Smart and Human
常翔学園
摂南大学

″Phase diagram of the GWW phase transition of the matrix quantum mechanics with a chemical potential″

Takehiro Azuma (Setsunan Univ.)
From Strings to LHC IV, Sinclairs Retreat, Chalsa
Mar. 10th 2017, 14:25-14:50
with Pallab Basu (ICTS,TIFR) and Prasant Samantray (IIT, Indore)

1. Introduction

Thermodynamic aspects of quantum gravity in AdS spacetime:

- Small blackhole (SBH):
  Unstable. Horizon radius smaller than AdS
- Big blackhole (BBH):
  Stable. Horizon radius comparable to AdS

Gross-Witten-Wadia (GWW) phase transition of the gauge theory and the blackhole phase transition

[L. Alvarez-Gaume, C. Gomez, H. Liu and S.R. Wadia, hep-th/0502227]

2. The model

Finite-temperature matrix quantum mechanics with a chemical potential

$S = S_b + S_f + S_g$, where (μ=1,2,….D, β=1/T)

$$S_{\rm b} = N\int_0^\beta {\rm tr}\left\{\frac{1}{2}\sum_{\mu=1}^{D}(D_t X_\mu(t))^2 - \frac{1}{4}\sum_{\mu,\nu=1}^{D}[X_\mu(t),X_\nu(t)]^2\right\}dt$$

$$D_t X_\mu(t) = \partial_t X_\mu(t) - i[A(t), X_\mu(t)]$$

$$S_{\rm f} = N\int_0^\beta {\rm tr}\left\{\sum_{\alpha=1}^{p}\bar{\psi}_\alpha(t)D_t\psi_\alpha(t) - \sum_{\mu=1}^{D}\sum_{\alpha,\eta=1}^{p}\bar{\psi}_\alpha(t)(\Gamma_\mu)_{\alpha\eta}[X_\mu(t),\psi_\eta(t)]\right\}dt$$

$$S_{\rm g} = N\mu({\rm tr}U + {\rm tr}U^\dagger) \qquad U = \mathcal{P}\exp\left(i\int_0^\beta A(t)dt\right)$$

- Bosonic ($S=S_b+S_g$): D is an arbitrary integer D=2,3,…
- Fermionic($S=S_b+S_f+S_g$): (D,p)=(3,2),(5,4),(9,16)

(For D=9, the fermion is Majorana-Weyl ($\bar{\Psi}\rightarrow\Psi$)

In the following, we focus on D=3.)

2. The model

A(t), X_μ(t), Ψ(t) : N × N Hermitian matrix

Boundary conditions:
$$A(t+\beta)=A(t),\quad X_\mu(t+\beta)=X_\mu(t)$$
$$\psi(t+\beta)=-\psi(t)$$

Static diagonal gauge:

$$A(t)=\frac{1}{\beta}\mathrm{diag}(\alpha_1,\alpha_2,\cdots,\alpha_N)\qquad -\pi\leqq\alpha_k<\pi$$

⇒ Add the gauge-fixing term $S_{\mathrm{g.f.}}=-\sum_{k,l=1,k\neq l}^{N}\log\left|\sin\frac{\alpha_k-\alpha_l}{2}\right|$

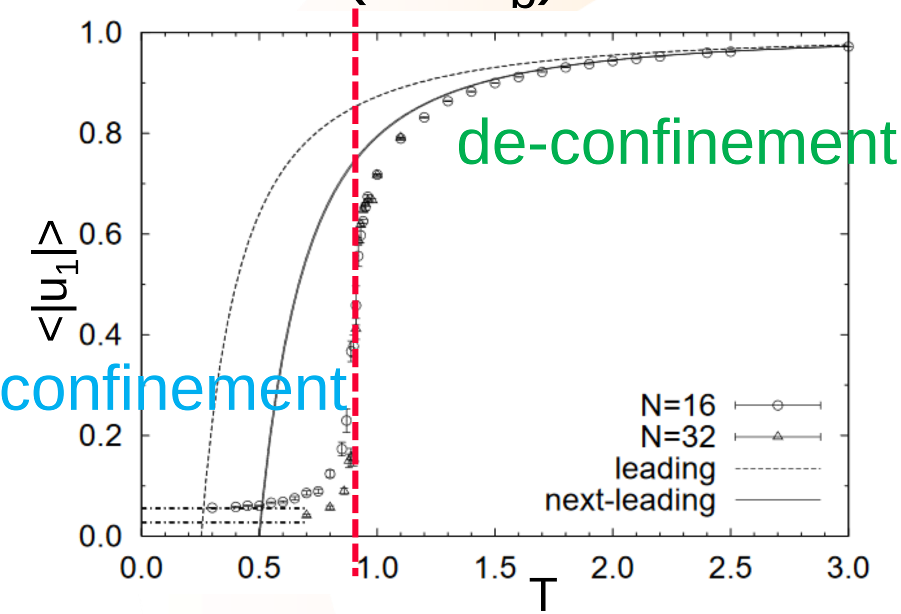
Under this gauge $u_n=\frac{1}{N}\mathrm{tr}U^n=\frac{1}{N}\sum_{k=1}^{N}e^{in\alpha_k}$

Supersymmetry for $S=S_b+S_f$ (μ=0), broken at μ≠0.

2. The model

Previous works for $\mu=0$ (without S_g)

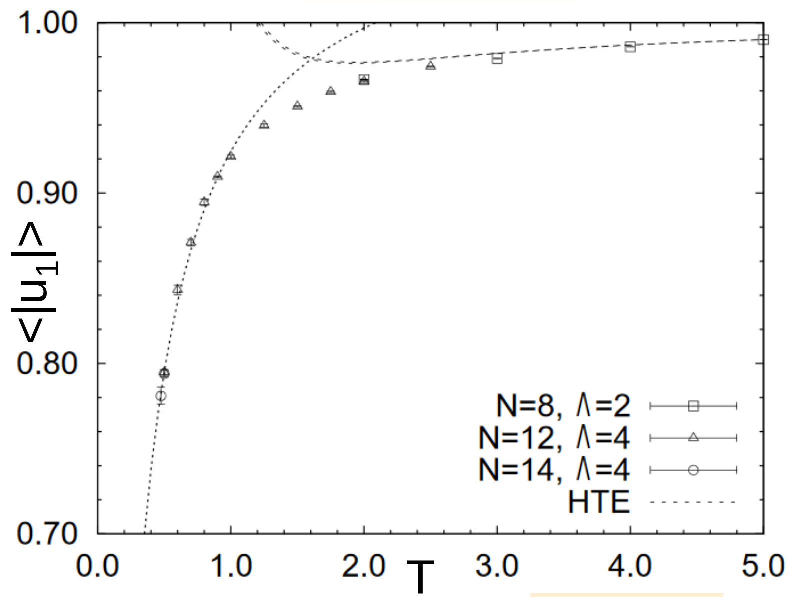
Bosonic ($S=S_b$)



[Quoted for D=9 from N. Kawahara, J. Nishimura and S. Takeuchi, arXiv:0706.3517]

Confinement-deconfinement phase transition at $T=T_{c0}$

SUSY ($S=S_b+S_f$)



[Quoted for D=9 from K.N. Anagnostopoulos, M. Hanada, J. Nishimura and S. Takeuchi, arXiv:0707.4454]

Absence of phase transition

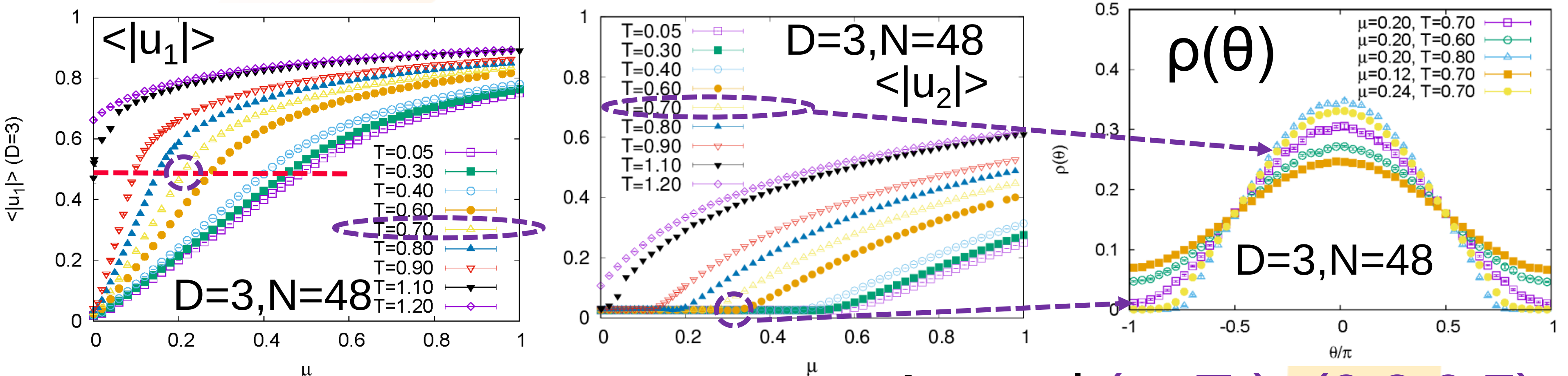
$\langle|u_1|\rangle = a_0 \exp(-a_1/T)$

3. Result of the bosonic model

Bosonic model without fermion $S=S_b+S_g$

[T. Azuma, P. Basu and S.R. Wadia, arXiv:0710.5873]

Result of D=3 (D=2,6,9 cases are similar)



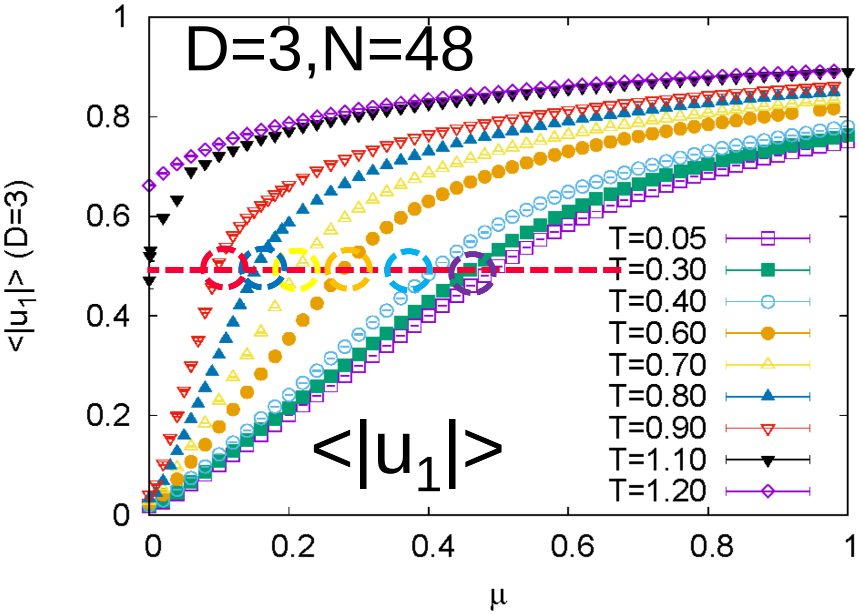
Critical points (μ_c, T_c) at $<|u_1|>=1/2$

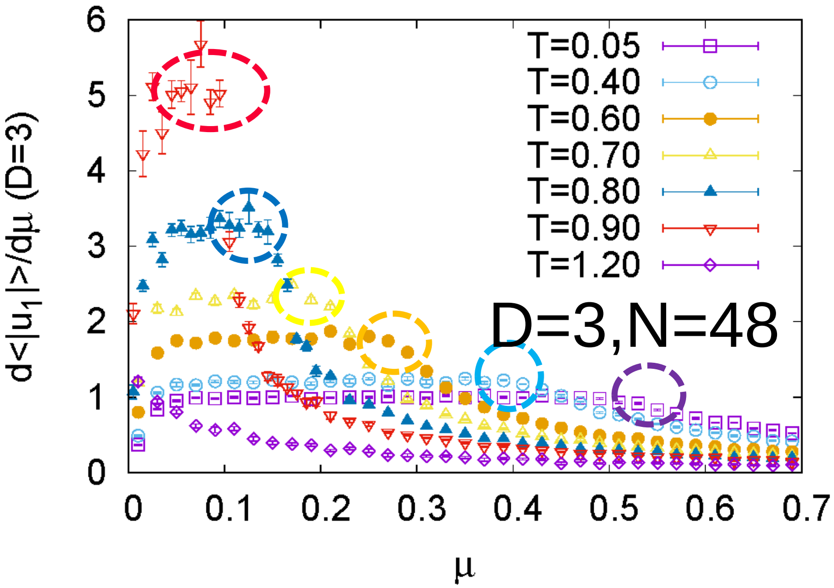
Around $(\mu_c,T_c)=(0.2,0.7)$, $\rho(\theta)=\frac{1}{N}\sum_{k=1}^{N}\langle\delta(\theta-\alpha_k)\rangle$ develops a gap.

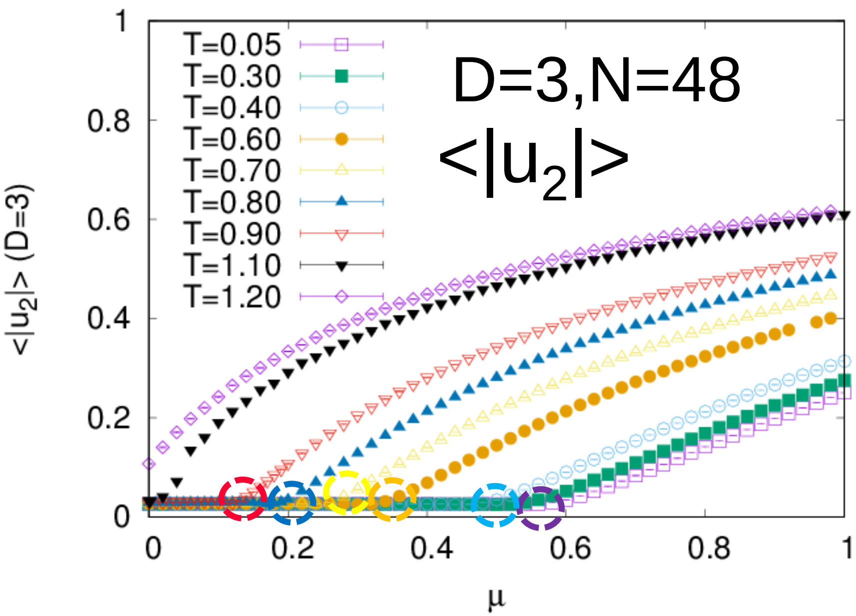
At (μ_c, T_c), $d<|u_{1,2}|>/d\mu$ and $d<|u_{1,2}|>/dT$ are not smooth ($d^2<|u_{1,2}|>/d\mu^2$ and $d^2<|u_{1,2}|>/dT^2$ are discontinuous)

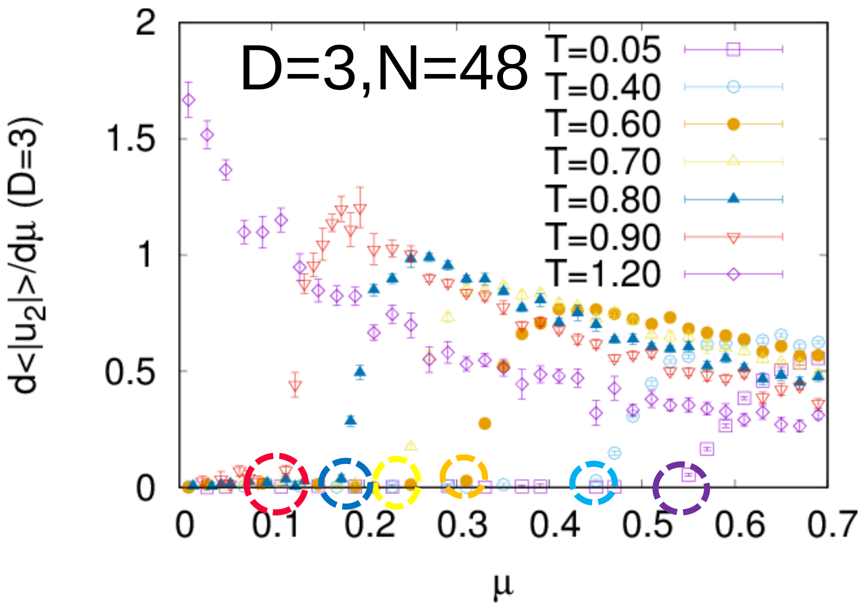
⇒ suggests third-order phase transition.

3. Result of the bosonic model

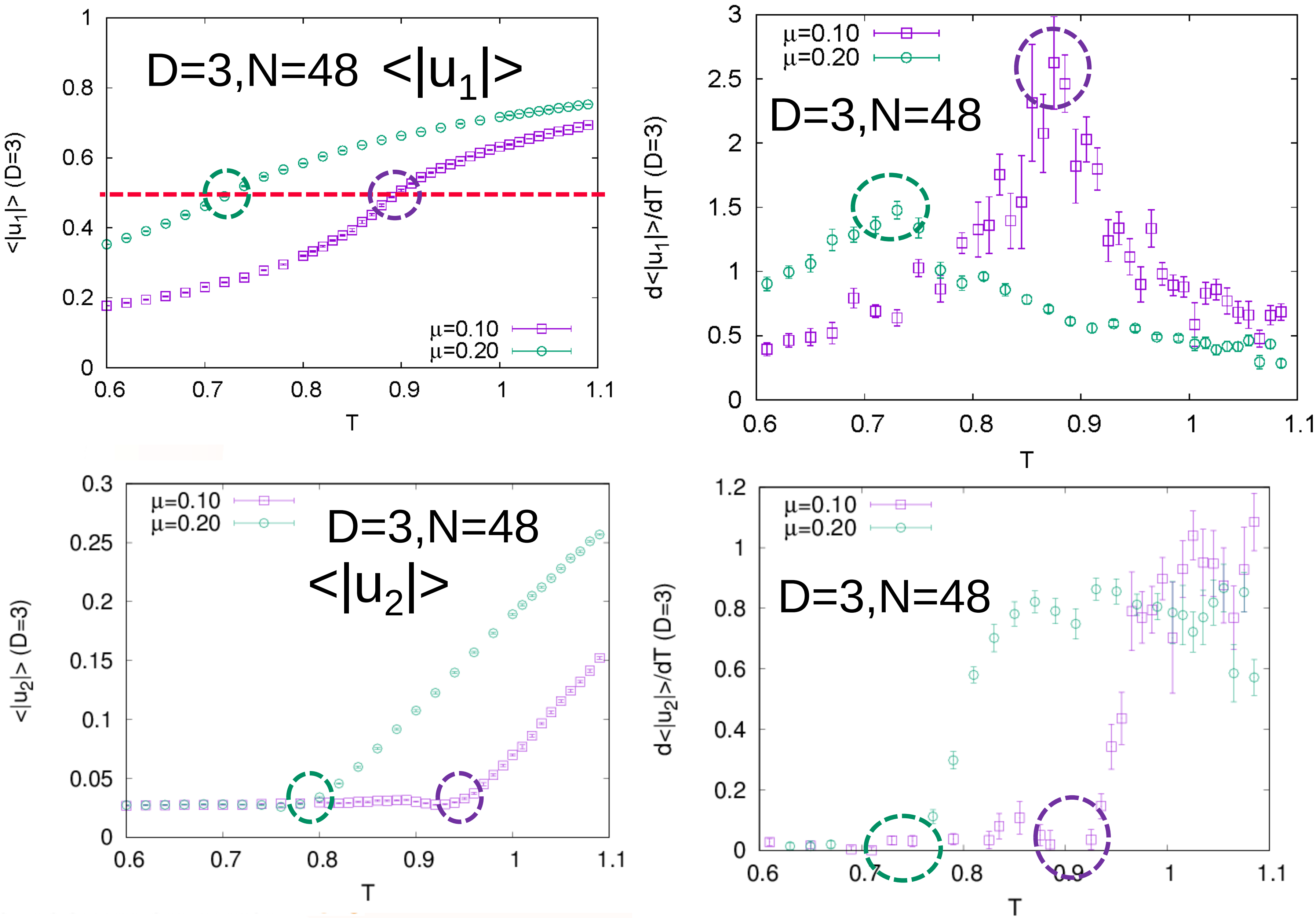








3. Result of the bosonic model



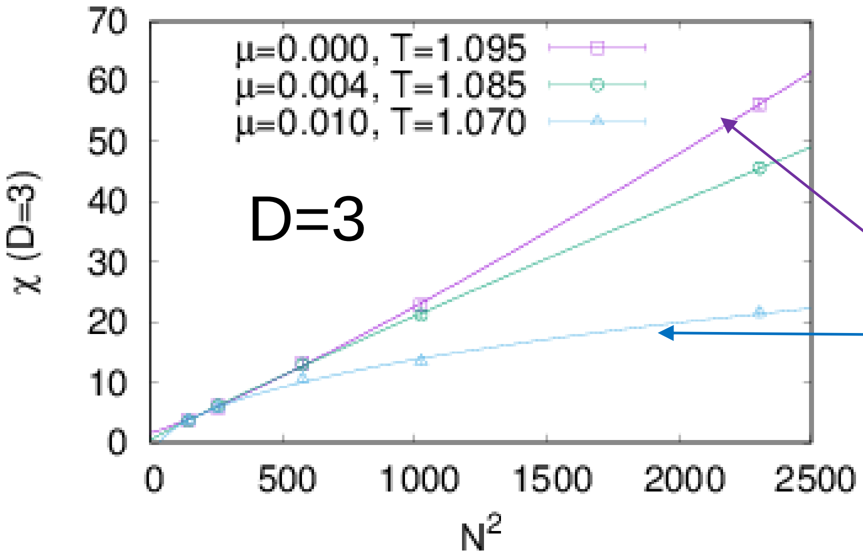
3. Result of the bosonic model

When μ=0, at the critical point T_{c0}=1.1,
there is a first-order phase transition at small D.
[T. Azuma, T. Morita and S. Takeuchi, arXiv:1403.7764]

We fit the susceptibility with (γ,p,c) as

$$\chi = N^2\{\langle|u_1|^2\rangle - (\langle|u_1|\rangle)^2\} = \gamma V^p + c \ (V = N^2)$$

p=1 ⇒ suggests first-order phase transition.
[M. Fukugita, H. Mino, M. Okawa and A. Ukawa, Phys. Rev. Lett.65, 816 (1990)]



μ_c	0.00	0.004	0.01
T_c	1.095	1.085	1.070
p	1.14(4)	0.94(3)	0.42(10)

4. Result of the fermionic model

The model with fermion (D=3) $S=S_b+S_f+S_g$

$\Gamma_\mu=\sigma_\mu$ (μ=1,2,3)

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Non-lattice simulation with Fourier expansion

[K.N. Anagnostopoulos, M. Hanada, J. Nishimura and S. Takeuchi, arXiv:0707.4454]

$$X^{kl}_\mu(t) = \sum_{n=-\Lambda}^{\Lambda} \tilde{X}^{kl}_{\mu,n} e^{i\omega n t}, \quad \psi^{kl}_\alpha(t) = \sum_{r=-\Lambda+\frac{1}{2}}^{\Lambda-\frac{1}{2}} \tilde{\psi}^{kl}_{\alpha,r} e^{i\omega r t}, \quad \bar{\psi}^{kl}_\alpha(t) = \sum_{r=-\Lambda+\frac{1}{2}}^{\Lambda-\frac{1}{2}} \tilde{\bar{\psi}}^{kl}_{\alpha,-r} e^{i\omega r t}. \qquad \left(\omega = \frac{2\pi}{\beta}\right)$$

$$S_{\mathrm{F,Fourier}} = N\beta \sum_{r=-\Lambda+\frac{1}{2}}^{\Lambda-\frac{1}{2}} \left\{ i \left\{ r\omega - \frac{\alpha_k - \alpha_l}{\beta} \right\} \tilde{\bar{\psi}}^{lk}_{\alpha,r} \tilde{\psi}^{kl}_{\alpha,r} - (\sigma_\mu)_{\alpha\eta} \mathrm{tr} \left\{ [\tilde{\bar{\psi}}_{\alpha,r} \left([\tilde{X}_\mu, \tilde{\psi}_\eta] \right)_r \right\} \right\}$$

$$\left(f^{(1)} \cdots f^{(p)}\right)_q = \sum_{k_1+\cdots+k_p=q} f^{(1)}_{k_1} \cdots f^{(p)}_{k_p}$$

Integrating out $\Psi \Rightarrow N_0 \times N_0$ matrix $\mathcal{M}$

($N_0 = 2 \times 2\Lambda \times (N^2-1)$) traceless adjoint

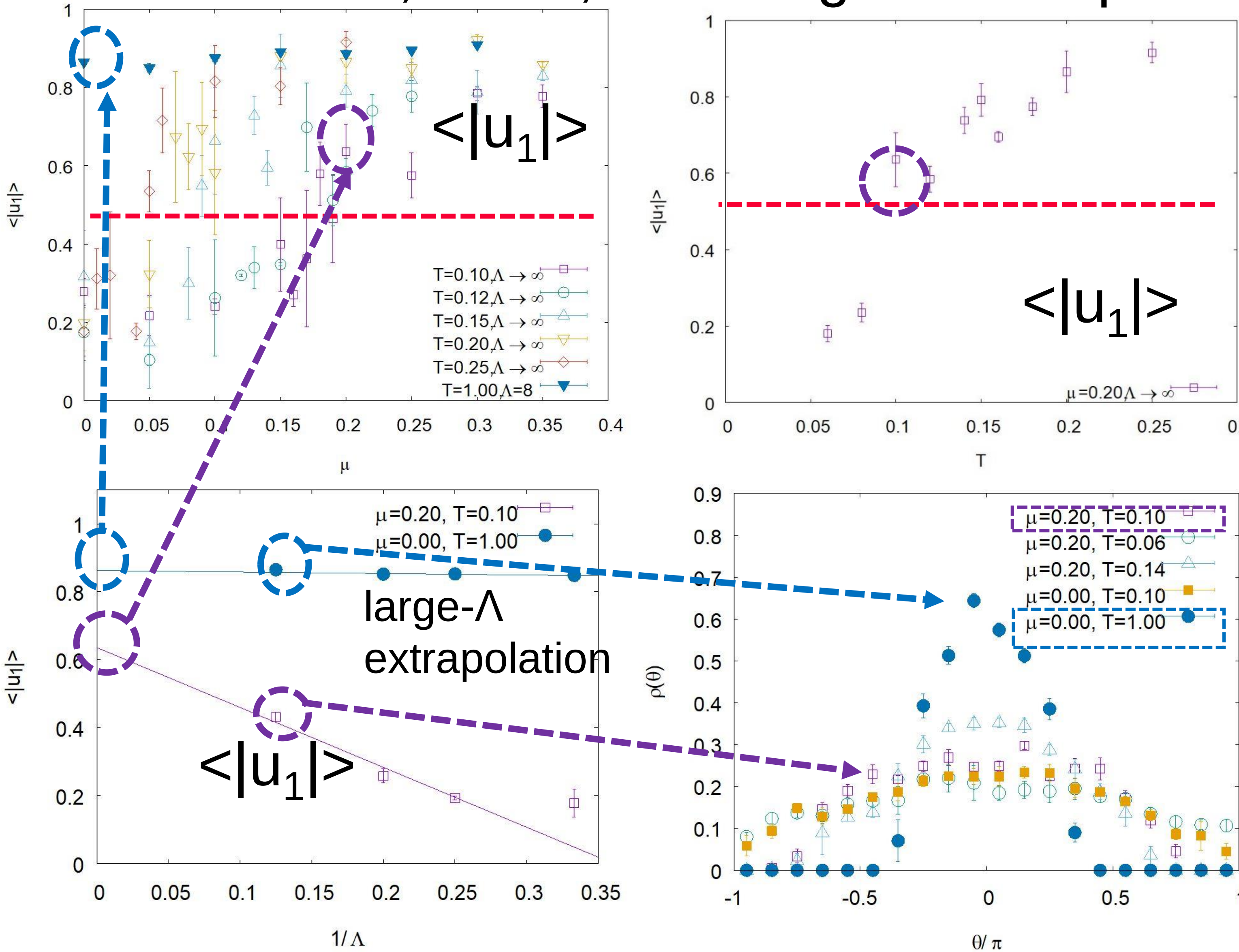
Gamma matrix [4(D=5), 16(D=9)] r=-Λ+1/2,…Λ-1/2

D=3: $\det \mathcal{M}$ is real ⇒ no sign problem.

4. Result of the fermionic model

SETSUDAI

Result of D=3, N=16, after large-Λ extrapolation:

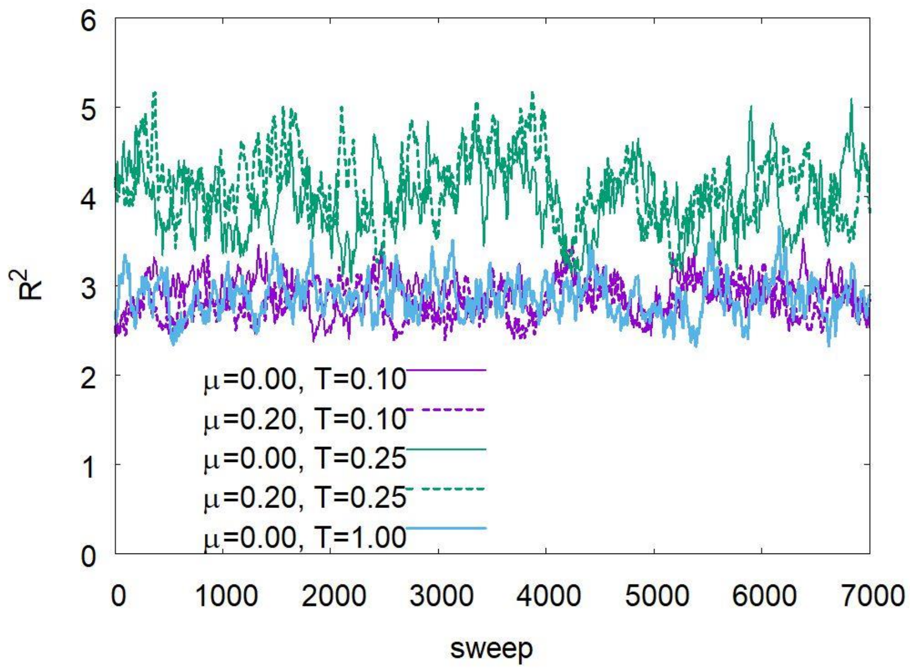


Gapped⇔ Ungapped

Possible phase transitions at (μ_c,T_c) where $\langle|u_1|\rangle=0.5$.

4. Result of the fermionic model

Result of D=3, N=16, after large-Λ extrapolation:



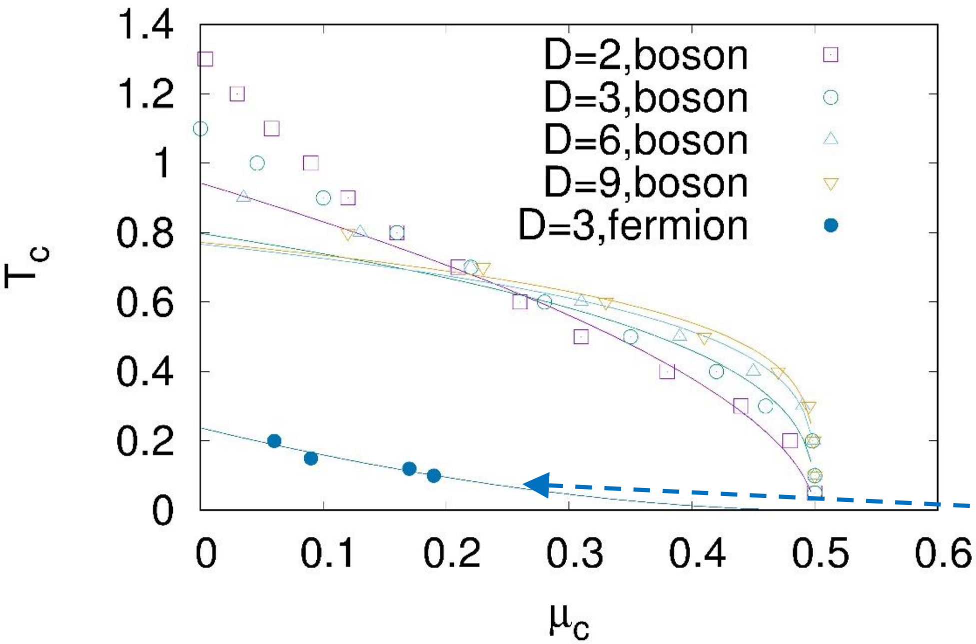
History of $R^2 = \frac{1}{N\beta}\int_0^\beta dt\, \mathrm{tr}X_\mu(t)^2$ at Λ=3

No instability in the typical (μ,T) region.

4. Result of the fermionic model

Phase diagram for D=2,3,6,9 (boson) and D=3(fermion) .

Some phase transitions at (μ_c, T_c) where $\langle|u_1|\rangle=0.5$



D=3 SUSY, μ=0:

$\langle|u_1|\rangle = a_0 \exp(-a_1/T)$

$a_0=1.03(1)$, $a_1=0.19(1)$

$\Rightarrow \langle|u_1|\rangle=0.5$ at $T=0.28$.

[M. Hanada, S. Matsuura, J. Nishimura and D. Robles-Llana, arXiv:1012.2913]

μ=0: $\langle|u_1|\rangle = 0.5$ at

$T_c=1.39\times 0.5^{2.30}\simeq 0.28$

Fitting of the critical point by $T_c=a(0.5-\mu_c)^b$.

D	2(boson)	3(boson)	6(boson)	9(boson)	3(fermion)
a	1.36(12)	1.01(15)	0.91(9)	0.90(8)	1.39(72)
b	0.55(6)	0.34(7)	0.25(4)	0.23(4)	2.30(59)

5. Summary

We have studied the matrix quantum mechanics with a chemical potential $S_{\mathrm{g}} = N\mu(\mathrm{tr}U + \mathrm{tr}U^{\dagger})$

・bosonic model ⇒ GWW-type third-order phase transition (except for very small μ)

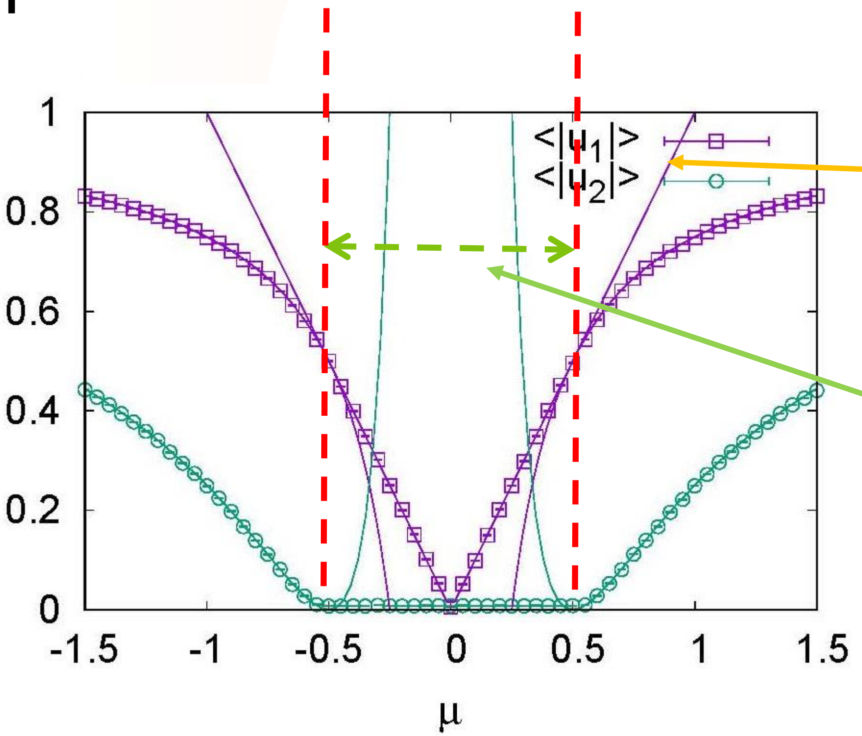
・phase diagram of the bosonic/fermionic model

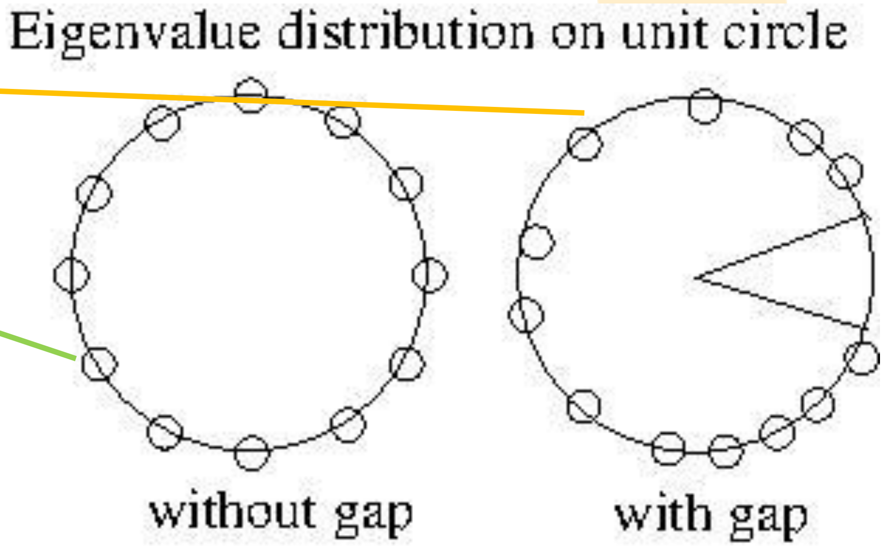
backup: GWW phase transition

$$S = \underbrace{2N\mu \sum_{k=1}^{N} \cos\alpha_k}_{=S_{\rm g}=N\mu({\rm tr}U+{\rm tr}U^{\dagger})} + \underbrace{\left(- \sum_{k,l=1,k\neq l}^{N} \log\sin\left|\frac{\alpha_k-\alpha_l}{2}\right|\right)}_{=S_{\rm g.f.}}.$$

[D.J. Gross and E. Witten, Phys. Rev. D21 (1980) 446, S.R. Wadia, Phys. Lett. B93 (1980) 403]

Gross-Witten-Wadia (GWW) type third-order phase transition





backup: GWW phase transition

SETSUDAI

Continuous at μ=1/2.

$$\langle|u_1|\rangle = \begin{cases} \mu & \left(0 \leqq \mu \leqq \frac{1}{2}\right) \\ 1 - \frac{1}{4\mu} & \left(\mu \geqq \frac{1}{2}\right) \end{cases}, \quad \langle|u_2|\rangle = \begin{cases} 0 & \left(0 \leqq \mu \leqq \frac{1}{2}\right) \\ \left(1 - \frac{1}{2\mu}\right)^2 & \left(\mu \geqq \frac{1}{2}\right) \end{cases}$$

$$\frac{d\langle|u_1|\rangle}{d\mu} = \begin{cases} 1 & \left(0 \leqq \mu \leqq \frac{1}{2}\right) \\ \frac{1}{4\mu^2} & \left(\mu \geqq \frac{1}{2}\right) \end{cases}, \quad \frac{d^2\langle|u_1|\rangle}{d\mu^2} = \begin{cases} 0 & \left(0 \leqq \mu \leqq \frac{1}{2}\right) \\ -\frac{1}{2\mu^3} & \left(\mu \geqq \frac{1}{2}\right) \end{cases}$$

Discontinuous at μ=1/2.

⇒For free energy, $d^3F/d\mu^3$ is discontinuous.

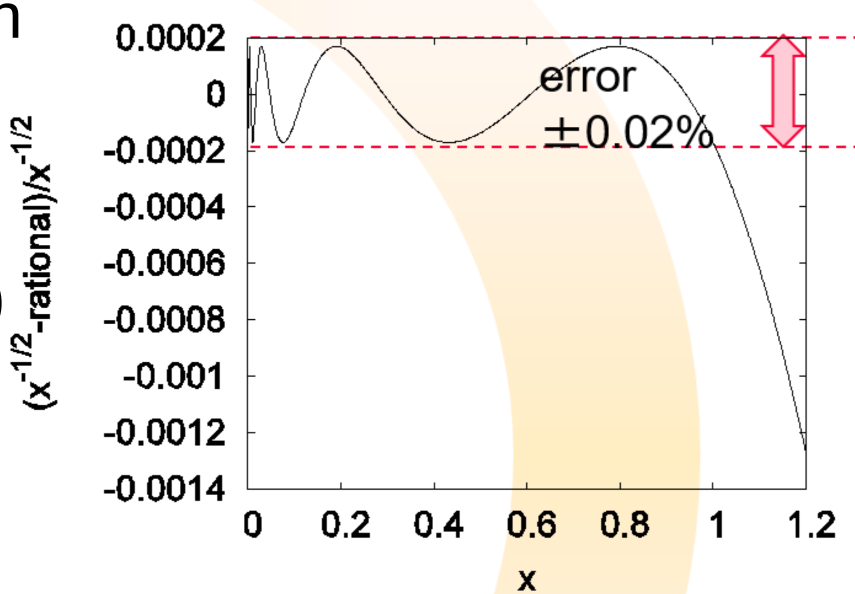
backup: RHMC

Simulation via Rational Hybrid Monte Carlo (RHMC) algorithm. [Chap 6,7 of B.Ydri, arXiv:1506.02567, for a review]

We exploit the rational approximation

$$x^{-1/2} \simeq a_0 + \sum_{k=1}^{Q} \frac{a_k}{x+b_k}$$

after a proper rescaling.
(typically Q=15⇒valid at $10^{-12}c<x<c$)
a_k, b_k come from Remez algorithm.
[M. A. Clark and A. D. Kennedy, https://github.com/mikeaclark/AlgRemez]



$$S_0 = S_{\rm b} + S_{\rm g} - \log|\det \mathscr{M}|$$

$$|\det \mathscr{M}| = (\det \mathscr{D})^{1/2} \simeq \int dF dF^* \exp\left(-F^* \mathscr{D}^{-1/2} F\right) \simeq \int dF dF^* e^{-S_{\rm PF}}$$

$$S_{\rm PF} = a_0 F^* F + \sum_{k=1}^{Q} a_k F^* (\mathscr{D}+b_k)^{-1} F, \quad (\text{where } \mathscr{D} = \mathscr{M}^\dagger \mathscr{M})$$

F: *bosonic* N_0-dim vector (called *pseudofermion*)

backup: RHMC

Hot spot (most time-consuming part) of RHMC:
⇒Solving $(\mathscr{D}+b_k)\chi_k = F \quad (k=1,2,\cdots,Q)$
by conjugate gradient (CG) method.

Multiplication $\mathscr{M}\chi_k$ ⇒
$\mathscr{M}$ is a very sparse matrix. No need to build $\mathscr{M}$ explicitly.
⇒CPU cost is $O(N^3)$ per CG iteration
The required CG iteration time depends on T.
(while direct calculation of $\mathscr{M}^{-1}$ costs $O(N^6)$.)

Multimass CG solver: [B. Jegerlehner, hep-lat/9612014]
Solve $(\mathscr{D}+b_k)\chi_k = F$ only for the smallest b_k
⇒The rest can be obtained as a byproduct,
which saves O(Q) CPU cost.

Conjugate Gradient (CG) method:
Iterative algorithm to solve the linear equation Ax=b
(A: symmetric, positive-definite n × n matrix)

Initial config. $\mathbf{x}_0 = 0 \quad \mathbf{r}_0 = \mathbf{b} - A\mathbf{x}_0 \quad \mathbf{p}_0 = \mathbf{r}_0$

(for brevity, no preconditioning on x_0 here)

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k \mathbf{p}_k \quad \mathbf{r}_{k+1} = \mathbf{r}_k - \alpha_k A \mathbf{p}_k \quad \alpha_k = \frac{(r_k, r_k)}{(p_k, Ap_k)}$$

$$\mathbf{p}_{k+1} = \mathbf{r}_{k+1} + \frac{(\mathbf{r}_{k+1}, \mathbf{r}_{k+1})}{(\mathbf{r}_k, \mathbf{r}_k)} \mathbf{p}_k$$

Iterate this until $\sqrt{\frac{(\mathbf{r}_{k+1}, \mathbf{r}_{k+1})}{(\mathbf{r}_0, \mathbf{r}_0)}} < (\text{tolerance}) \simeq 10^{-4}$

The approximate answer of Ax=b is $x = x_{k+1}$.