha})^3\right) + \log f(\tilde{\alpha}) \\
&= -\frac{\tilde{\alpha}^4}{24} + \log\tilde{\alpha} - \frac{1}{\tilde{\alpha}^4} - \frac{14}{3\tilde{\alpha}^8} - \frac{110}{3\tilde{\alpha}^{12}} - \frac{364}{\tilde{\alpha}^{16}} - \cdots, \\
\lim_{N\to\infty}\frac{1}{N^2}\langle S\rangle &= \frac{3}{4} - \frac{1}{24}\tilde{\alpha}f(\tilde{\alpha})^3 \\
&= -\frac{\tilde{\alpha}^4}{24} + 1 \underbrace{+\frac{1}{\tilde{\alpha}^4}}_{\text{agrees with two-loop calculation!}} + \frac{28}{3\tilde{\alpha}^8} + \frac{110}{\tilde{\alpha}^{12}} + \frac{1456}{\tilde{\alpha}^{16}} + \cdots.
\end{aligned}$$

All order calculation of generic observables $\mathcal{O}$

Consider the action $S_\epsilon = S + \epsilon\mathcal{O}$.

Corresponding free energy:

$$\begin{aligned} W_\epsilon &= -\log\left(\int d\tilde{A} e^{-(S+\epsilon\mathcal{O})}\right) = -\log\left(\int d\tilde{A} e^{-S}\right) + \epsilon \frac{\int d\tilde{A}\mathcal{O}e^{-S}}{\int d\tilde{A}e^{-S}} + \mathrm{O}(\epsilon^2) \\ &= W + \epsilon\langle\mathcal{O}\rangle + \mathrm{O}(\epsilon^2). \end{aligned}$$

One-loop effective action (take only 1PI diagrams into account)

$$\Gamma_\epsilon(\tilde{\beta}) = \Gamma(\tilde{\beta}) + \epsilon\Gamma_1(\tilde{\beta}) + \mathrm{O}(\epsilon^2).$$

Its saddle point:

$$\frac{\partial}{\partial\tilde{\beta}}\Gamma_\epsilon(\tilde{\beta}) = 0, \quad \Rightarrow \tilde{\beta} = f(\tilde{\alpha}) + \epsilon g(\tilde{\alpha}) + \mathrm{O}(\epsilon^2).$$

Plugging this solution, we obtain the free energy as

$$W_\epsilon = \Gamma_\epsilon(f(\tilde{\alpha}) + \epsilon g(\tilde{\alpha}) + \cdots) = \Gamma(f(\tilde{\alpha})) + \epsilon\left(\Gamma_1(f(\tilde{\alpha})) + g(\tilde{\alpha})\underbrace{\left(\frac{\partial\Gamma}{\partial\tilde{\beta}}\right)\Big|_{\tilde{\beta}=f(\tilde{\alpha})}}_{=0}\right) + \mathrm{O}(\epsilon^2).$$

We thus obtain $\langle O\rangle = \Gamma_1(f(\tilde{\alpha}))$.

All order calculation of the spacetime content:

$$\lim_{N\to\infty} \frac{1}{N}\langle \frac{1}{N}\mathrm{tr}\, A_\mu^2 \rangle = \frac{\tilde{\alpha}^2}{4} \underbrace{-\frac{1}{\tilde{\alpha}^2}}_{\text{one-loop}} .$$

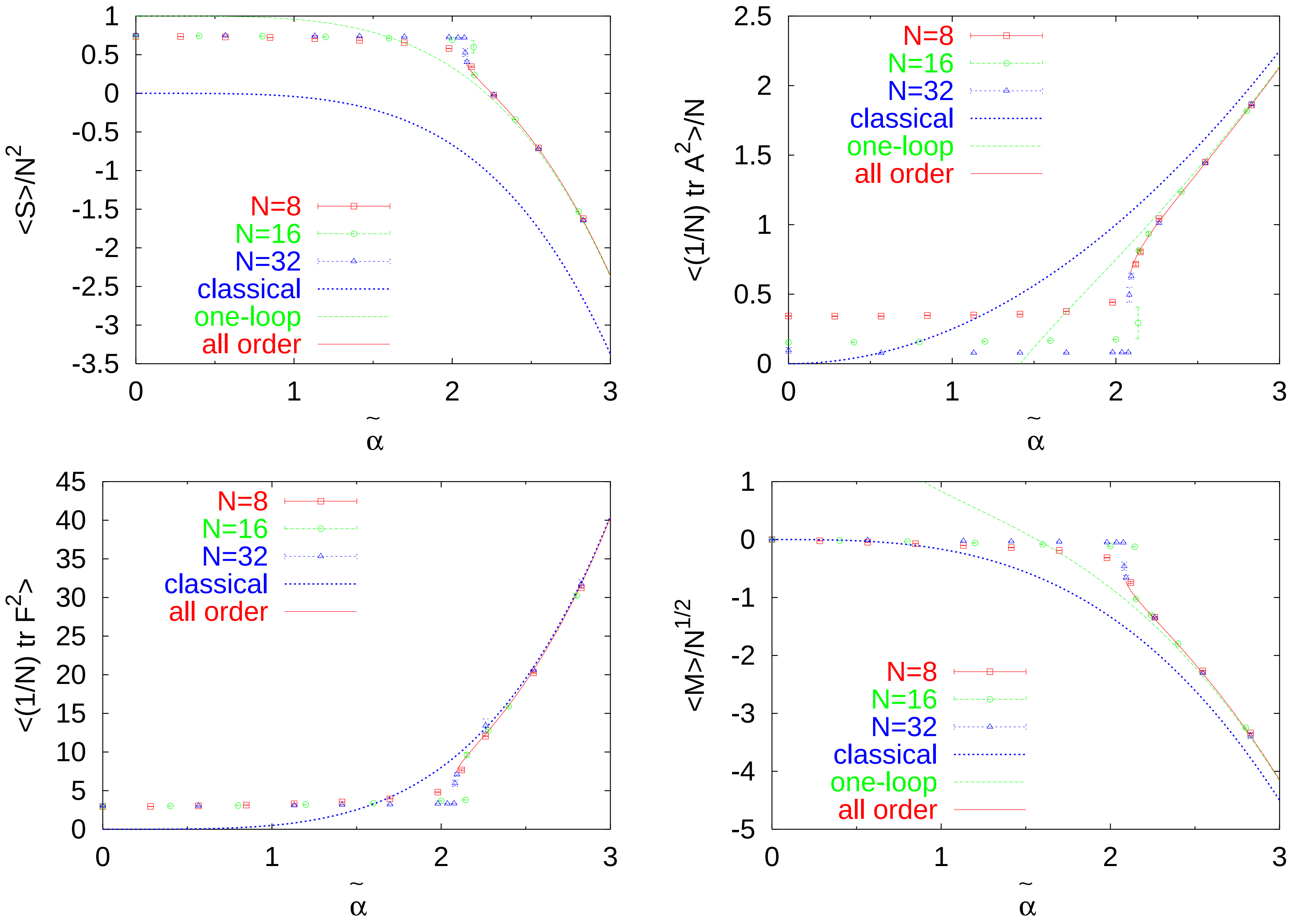
The one-loop effect comes from tadpole diagrams.

$$\frac{1}{N}\langle \frac{1}{N}\mathrm{tr}\, A^2 \rangle = \frac{1}{4} f(\tilde{\alpha})^2 = \frac{1}{4}\tilde{\alpha}^2 - \frac{1}{\tilde{\alpha}^2} - \frac{5}{\tilde{\alpha}^6} - \frac{48}{\tilde{\alpha}^{10}} - \frac{572}{\tilde{\alpha}^{14}} - \cdots .$$

Other observables:

$$\lim_{N\to\infty} \frac{1}{\sqrt{N}} \langle M \rangle = -\frac{1}{6} f(\tilde{\alpha})^3 = -\frac{1}{6}\tilde{\alpha}^3 + \frac{1}{\tilde{\alpha}} + \frac{4}{\tilde{\alpha}^5} + \frac{112}{3\tilde{\alpha}^9} + \frac{440}{\tilde{\alpha}^{13}} + \cdots ,$$

$$\lim_{N\to\infty} \left\langle \frac{1}{N}\mathrm{tr}\,(F_{\mu\nu})^2 \right\rangle = 3 + \frac{1}{2}\tilde{\alpha} f(\tilde{\alpha})^3 = \frac{1}{2}\tilde{\alpha}^4 - \frac{12}{\tilde{\alpha}^4} - \frac{112}{\tilde{\alpha}^8} - \frac{1320}{\tilde{\alpha}^{12}} - \cdots .$$



<S>/N^2
<(1/N) tr A^2>/N
<(1/N) tr F^2>
<M>/N^{1/2}
α̃
N=8
N=16
N=32
classical
one-loop
all order

5 Dynamical generation of gauge group

Expansion around k coincident fuzzy spheres $A_\mu = X_\mu + \tilde{A}_\mu$, where

$$X_\mu = \alpha L^{(n)}_\mu \otimes 1_k.$$

Quantum field theory with U(k) gauge group.

Fuzzy sphere is a compact manifold.

It is realized by the finite $N = nk$ matrices.

It facilitates the numerical treatment of the gauge group.

Simulation from zero start $A^{(0)}_\mu = 0$ for $N = 16$, $\alpha = 2.0$.

Metastability of multi-fuzzy-sphere state.

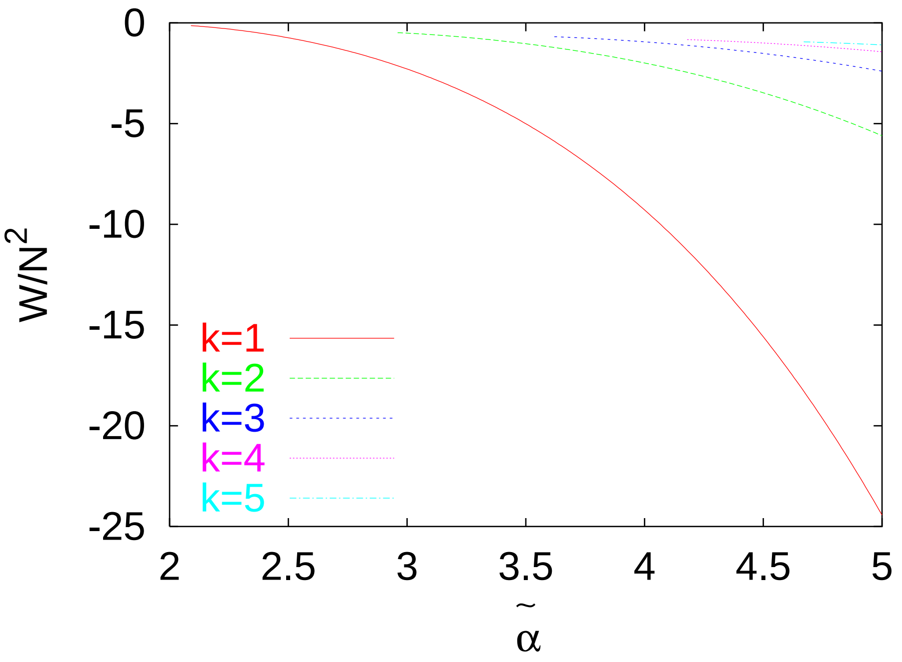
$$\underbrace{A^{(0)}_\mu = 0}_{\text{initial state}} \rightarrow \cdots \rightarrow \underbrace{A_\mu = \alpha \begin{pmatrix} L^{(6\to5\to4\to3\to2\to1)}_\mu & 0 \\ 0 & L^{(10\to11\to12\to13\to14\to15)}_\mu \end{pmatrix}}_{\text{metastable vacuum}} \rightarrow \underbrace{A_\mu = \alpha L_\mu}_{\text{stable vacuum}}.$$

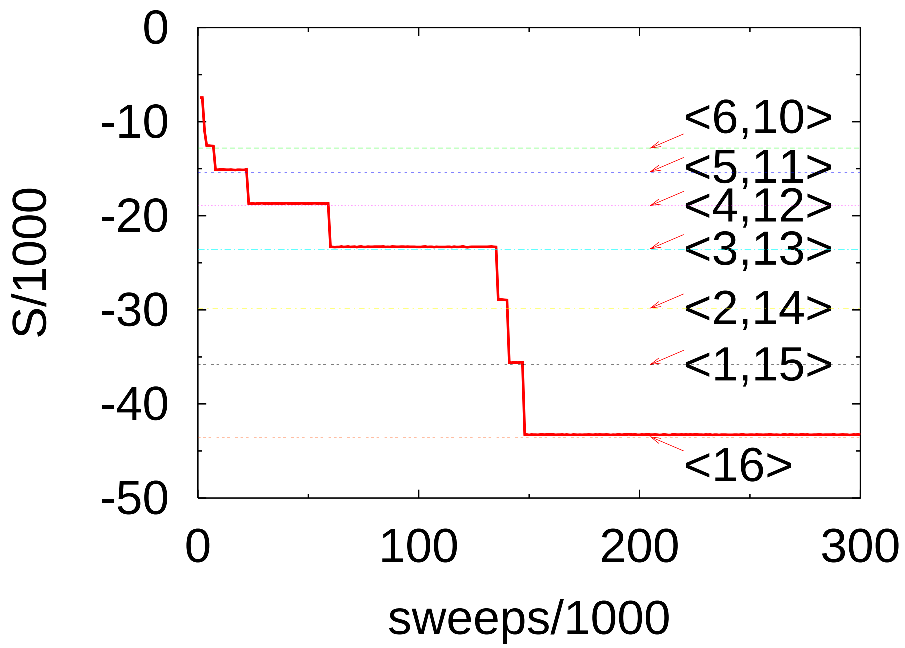
Analytical results

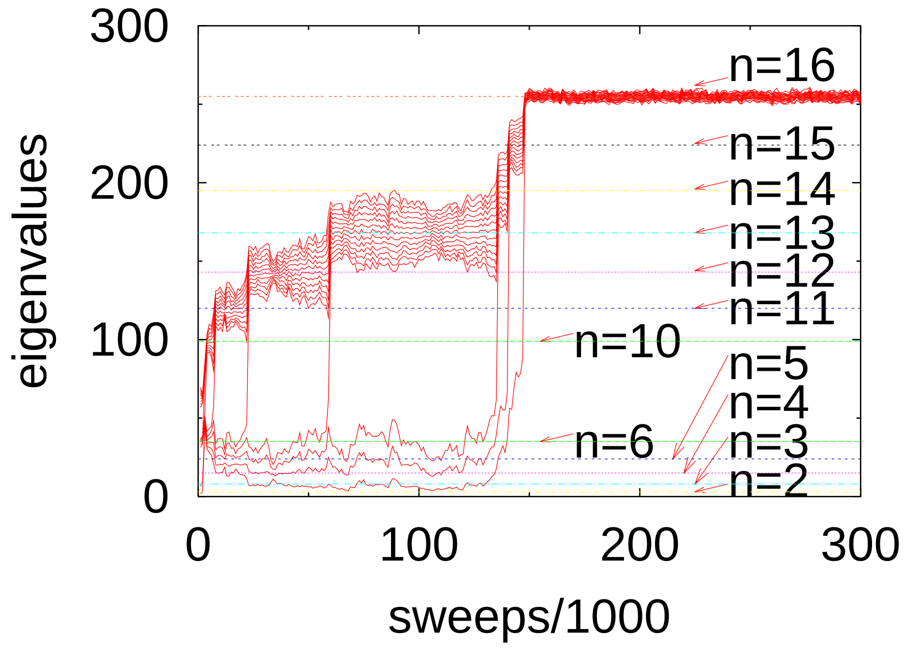
Calculation of the free energy

$$W = -\log\left(\int d\tilde{A} e^{-S}\right).$$

$k = 1$ **has the lowest free energy to all order of perturbation.**







6 Conclusion

- **In this talk, we have scrutinized the perturbative dynamics of the 3d YMCS model.**
- **We have obtained the all order results for generic observables at large N.**
- **We have refined the analytical argument of the dynamical generation of the gauge group. Finally, U(1) gauge group appears.**

Future direction

- **Extension of this technique to the 4-dimensional fuzzy manifolds:**
  fuzzy $\mathbf{CP}^2$ (hep-th/0405277), fuzzy $\mathbf{S}^2 \times \mathbf{S}^2$ (hep-th/0503*).**