Monte Carlo studies of the large-N reduced models and dynamical generation of the spacetime

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Contents

1 Introduction

Matrix models as a constructive definition of superstring theory

IKKT model (IIB matrix model)

⇒ Promising candidate for the constructive definition of superstring theory.
Ishibashi, Kawai, Kitazawa and Tsuchiya, hep-th/9612115.

$$S = N\left(-\frac{1}{4}\mathrm{tr}\,[A_\mu, A_\nu]^2 + \frac{1}{2}\mathrm{tr}\,\bar{\psi}_\alpha(\Gamma_\mu)_{\alpha\beta}[A_\mu, \psi_\beta]\right).$$

- A_μ (10d vector) and ψ_α (10d Majorana-Weyl spinor) ⇒ $N \times N$ matrices .
- Evidences for spontaneous breakdown of SO(10) symmetry to SO(4).
  Nishimura and Sugino, hep-th/0111102, Kawai, et. al. hep-th/0204240,0211272,0602044,0603146.
- Complex fermion determinant:
  - Crucial for rotational symmetry breaking.
    Nishimura and Vernizzi, hep-th/0003223.
  - Difficulty of Monte Carlo simulation.

Brief History of the Monte Carlo simulation of large-N reduced models

Bosonic models

- **Simulation of bosonic Yang-Mills model** T. Hotta, J. Nishimura and A. Tsuchiya, hep-th/9811220
- **Simulation of bosonic Yang-Mills-Chern-Simons models**

  $\Rightarrow$ **Properties of fuzzy manifolds (fuzzy S^2, S^4, CP^2, $S^2 \times S^2$).**

  T. Azuma, S. Bal, K. Nagao and J. Nishimura hep-th/0401038,0405096,0405277,0506205
- **Simulation of finite-temperature BFSS-type (0+1)d models.**

  N. Kawahara, J. Nishimura and S. Takeuchi, arXiv:0704.3183, 0706.3517, 0710.2188,
  O. Aharony, J. Marsano, S. Minwalla, K. Papadodimas and M. Raamsdonk, hep-th/0508077,
  T. Azuma, P. Basu and S.R. Wadia, arXiv:0710.5873.

Supersymmetric models

Simulation of IIB matrix model is difficult due to fermion determinant.

- **hybrid Monte Carlo simulation of the 4d supersymmetric model (fermion determinant is real positive, O(N^6) CPU times).**

  J. Ambjorn, K. N. Anagnostopoulos, W. Bietenholz, T. Hotta and J. Nishimura, hep-th/0003208,
  K. N. Anagnostopoulos, T. Azuma, K. Nagao and J. Nishimura, hep-th/0506062.

- **hybrid Monte Carlo simulation of the one-loop effective action of the quenched 10d IIB matrix model, (O(N^3) CPU time).**

  J. Ambjorn, K. N. Anagnostopoulos, W. Bietenholz, T. Hotta and J. Nishimura, hep-th/0005147.

Complex action is important in spontaneous breakdown of Lorentz symmetry:

- **Factorization method to simulate a complex action system.**

  K. N. Anagnostopoulos and J. Nishimura, hep-th/0108041,
  J. Ambjorn, K. N. Anagnostopoulos, J. Nishimura and J. J. M. Verbaarschot, hep-lat/0208025.

- **Non-lattice simulation of (0+1)d supersymmetric matrix model.**

  M. Hanada, J. Nishimura and S. Takeuchi arXiv:0706.1647,
  K. N. Anagnostopoulos, M. Hanada, J. Nishimura and S. Takeuchi, arXiv:0707.4454.

2 Gaussian toy model

Simplified model with spontaneous rotational symmetry breakdown

Nishimura, hep-th/0108070.

$$S = \underbrace{\frac{N}{2}\mathrm{tr}\, A_\mu^2}_{=S_b} \underbrace{-\bar{\psi}_\alpha^f (\Gamma_\mu)_{\alpha\beta} A_\mu \psi_\beta^f}_{=S_f}$$

$$\Gamma_1 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \Gamma_2 = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \Gamma_3 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \Gamma_4 = i\sigma_4 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}.$$

- A_μ: $N \times N$ hermitian matrices ($\mu = 1, \cdots, 4$)
  $\bar{\psi}_\alpha^f, \psi_\alpha^f$: N-dim vector ($\alpha = 1, 2, f = 1, \cdots, N_f$)
  $N_f =$ (number of flavors)
- SO(4) rotational symmetry.
- No supersymmetry.

- **Partition function:**

$$Z = \int dA e^{-S_B} (\det \mathcal{D})^{N_f} = \int dA e^{-S_0} e^{i\Gamma}, \text{ where}$$

$$\mathcal{D} = \Gamma_\mu A_\mu = \begin{pmatrix} A_3 + iA_4 & A_1 - iA_2 \\ A_1 + iA_2 & -A_3 + iA_4 \end{pmatrix} = (2N \times 2N \text{ matrices}),$$

Phase-quenched partition function

$$Z_0 = \int dA e^{-S_0} = \int dA e^{-S_B} |\det \mathcal{D}|^{N_f}.$$

$(\Gamma_4)^\dagger = -\Gamma_4$, $(\Gamma_i)^\dagger = \Gamma_i$ $(i = 1, 2, 3)$

$\Rightarrow$ $\mathcal{D}$ **becomes complex conjugate under**

$$A_i^P = A_i (i = 1, 2, 3), \quad A_4^P = -A_4.$$

In general, $\det \mathcal{D}$ is complex, while $\det \mathcal{D}$ is real when $A_4 = 0$.

(example):

$$A_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & -2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & -3 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 1-i & 1-2i \\ 1+i & 2 & 2-3i \\ 1+2i & 2+3i & -3 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 2 & 3+i & 3+2i \\ 3-i & 2 & 4+3i \\ 3-2i & 4-3i & -4 \end{pmatrix}$$

$$\Rightarrow \mathcal{D} = \Gamma_\mu A_\mu = \begin{pmatrix} A_3 + iA_4 & A_1 - iA_2 \\ A_1 + iA_2 & -A_3 + iA_4 \end{pmatrix}$$

$$= \left(\begin{array}{ccc|ccc} 1+2i & (1-i)+i(3+i) & (1-2i)+i(3+2i) & 1-2i & 1-2i & 1-2i \\ (1+i)+i(3-i) & 2+2i & (2-3i)+i(4+3i) & 1-2i & 1-i & 1-2i \\ (1+2i)+i(3-2i) & (2+3i)+i(4-3i) & -3-4i & 1-2i & 1-2i & -2+3i \\ \hline 1+2i & 1+2i & 1+2i & -1+2i & -(1-i)+i(3+i) & -(1-2i)+i(3+2i) \\ 1+2i & 1+i & 1+2i & -(1+i)+i(3-i) & -2+2i & -(2-3i)+i(4+3i) \\ 1+2i & 1+2i & -2-3i & -(1+2i)+i(3-2i) & -(2+3i)+i(4-3i) & 3-4i \end{array}\right)$$

$$\Rightarrow \det \mathcal{D} = -5572 + 304i \notin R$$

If $A_4 = 0$, we have

$$\mathcal{D} = \Gamma_\mu A_\mu = \begin{pmatrix} A_3 & A_1 - iA_2 \\ A_1 + iA_2 & -A_3 \end{pmatrix} = \left(\begin{array}{ccc|ccc} 1 & 1-i & 1-2i & 1-2i & 1-2i & 1-2i \\ 1+i & 2 & 2-3i & 1-2i & 1-i & 1-2i \\ 1+2i & 2+3i & -3 & 1-2i & 1-2i & -2+3i \\ \hline 1+2i & 1+2i & 1+2i & -1 & -(1-i) & -(1-2i) \\ 1+2i & 1+i & 1+2i & -(1+i) & -2 & -(2-3i) \\ 1+2i & 1+2i & -2-3i & -(1+2i) & -(2+3i) & 3 \end{array}\right)$$

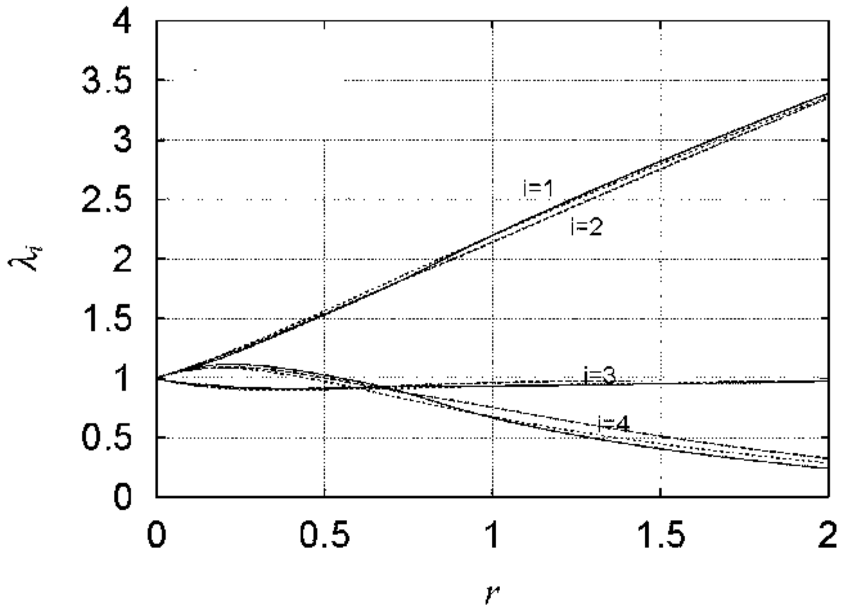
$$\Rightarrow \det \mathcal{D} = -670 \in R$$

(*) $N = 2 \Rightarrow (N^2 - 1) < D = 4 \Rightarrow$ **we can set** $A_4 = 0$ **by SO(4) transformation** $\Rightarrow \det \mathcal{D} \in R$.

Gaussian expansion analysis up to 9th order:
Okubo, Nishimura and Sugino, hep-th/0412194.

Observable for probing dimensionality : $T_{\mu\nu} = \frac{1}{N}\mathrm{tr}\,(A_\mu A_\nu)$.

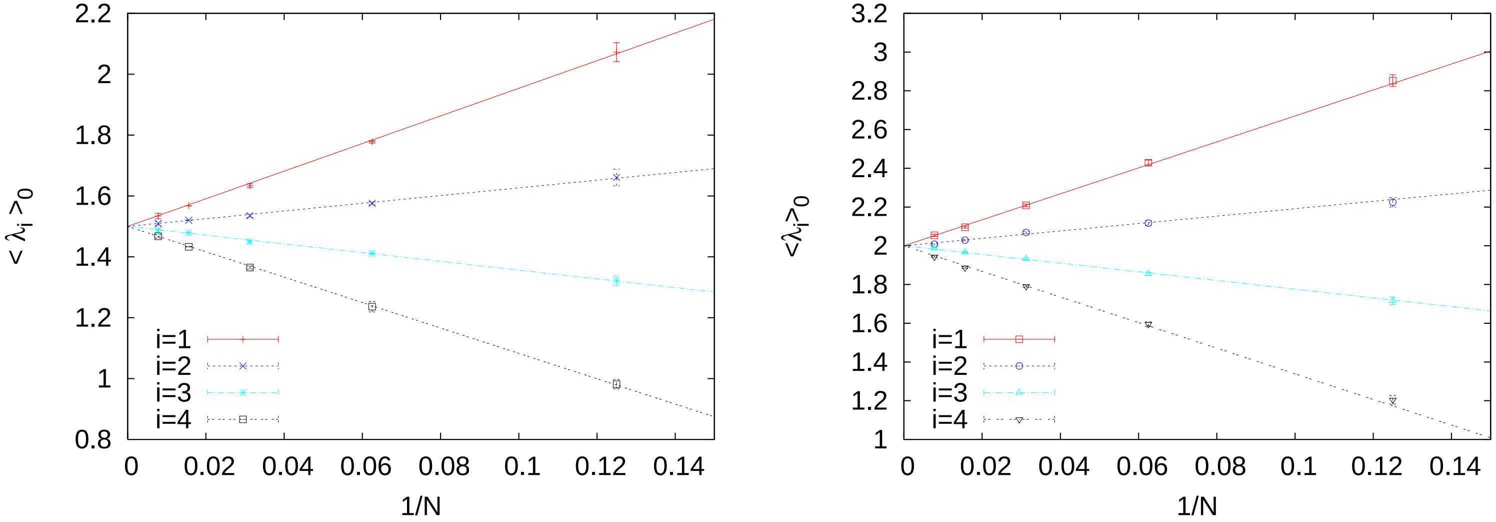
λ_i $(i = 1, 2, 3, 4)$ **: eigenvalues of** $T_{\mu\nu}$ $(\lambda_1 \geqq \lambda_2 \geqq \lambda_3 \geqq \lambda_4)$



Spontaneous breakdown of SO(4) to SO(2) at finite $r \left(= \frac{N_f}{N}\right)$.

Hybrid Monte Carlo (HMC) simulation of the phase-quenched model

HMC simulation of the partition function Z_0 with the phase omitted.



Results for $r = 1$ (left) and $r = 2$ (right).

$$\lambda_1 = \cdots = \lambda_4 \to 1 + \frac{r}{2} \text{ (as } N \to \infty).$$

The effect of the phase is crucial for the spontaneous rotational symmetry breaking.

Factorization method

An approach to the complex action problem in Monte Carlo simulation.

K. N. Anagnostopoulos and J. Nishimura, hep-th/0108041,
J. Ambjorn, K. N. Anagnostopoulos, J. Nishimura and J. J. M. Verbaarschot, hep-lat/0208025.

Overlap problem: Discrepancy of a distribution function between the phase-quenched model Z_0 and the full model Z.

Standard reweighting method:

$$\langle \lambda_i \rangle = \frac{\langle \lambda_i e^{i\Gamma} \rangle_0}{\langle e^{i\Gamma} \rangle_0} = \frac{\langle \lambda_i \cos \Gamma \rangle_0}{\langle \cos \Gamma \rangle_0},$$ where $\langle * \rangle_0 =$ (V.E.V. for the phase-quenched model Z_0).

Under parity transformation $A_i^P = A_i (i = 1, 2, 3),\ A_4^P = -A_4 \Rightarrow$

- λ_i (eigenvalues of $T_{\mu\nu} = \frac{1}{N} \mathrm{tr}\, A_\mu A_\nu$) are invariant.
- Partition function Z is complex conjugate.

(Number of configurations required) $\simeq e^{O(N^2)}$. $\Rightarrow$ complex-action problem.

$\tilde{\lambda}_i \stackrel{\text{def}}{=} \lambda_i / \langle \lambda_i \rangle_0$: deviation from 1 $\Rightarrow$ effect of the phase.

Distribution function

$$\rho_i(x) \stackrel{\text{def}}{=} \langle \delta(x - \tilde{\lambda}_i) \rangle = \frac{\langle \delta(x - \tilde{\lambda}_i) \cos \Gamma \rangle_0}{\langle \cos \Gamma \rangle_0} = \frac{\langle \delta(x - \tilde{\lambda}_i) \rangle_0 \langle \cos \Gamma \rangle_{i,x}}{\langle \cos \Gamma \rangle_0} = \frac{1}{C} \rho_i^{(0)}(x) w_i(x),$$

where

$$C = \langle \cos \Gamma \rangle_0, \quad \rho_i^{(0)}(x) = \langle \delta(x - \tilde{\lambda}_i) \rangle_0, \quad w_i(x) = \langle \cos \Gamma \rangle_{i,x},$$

$$\langle * \rangle_{i,x} = [\text{V.E.V. for the partition function } Z_{i,x} = \int dA e^{-S_0} \delta(x - \tilde{\lambda}_i)].$$

Resolution of the overlap problem:

The system is forced to visit the configurations where $\rho_i(x)$ is important.

In practice, we approximate the partition function $Z_{i,x}$ by

$$Z_{i,V} = \int dA e^{-S_0} e^{-V(\lambda_i)}, \text{ where } V(x) = \frac{\gamma}{2}(x-\xi)^2, \quad \gamma, \xi = \text{(parameters)}.$$

Monte Carlo evaluation of $\rho_i^{(0)}(x)$ and $w_i(x)$:

$$\rho_{i,V}(x) \stackrel{\text{def}}{=} \langle \delta(x - \tilde{\lambda}_i) \rangle_{i,V} \propto \rho_i^{(0)}(x) \exp(-V(\langle \lambda_i \rangle_0 x)).$$

The position of the peak x_p for the distribution function $\rho_{i,V}(x)$:

$$0 = \frac{\partial}{\partial x} \log \rho_{i,V}(x) = f_i^{(0)}(x) - \langle \lambda_i \rangle_0 V'(\langle \lambda_i \rangle_0 x), \text{ where } f_i^{(0)}(x) \stackrel{\text{def}}{=} \frac{\partial}{\partial x} \log \rho_i^{(0)}(x).$$

- **Determination of x_p: $\rho_{i,V}(x)$ has a sharp peak for large γ**
  $\Rightarrow x_p$ is approximated as $x_p \simeq \langle \tilde{\lambda}_i \rangle_{i,V}$.
- **Determination of $\rho_i^{(0)}(x)$: Vary ξ, and calculate $f_i^{(0)}(x_p)$ for different x_p.**
  Then, evaluate $\rho_i^{(0)}(x) = \exp\left\{ \int_0^x dz f_i^{(0)}(z) + \text{const.} \right\}$.

Monte Carlo evaluation of $\langle \tilde{\lambda}_i \rangle$

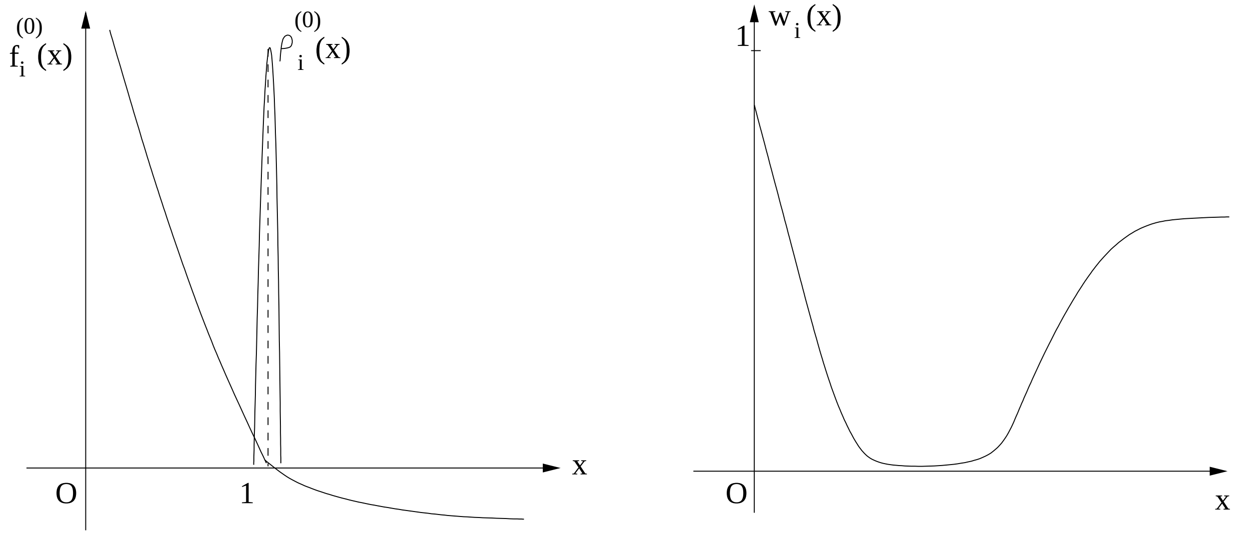
$w_i(x) > 0 \Rightarrow \langle \tilde{\lambda}_i \rangle$ **is the minimum of** $\mathcal{F}_i(x)$:

$$\mathcal{F}_i(x) = (\text{free energy density}) = -\frac{1}{N^2} \log \rho_i(x).$$

We solve $\mathcal{F}'_i(x) = 0$, **namely** $\frac{1}{N^2} f_i^{(0)}(x) = -\frac{d}{dx}\left\{\frac{1}{N^2}\log w_i(x)\right\}$.

Both $\frac{1}{N^2}\log w_i(x)$ **and** $\frac{1}{N^2} f_i^{(0)}(x)$ **scale at large** N **as**

$$\frac{1}{N^2}\log w_i(x) \to \Phi_i(x), \quad \frac{1}{N^2} f_i^{(0)}(x) \to F_i(x)$$



Behavior of $\Phi_i(x)$

Asymptotic behavior of $\Phi_i(x) = \dfrac{1}{N^2}\log w_i(x)$ at $x \ll 1$ and $x \gg 1$.

When we fix the i-th largest eigenvalue $\rightarrow$

- **$x \ll 1$ $(i = 2, 3, 4)$: $(5 - i)$ directions are shrunk**
  $\Rightarrow$ $(i - 1)$-dimensional configuration

- **$x \gg 1$ $(i = 1, 2, 3)$: $(4 - i)$ directions are shrunk**
  $\Rightarrow$ i-dimensional configuration

Fermion determinant $\mathcal{D}$ is complex conjugate under

$$A_i^P = A_i (i = 1, 2, 3), \;\; A_4^P = -A_4$$

$$\Omega_d = \{\{A_\mu\}; n_\mu^{(i)} A_\mu = 0 \text{ for } \exists n_\mu^{(i)} (i = 1, \cdots 4 - d)\}$$

3-dimensional configuration Ω_3 $\Rightarrow$ Fermion determinant is real.

J. Nishimura and G. Vernizzi, hep-th/0003223.

For d-dimensional configuration Ω_d,

$$\frac{\partial^n \Gamma}{\partial A^{a_1}_{\mu_1} \cdots \partial A^{a_n}_{\mu_n}} = 0 \text{ for } n = 1, \cdots 3 - d$$

(Up to $(3-d)$-order perturbation $\Rightarrow$ configuration $\in \Omega_3$)

Expected power behaviors:

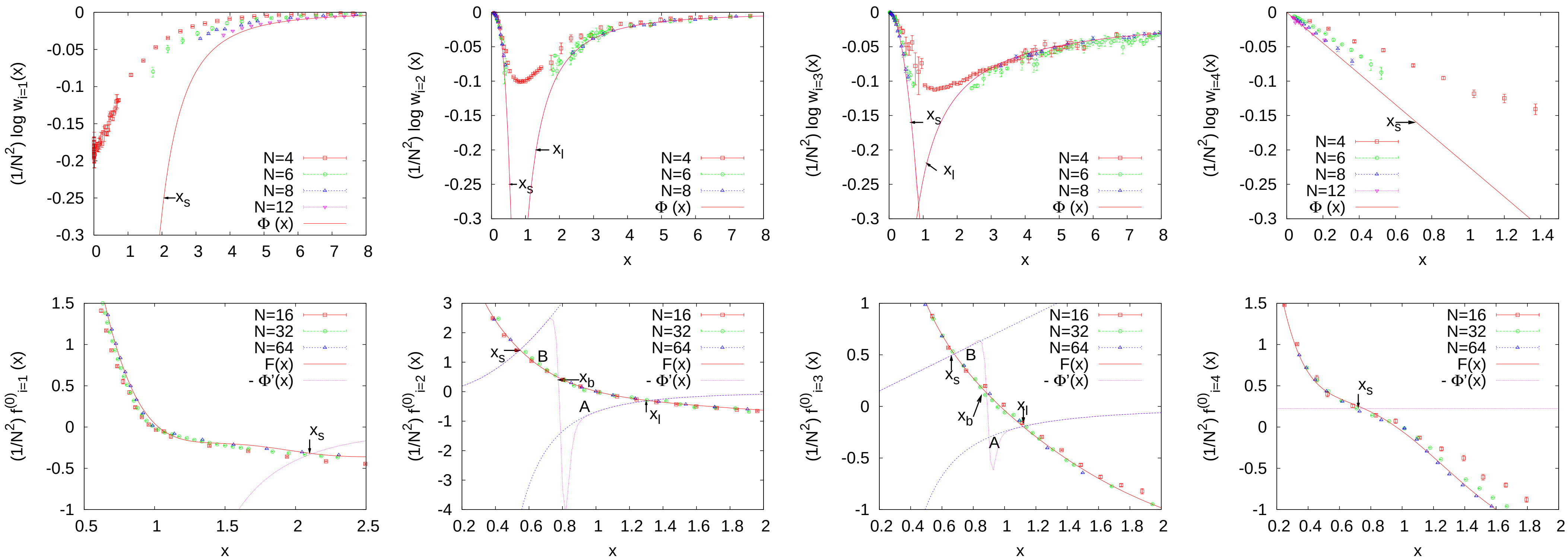
$$\Phi_i(x) \propto \begin{cases} c_{i,0} x^{5-i} + \cdots & (x \ll 1, i = 2, 3, 4) \\ \frac{d_{i,0}}{x^{4-i}} + \cdots & (x \gg 1, i = 1, 2, 3) \end{cases}$$

(*) x has the order of the eigenvalues of $T_{\mu\nu} = \frac{1}{N} \mathrm{tr}\,(A_\mu A_\nu)$.

Simulation for $r = 1$

Evaluation of $\langle \tilde{\lambda}_i \rangle$ at the leading order.

$i = 1$ $\qquad i = 2$ $\qquad i = 3$ $\qquad i = 4$



$\langle \tilde{\lambda}_{i=2} \rangle = 1.3$, $\langle \tilde{\lambda}_{i=3} \rangle = 0.7 \Rightarrow$ **Rotational symmetry breaking SO(4) → SO(2).**

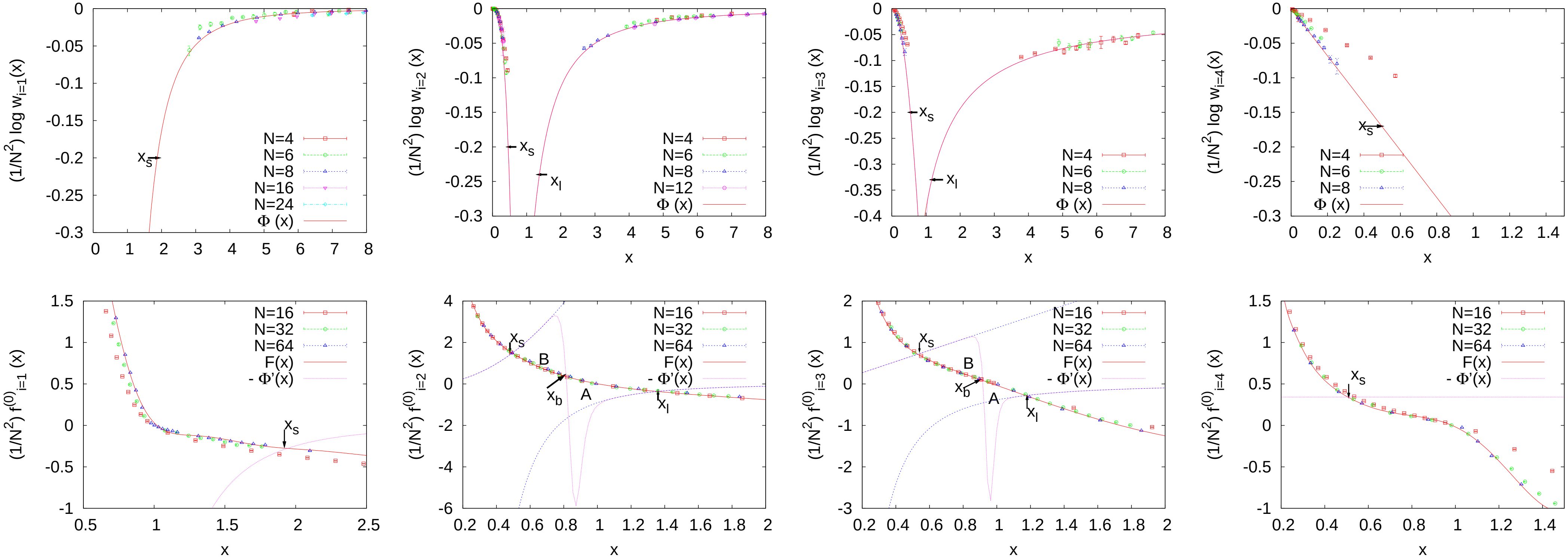
Result of 9th-order Gaussian expansion:

$$\tilde{\lambda}_{i=1} \simeq 1.4,\ \tilde{\lambda}_{i=2} \simeq 1.4,\ \tilde{\lambda}_{i=3} \simeq 0.7,\ \tilde{\lambda}_{i=4} \simeq 0.5.$$

Simulation for $r = 2$

Evaluation of $\langle \tilde{\lambda}_i \rangle$ at the leading order.

$i = 1$ $\quad$ $i = 2$ $\quad$ $i = 3$ $\quad$ $i = 4$



$\langle \tilde{\lambda}_{i=2} \rangle = 1.4$, $\langle \tilde{\lambda}_{i=3} \rangle = 0.5 \Rightarrow$ **Rotational symmetry breaking SO(4) → SO(2).**

Result of 9th-order Gaussian expansion:

$$\tilde{\lambda}_{i=1} \simeq 1.7,\ \tilde{\lambda}_{i=2} \simeq 1.7,\ \tilde{\lambda}_{i=3} \simeq 0.5,\ \tilde{\lambda}_{i=4} \simeq 0.1.$$

Double-peak structure of $\rho_i(x)$ for $i = 2, 3$

Extrapolation of $\Phi_i(x)$:

$$\Phi_i(x) \simeq \begin{cases} \phi_{i,s}(x) = c_{i,0} x^{5-i} + \cdots, & (x \ll 1), \\ \phi_{i,l}(x) = \frac{d_{i,0}}{x^{4-i}} + \cdots, & (x \gg 1), \\ \frac{\phi_{i,s}(x) e^{-\mathcal{C}(x-\alpha)} + \phi_{i,l}(x) e^{\mathcal{C}(x-\alpha)}}{e^{-\mathcal{C}(x-\alpha)} + e^{\mathcal{C}(x-\alpha)}}, & (\text{intermediate } x). \end{cases}$$

At $x = \alpha$, $\phi_{i,s}(x) = \phi_{i,l}(x)$.

Three solutions of $\mathcal{F}'_i(x) = 0$ $(x_s < x_b < x_l)$.

Which peak is higher?

- $\frac{1}{N^2}(\log \rho_i(x_l) - \log \rho_i(x_b)) = \int_{x_b}^{x_l} dx (F_i(x) + \Phi'_i(x)) = (\text{A's area}).$
- $\frac{1}{N^2}(\log \rho_i(x_s) - \log \rho_i(x_b)) = -\int_{x_s}^{x_b} dx (F_i(x) + \Phi'_i(x)) = (\text{B's area}).$

Difference of the height:

$$\Delta_i = \frac{1}{N^2}(\log \rho_i(x_l) - \log \rho_i(x_s)) = (\Phi_i(x_l) - \Phi_i(x_s)) + \int_{x_s}^{x_l} dx F_i(x) = (\text{A's area})\text{-}(\text{B's area})$$

$$\simeq \begin{cases} +0.24\cdots > 0, \ (i=2), \ -0.01\cdots < 0, \ (i=3), \ [r=1] \\ +0.23\cdots > 0, \ (i=2), \ -0.02\cdots < 0, \ (i=3), \ [r=2] \end{cases}$$

Behavior of $\frac{1}{N^2} f_i^{(0)}(x)$

Small x ($x \ll 1$) $\to$ $(5-i)$ directions are shrunk.

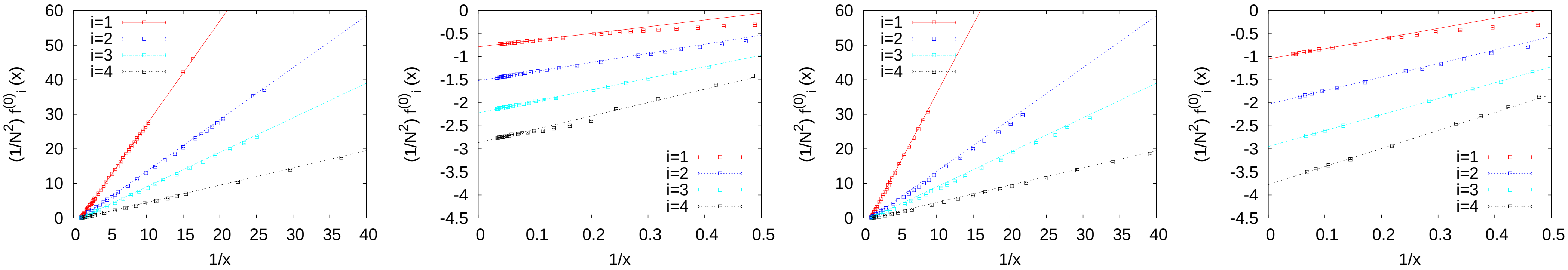
- $i = 2, 3, 4$: $\rho_i^{(0)}(x) \simeq (\sqrt{x})^{N^2(5-i)} \Rightarrow \frac{1}{N^2} f_i^{(0)}(x) = \left(\frac{5-i}{2x}\right) + e_i$
- $i = 1$: Eigenvalues of A_μ are collapsed to zero.

  $\Rightarrow$ Add the effect of fermionic determinant (polynomial of A_μ with degree $2N^2 r$).

  $\Rightarrow \rho_{i=1}^{(0)}(x) \simeq (\sqrt{x})^{2N^2(1+r)} \Rightarrow \frac{1}{N^2} f_i^{(0)}(x) = \left(\frac{2+r}{x}\right) + e_{i=1}$

Large x ($x \gg 1$): $\frac{1}{N^2} f_i^{(0)}(x) \xrightarrow{x \to +\infty} g_{i,0} + \frac{g_{i,1}}{x} + \cdots, \quad g_{i,0} = -\frac{i}{2}\langle \lambda_0 \rangle \xrightarrow{\text{large } N} -\frac{i}{2}\left(1 + \frac{r}{2}\right)$

Simulation for $r = 1, N = 64$ **$r = 2, N = 64$**



3 6d IKKT model

T .Aoyama, T. Azuma, M. Hanada and J. Nishimura

$$\boxed{S = \underbrace{-\frac{N}{4}\mathrm{tr}\,[A_\mu, A_\nu]^2}_{=S_B} + \underbrace{\frac{N}{2}\mathrm{tr}\,\bar{\psi}\Gamma_\mu[A_\mu, \psi]}_{=S_F}.}$$

- A_μ (6d vector) and ψ (6d Weyl spinor) are $N \times N$ matrices .

  $\Gamma_1 = i\sigma_1 \otimes \sigma_2,\ \Gamma_2 = i\sigma_2 \otimes \sigma_2,\ \Gamma_3 = i\sigma_3 \otimes \sigma_2,\ \Gamma_4 = i1 \otimes \sigma_1,\ \Gamma_5 = i1 \otimes \sigma_3,\ \Gamma_6 = 1 \otimes 1.$

- SO(6) rotational symmetry and SU(N) gauge symmetry.

- Presence of $\mathcal{N} = 2$ supersymmetry.

- $Z = \int dA e^{-S_B}(\det \mathcal{M}) = \int dA e^{-S_0} e^{i\Gamma}$. CPU cost is $O(N^6)$.

  4d $\rightarrow$ det $\mathcal{M}$ is real positive

  6d and 10d $\rightarrow$ det $\mathcal{M}$ is complex.

  Complex phase is important in SO(6) breakdown.

- **Previous works on this model:**

  * **Simulation of phase-quenched 6d and 10d IKKT model**

    ⇒ no symmetry breakdown of SO(6) (and SO(10)).

    J. Ambjorn, K. N. Anagnostopoulos, W. Bietenholz, T. Hotta and J. Nishimura, hep-th/0005147

  * **Simulation of one-loop effective action (CPU cost is O(N^3)).**

    K.N. Anagnostopoulos and J. Nishimura, hep-th/0108041.

  * **Gaussian expansion method ⇒ symmetry breakdown of SO(6) to SO(3).**

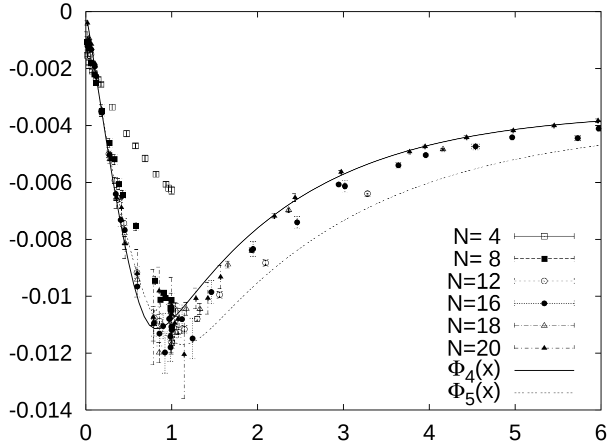
    T. Aoyama, J. Nishimura and T. Okubo

Simulation of the one-loop model

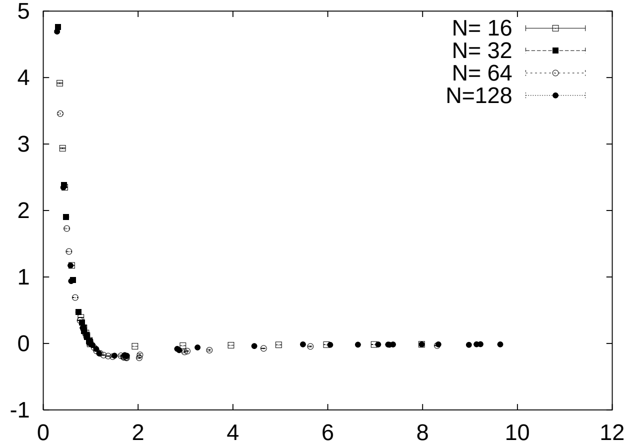
K. N. Anagnostopoulos and J. Nishimura, hep-th/0108041.

Peak of $\rho_{i=4}(x)$ similarly to Gaussian toy model.

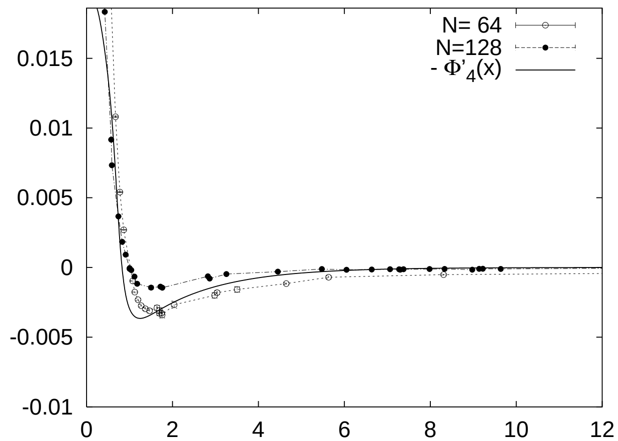
$\frac{1}{N^2}\log w_{i=4}(x)$



$\frac{1}{N}f^{(0)}_{i=4}(x)$



$\frac{1}{N^2}f^{(0)}_{i=4}(x)$ and $-\Phi'_{i=4}(x)$

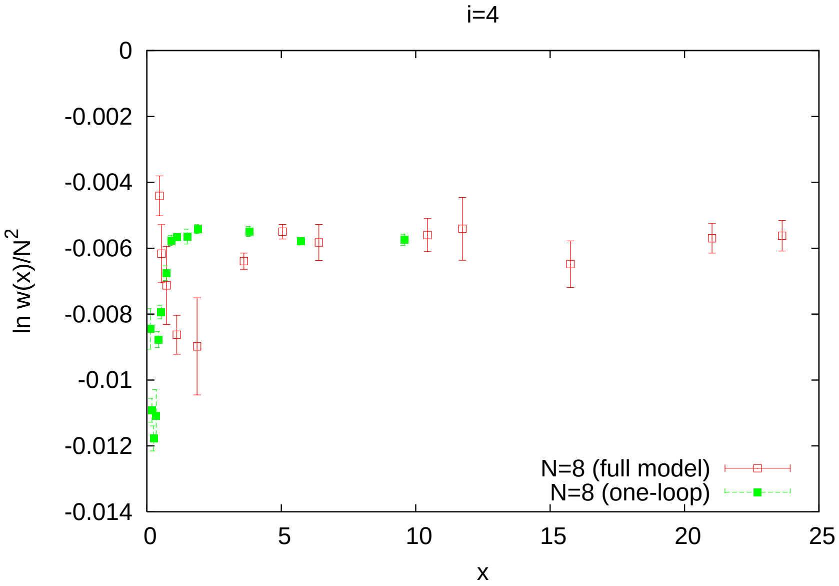


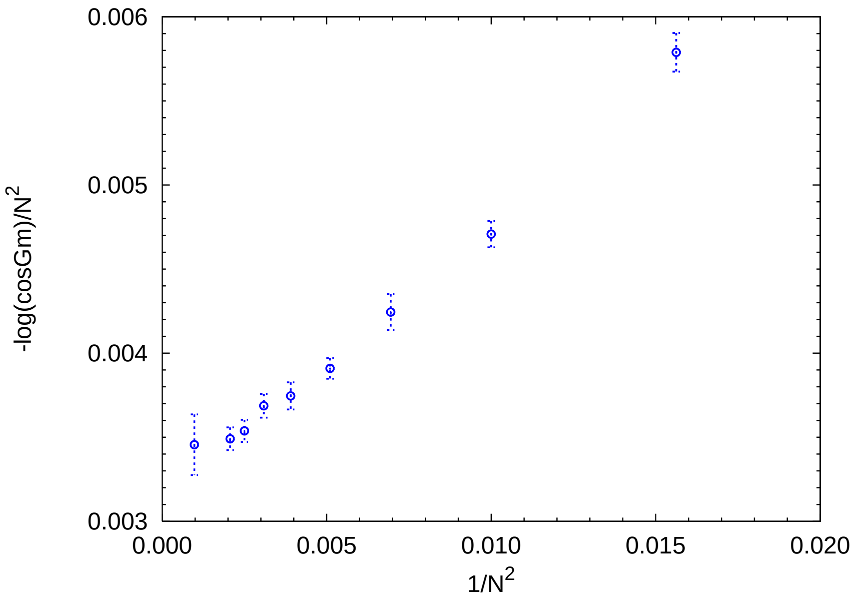
In the equaton $\frac{1}{N^2}f^{(0)}_i(x) = -\frac{d}{dx}(\frac{1}{N^2}\log w_i(x))$,

$\Phi_{i=4}(x) = \frac{1}{N^2}\log w_{i=4}(x)$ **scales, but** $\frac{1}{N}f^{(0)}_{i=4}(x)$ **(not** $\frac{1}{N^2}f^{(0)}_{i=4}(x)$**) scales!**

The peak of $\rho_{i=4}(x) = \langle\delta(x-\tilde{\lambda}_{i=4})\rangle$ **at** $x<1$ **dominates.** $\Rightarrow$ **The spacetime is** $d \le 3$**?**

Then, how about in the full model?





The full model is close to one-loop model at large x.
(comparison of $\frac{1}{N^2}\log w_{i=4}(x)$ at large x for $N = 8$)

For one-loop model, $\frac{1}{N^2}\log w_{i=4}(x) = a_{i,0}(x) + \frac{1}{N^2}a_{i,1}(x) + \cdots$ at large x.

Behavior of $\Phi_i(x)$

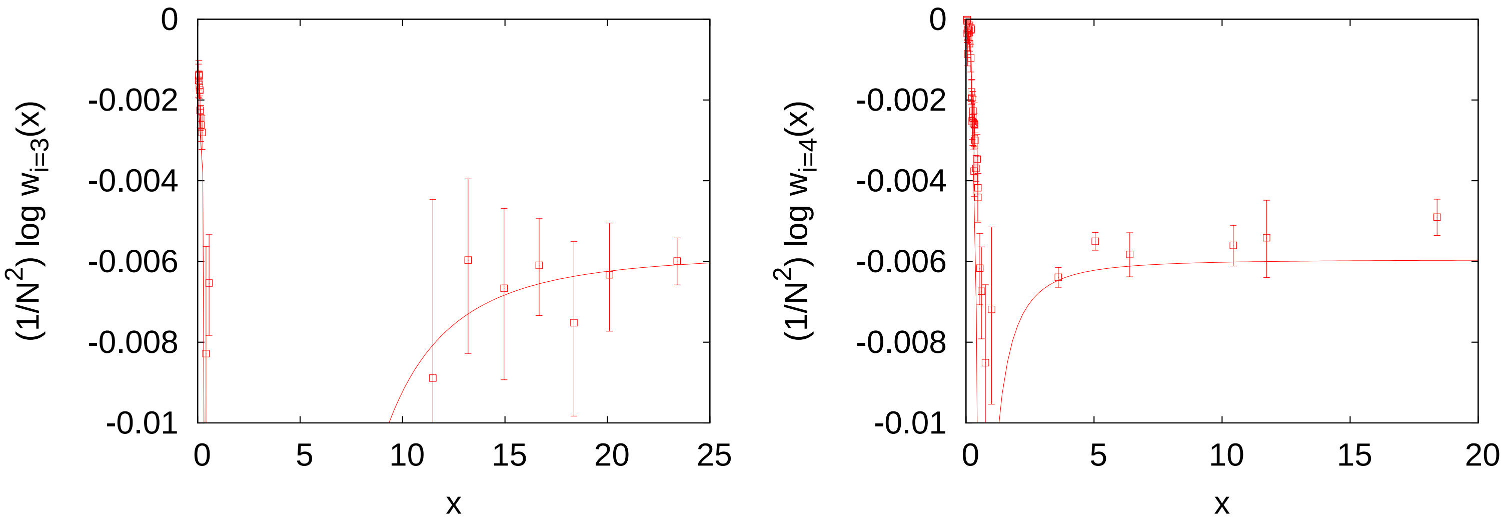
Asymptotic behavior of $\Phi_i(x) = \frac{1}{N^2} \log w_i(x)$ at $x \ll 1$ and $x \gg 1$.

When we fix the i-th largest eigenvalue →

- $x \ll 1$ $(i = 2, \cdots, 6)$: $(7 - i)$ **directions are shrunk** $\Rightarrow$ $(i - 1)$**-dim. configuration**
- $x \gg 1$ $(i = 1, \cdots, 5)$: $(6 - i)$ **directions are shrunk** $\Rightarrow$ i**-dim. configuration**

Expected power behaviors:

$$\Phi_i(x) \propto \begin{cases} c_{i,0} x^{7-i} + \cdots & (x \ll 1, i = 2, 3, 4) \\ \frac{d_{i,0}}{x^{6-i}} + \cdots & (x \gg 1, i = 1, 2, 3) \end{cases}$$



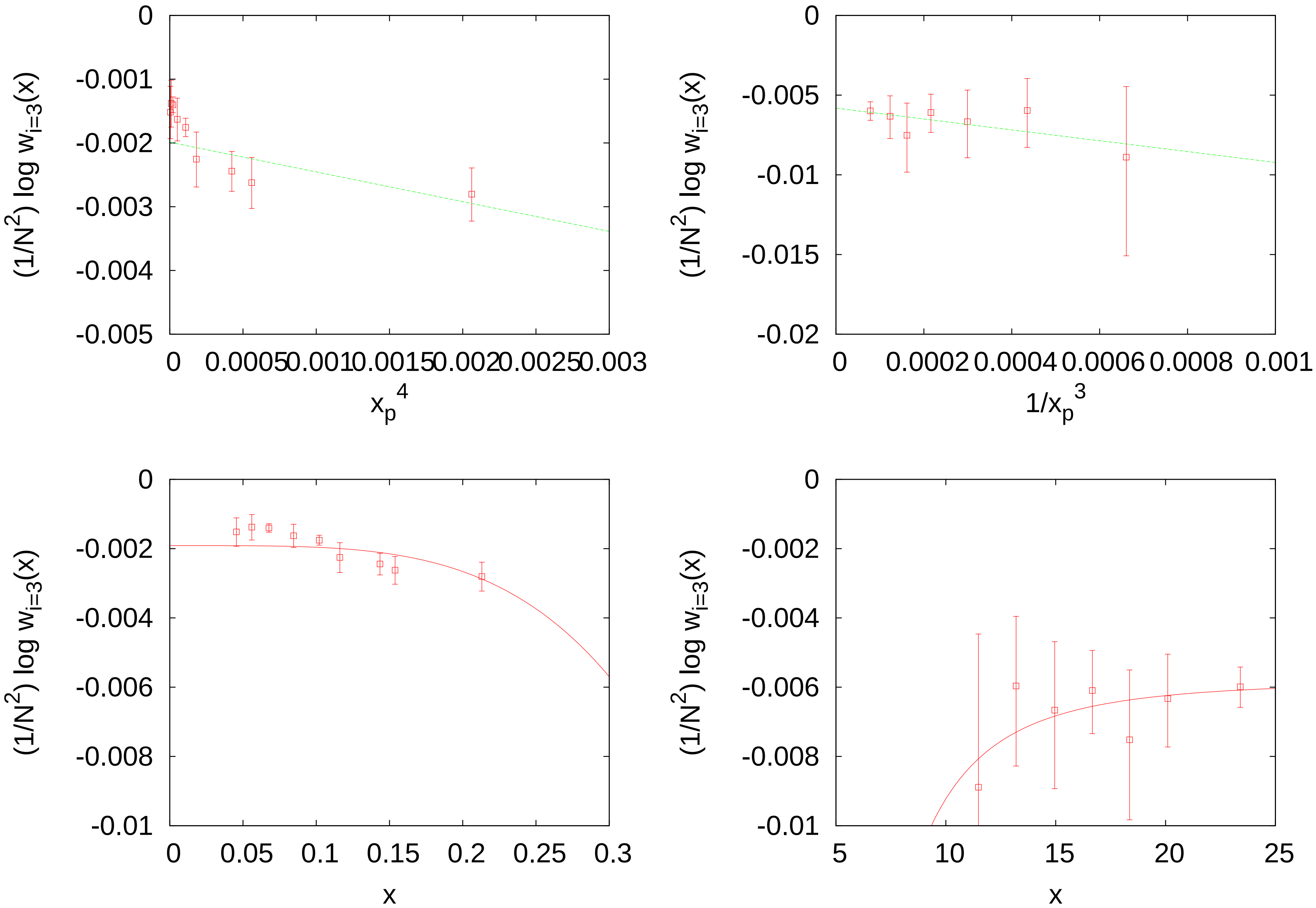


Figure 1: $\frac{1}{N} \log w_{i=3}(x)$ for $N = 8$

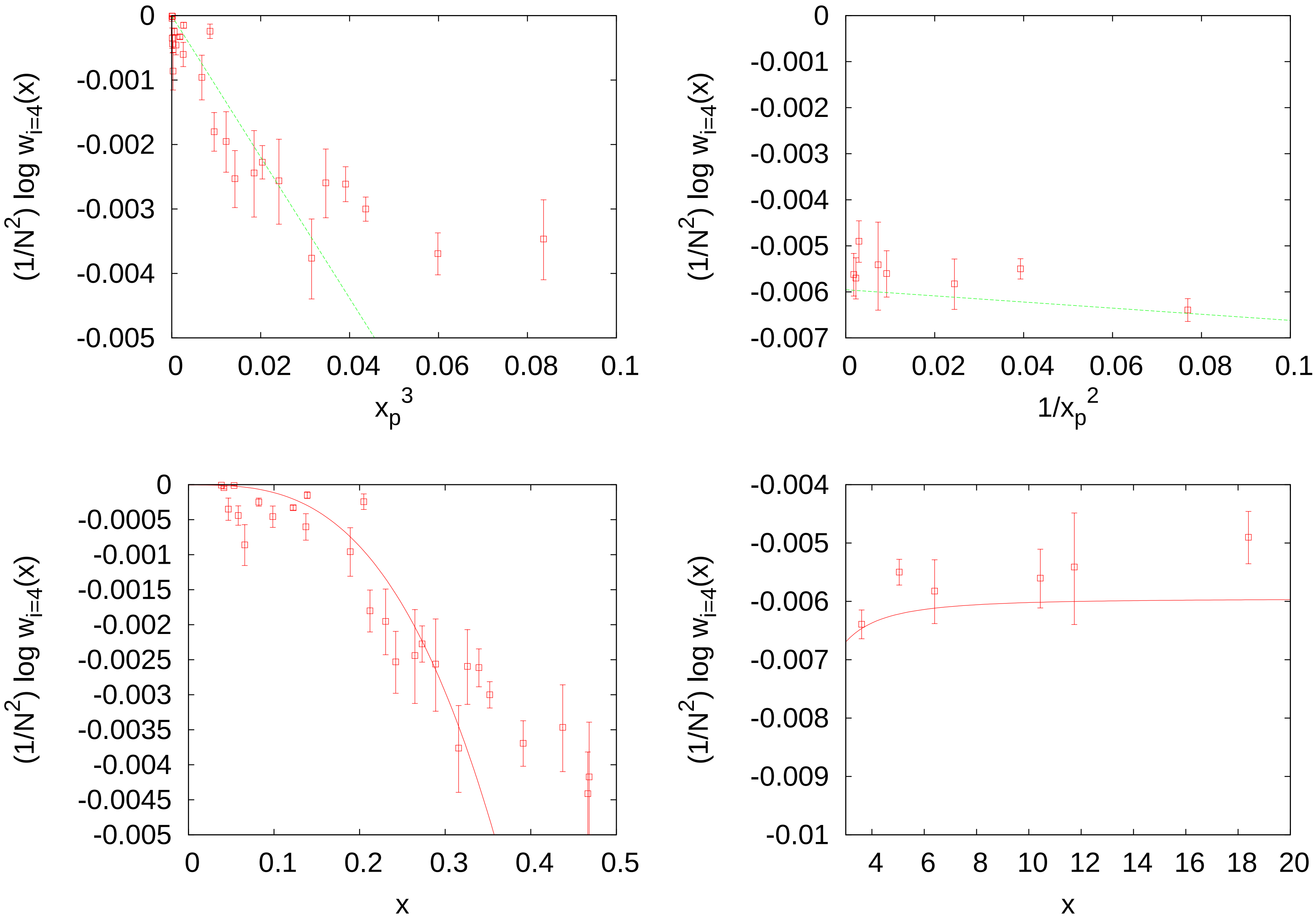


Figure 2: $\frac{1}{N} \log w_{i=4}(x)$ for $N = 8$

Behavior of $\frac{1}{N^2} f_i^{(0)}(x)$

Leading behavior at small x ($x \ll 1$) $\to$ $(7 - i)$ directions are shrunk.

- $i = 2, \cdots, 6$: $\rho_i^{(0)}(x) \simeq (\sqrt{x})^{N^2(7-i)} \Rightarrow \frac{1}{N^2} f_i^{(0)}(x) = \frac{7-i}{2x}$
- $i = 1$: **Eigenvalues of A_μ are collapsed to zero.**

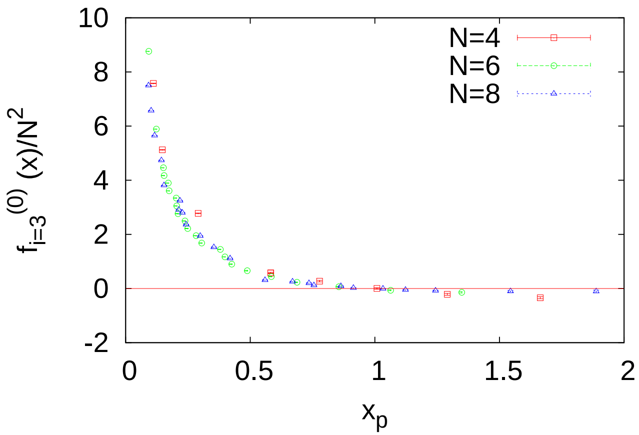
  $\Rightarrow$ **Add the effect of fermionic determinant (polynomial of A_μ with degree $4N^2$).**

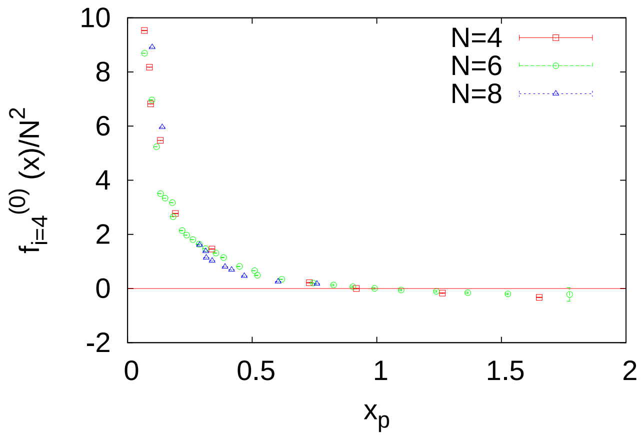
  $\Rightarrow \rho_{i=1}^{(0)}(x) \simeq (\sqrt{x})^{10N^2} \Rightarrow \frac{1}{N^2} f_i^{(0)}(x) = \frac{5}{x}$

At large x: $\frac{1}{N^2} f_i^{(0)}(x) \to 0$.

Ansatz for all x: $\frac{1}{N^2} f_i^{(0)}(x) = \begin{cases} \frac{5}{x} \exp(-b_{i=1} x) & i = 1 \\ \frac{7-i}{2x} \exp(-b_i x) & i = 2, \cdots, 6 \end{cases}$

For $N = 8$ numerical data, we have $b_{i=3,4} \simeq 5$.





Solutions of the equation $\frac{1}{N^2} f_i^{(0)}(x) = -\frac{d}{dx}\left\{\frac{1}{N^2}\log w_i(x)\right\}$ **:**

Double-peak structure for $i = 3, 4 \rightarrow$ **two solutions** x_s **and** x_l $(x_s < x_l = +\infty)$

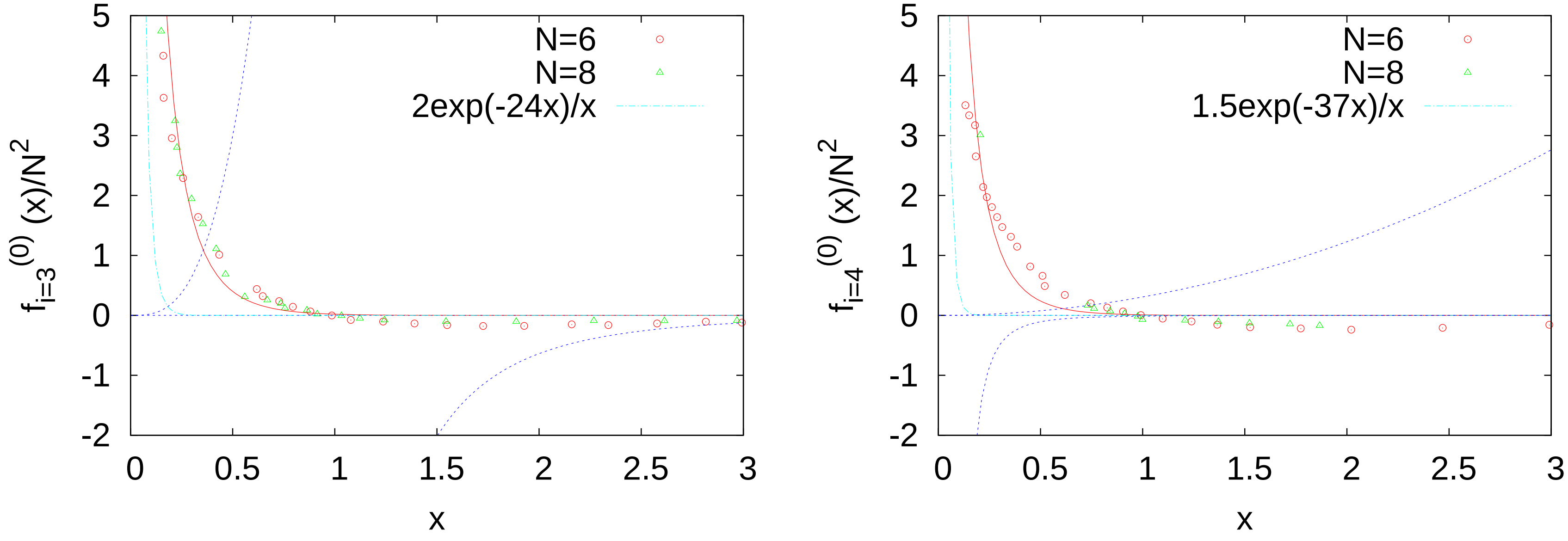
Result of Gaussian Expansion Method (GEM) **T. Aoyama, J. Nishimura and T. Okubo**

- **Symmetry breakdown SO(6) → SO(3):**

  x_l **should dominate for** $i = 3$. x_s **should dominate for** $i = 4$.

- **For collapsed directions,** $x_s = 0.18$.

  To have this solution, the coeffcient b_i **should be** $b_{i=3} \simeq 24, \quad b_{i=4} \simeq 37$.



- **Strong finite-N effect $\rightarrow$ For $i = 4$ we expect $x_s \simeq 0.18$ ($b_{i=4} \simeq 37$) at large N.**

- **For $b_{i=3} \simeq 24, b_{i=4} \simeq 37$ (expected behavior at large N) $\rightarrow$ x_l dominates for $i = 3$ while x_s dominates for $i = 4$.**

Evidence for symmetry breakdown SO(6) $\rightarrow$ SO(3)

4 Conclusion

Numerical simulation of the large-N reduced models $\rightarrow$ the spontaneous breakdown of SO(D) symmetry.

Can we understand the emergence of our four-dimensional world?

- **Simulation of the Gaussian toy model and the 6d IKKT model**

Future works

- **Does the 10-dimensional spacetime break to 4-dimension in IKKT model?**
- **Does the SO(D) symmetry breakdon occur in BFSS or DVV matrix model?**

Detailed definition of $\mathcal{M}$

Introduce a complete basis for general complex $N \times N$ matrices:

$$t^a = E_{i_a j_a}, \ \text{where } a = (i_a - 1)N + j_a, \ E_{i_a j_a} = \begin{cases} 1, & (i_a, j_a) \text{ component} \\ 0, & \text{otherwise} \end{cases}$$

Decomposition with respect to this basis:

$$(\psi_\alpha)_{i_a j_a} = \sum_{a=1}^{N^2} (\psi_{a,\alpha})(t^a)_{i_a j_a}, \ (\bar{\psi}_\alpha)_{i_a j_a} = \sum_{a=1}^{N^2} (\bar{\psi}_{a,\alpha})(t^a)_{i_a j_a}.$$

Tracelessness condition:

$$\underbrace{\psi_{1,\alpha}}_{(i_a,j_a)=(1,1)} + \underbrace{\psi_{N+2,\alpha}}_{(i_a,j_a)=(2,2)} + \cdots + \underbrace{\psi_{N^2,\alpha}}_{(i_a,j_a)=(N,N)} = 0, \ \psi_{1,\alpha} + \psi_{N+2,\alpha} + \cdots + \psi_{N^2,\alpha} = 0.$$

Integrate out $\psi_{N^2,\alpha}$, $\bar{\psi}_{N^2,\alpha}$:

$$\text{tr}\,(N\bar{\psi}_\alpha(\Gamma_\mu)_{\alpha\beta}[A_\mu, \psi_\beta]) = \sum_{a=1}^{N^2}\sum_{b=1}^{N^2}\sum_{\alpha,\beta=1}^{4} \bar{\psi}_{a,\alpha}\mathcal{M}'_{a\alpha,b\beta}\psi_{b,\beta} = \sum_{a=1}^{N^2-1}\sum_{b=1}^{N^2-1}\sum_{\alpha,\beta=1}^{4} \bar{\psi}_{a,\alpha}\mathcal{M}_{a\alpha,b\beta}\psi_{b,\beta}, \ \text{where}$$

$$\mathcal{M}'_{a\alpha,b\beta} = N(\Gamma_\mu)_{\alpha\beta}\text{tr}\,(t^a[A_\mu, t^b]) = N(\delta_{i_a j_b}(A_\mu)_{j_a i_b} - \delta_{i_b j_a}(A_\mu)_{j_b i_a}),$$

$$\mathcal{M}_{a\alpha,b\beta} = \mathcal{M}'_{a\alpha,b\beta} - \delta_{i_a j_a}\mathcal{M}'_{N^2\alpha,b\beta} - \delta_{i_b j_b}\mathcal{M}'_{a\alpha,N^2\beta}.$$