Curved-space Classical Solution of a Massive Supermatrix Model

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1. Constructive definition of superstring theory

A large N reduced model has been proposed as a constructive definition (nonperturbative formulation) of the superstring theory:
N.Ishibashi, H.Kawai, Y.Kitazawa and A.Tsuchiya, hep-th/9612115.

$$S = \frac{1}{g^2} Tr_{N\times N}\left(\frac{1}{4}\sum_{\mu,\nu=0}^{9}[A_\mu, A_\nu]^2 + \frac{1}{2}\bar{\psi}\sum_{\mu=0}^{9}\Gamma^\mu[A_\mu,\psi]\right),$$

$$(\text{ where } Z = \int dA d\psi e^{+S}).$$

- Dimensional reduction of $\mathcal{N}=1$ 10-dimensional SYM theory to 0 dimension.
  A_μ and ψ are $N\times N$ Hermitian matrices.
  - * A_μ: 10-dimensional vectors
  - * ψ: 10-dimensional Majorana-Weyl (i.e. 16-component) spinors
- Matrix regularization of the Schild action of the type IIB superstring theory.
- $SU(N)$ gauge symmetry and $SO(9,1)$ Lorentz symmetry ($SO(9,1)\times U(N)$).
- $\mathcal{N}=2$ SUSY: This theory must contain spin-2 gravitons if it contains massless particles.
  - * homogeneous: $\delta^{(1)}_\epsilon A_a = i\bar{\epsilon}\Gamma_a\psi, \delta^{(1)}_\epsilon\psi = \frac{i}{2}\Gamma^{ab}[A_a, A_b]\epsilon$.
  - * inhomogeneous: $\delta^{(2)}_\xi A_a = 0, \delta^{(2)}_\xi\psi = \xi$.

  This gives a shift of the bosonic variables for
  $\tilde{\delta}^{(1)} = \delta^{(1)} + \delta^{(2)}$, $\tilde{\delta}^{(2)} = i(\delta^{(1)} - \delta^{(2)})$: $(\alpha,\beta = 1,2)$

$$[\tilde{\delta}^{(\alpha)}_\epsilon, \tilde{\delta}^{(\beta)}_\xi]\psi = 0,$$
$$[\tilde{\delta}^{(\alpha)}_\epsilon, \tilde{\delta}^{(\beta)}_\xi]A_\mu = -2i\delta^{\alpha\beta}\bar{\epsilon}\Gamma_\mu\xi.$$

2. How can we realize the curved-space background in terms of a large N reduced model?

The classical equation of motion of IIB matrix model:

$$[A^\nu, [A_\mu, A_\nu]] = 0.$$

This has only a flat non-commutative background as a classical solution.

$$[A_\mu, A_\nu] = ic_{\mu\nu}\mathbf{1}_{N\times N}.$$

There are three approaches to surmount this difficulty:

1. IIB matrix model itself is nevertheless an authentic framework, since it effectively induces the curved-space background by the graviton condensation.
2. We enlarge the symmetry of the matrix model so that it may incorporate local Lorentz invariance.
3. We add a mass term so that the model has a curved-space background from the outset.

Here, we follow the third approach, and scrutinize how the matrix model incorporates a curved-space classical solution.

[Example] IIB matrix model with a tachyonic mass term:
Y. Kimura, Prog. Theor. Phys. **106** (2001) 445, [hep-th/0103192].

$$S = \frac{1}{g^2}Tr\left(\frac{1}{4}[A_a, A_b]^2 + \lambda^2 A_a A_a\right),$$
$$\text{EOM: } [A_b, [A_a, A_b]] + 2\lambda^2 A_a = 0.$$

- $SO(4)\times U(N)$ symmetry.
  a, b runs over $1,2,3,4$ in the Euclidean space.
- Classical solutions of compact curved spacetime:
  - * $SO(3)$ fuzzy sphere:
    $[A_i, A_j] = i\lambda\epsilon_{ijk}A_k$ $(i,j,k = 1,2,3)$, $A_4 = 0$.
  - * two-dimensional fuzzy torus:

$$A_1 = \frac{r}{2}(U + U^\dagger),\ A_2 = \frac{r}{2i}(U - U^\dagger),\ A_3 = \frac{r}{2}(V + V^\dagger),\ A_4 = \frac{r}{2i}(V - V^\dagger),$$

$$U = \begin{pmatrix} 1 & & & & \\ & \omega & & & \\ & & \omega^2 & & \\ & & & \ddots & \\ & & & & \omega^{N-1}\end{pmatrix},\ V = \begin{pmatrix} 0 & & & & 1 \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0\end{pmatrix},$$

$$\omega = e^{i\theta},\quad \theta = \frac{2\pi}{N},\quad UV = e^{i\theta}VU,\quad \lambda^2 = r^2(1-\cos\theta).$$

3. $osp(1|32,R)$ supermatrix model

We investigate a massive matrix model based on super Lie algebra $osp(1|32,R)$.

$osp(1|32,R)$ was first mentioned on 11-dimensional supergravity.
E. Cremmer, B. Julia, J. Scherk Phys.Lett. **B76** ,409 (1978)

This has attracted a new attention as the unified super Lie algebra for M-theory.

The matrix model based on $osp(1|32,R)$ would be a natural extension of IIB matrix model.

$$M \in osp(1|32,R) \stackrel{\text{def}}{\Rightarrow} {}^T MG + GM = 0,\ \text{where } G = \begin{pmatrix}\Gamma^0 & 0 \\ 0 & i\end{pmatrix}.$$

$\Downarrow$

$$M = \begin{pmatrix} m & \psi \\ i\bar{\psi} & 0\end{pmatrix},\ \text{where } {}^T m\Gamma^0 + \Gamma^0 m = 0;\ \text{i.e. } m \in sp(32),$$
$$m = u_{A_1}\Gamma^{A_1} + \frac{1}{2!}u_{A_1A_2}\Gamma^{A_1A_2} + \frac{1}{5!}u_{A_1\cdots A_5}\Gamma^{A_1\cdots A_5}.$$

We add a mass term to the pure cubic action:

$$\begin{aligned}
S &= Tr\left[str\left(-3\mu M^2 + \frac{i}{g^2}M[M,M]\right)\right] \\
&= Tr\left[-3\mu\left\{\left(\sum_{p=1}^{32} M_p{}^Q M_Q{}^p\right) - M_{33}{}^Q M_Q{}^{33}\right\}\right. \\
&\qquad \left. + \frac{i}{g^2}\left\{\left(\sum_{p=1}^{32} M_p{}^Q[M_Q{}^R, M_R{}^p]\right) - M_{33}{}^Q[M_Q{}^R, M_R{}^{33}]\right\}\right] \\
&= Tr\left[3\mu(-tr(m^2) + 2i\bar{\psi}\psi) + \frac{i}{g^2}\left(m_p{}^q[m_q{}^r, m_r{}^p] - 3i\bar{\psi}^p[m_p{}^q, \psi^q]\right)\right].
\end{aligned}$$

- Each component of the 33×33 supermatrices is promoted to a large N hermitian matrix.
- $osp(1|32,R)$ symmetry and $u(N)$ gauge symmetry are decoupled (i.e. $Osp(1|32,R)\times U(N)$ symmetry).

In order to see the correspondence of the fields with IIB matrix model, we express the bosonic 32×32 matrices in terms of the 10-dimensional indices.
$(\mu, \nu, \cdots = 0, 1, \cdots, 9, \sharp = 10)$.

$$W = m_{\sharp}, A_{\mu} = m_{\mu}, B_{\mu} = m_{\mu\sharp}, C_{\mu_1\mu_2} = m_{\mu_1\mu_2},$$
$$H_{\mu_1\cdots\mu_4} = m_{\mu_1\cdots\mu_4\sharp}, Z_{\mu_1\cdots\mu_5} = m_{\mu_1\cdots\mu_5}.$$

Then, the action is decomposed as

$$\begin{aligned} S &= 96\mu Tr\left(-W^2 - A_{\mu}A^{\mu} + B_{\mu}B^{\mu} + \frac{1}{2}C_{\mu_1\mu_2}C^{\mu_1\mu_2}\right. \\ &\left. -\frac{1}{4!}H_{\mu_1\cdots\mu_4}H^{\mu_1\cdots\mu_4} - \frac{1}{5!}Z_{\mu_1\cdots\mu_5}Z^{\mu_1\cdots\mu_5} + \frac{i}{16}\bar{\psi}\psi\right) \\ &+ 32iTr\left(3C_{\mu_1\mu_2}[B^{\mu_1}, B^{\mu_2}] + C_{\mu_1\mu_2}[C^{\mu_2}{}_{\mu_3}, C^{\mu_3\mu_1}]\right) \\ &+ [\text{cubic interactions involving } (W, A, H, Z, \psi)]. \end{aligned}$$

- The rank-1 and rank-5 fields (in 11 dimensions) have a positive mass, while the rank-2 fields are tachyonic.

$$\underbrace{\Gamma_A\Gamma^A}_{\text{no sum}} = \underbrace{\Gamma_{A_1\cdots A_5}\Gamma^{A_1\cdots A_5}}_{\text{no sum}} = +\mathbf{1}_{32\times 32},$$
$$\underbrace{\Gamma_{A_1A_2}\Gamma^{A_1A_2}}_{\text{no sum}} = -\mathbf{1}_{32\times 32}.$$

- The rank-1 and rank-5 fields has a stable trivial commutative classical solution:

$$W = A_{\mu} = H_{\mu_1\cdots\mu_4} = Z_{\mu_1\cdots\mu_5} = 0.$$

- For the rank-2 tachyonic fields $B_{\mu}, C_{\mu_1\mu_2}$, the trivial solution $B_{\mu} = C_{\mu_1\mu_2} = 0$ is unstable.
$\Rightarrow$ They may incorporate an interesting stable non-commutative solution!

4. Resolution of the equations of motion

From now on, we set the fermions and the positive-mass-square bosonic fields to zero:

$$\begin{aligned} S &= 96\mu Tr\left(B_{\mu}B^{\mu} + \frac{1}{2}C_{\mu_1\mu_2}C^{\mu_1\mu_2}\right) \\ &+ 32iTr\left(3C_{\mu_1\mu_2}[B^{\mu_1}, B^{\mu_2}] + C_{\mu_1\mu_2}[C^{\mu_2}{}_{\mu_3}, C^{\mu_3\mu_1}]\right). \end{aligned}$$

The equations of motion:

$$\begin{aligned} B_{\mu} &= -i\mu^{-1}[B^{\nu}, C_{\mu\nu}], \\ C_{\mu_1\mu_2} &= -i\mu^{-1}([B_{\mu_1}, B_{\mu_2}] + [C_{\mu_1}{}^{\rho}, C_{\mu_2\rho}]). \end{aligned}$$

We integrate out the rank-2 fields (in 10 dimensions) $C_{\mu_1\mu_2}$ by solving the latter equation of motions iteratively.

$$\begin{aligned} C_{\mu_1\mu_2} &= -i\mu^{-1}[B_{\mu_1}, B_{\mu_2}] + i\mu^{-3}[[B_{\mu_1}, B_{\rho}], [B_{\mu_2}, B^{\rho}]] \\ &- 2i\mu^{-5}[[B_{[\mu_1}, B_{\rho}], [[B_{\mu_2]}, B_{\chi}], [B^{\rho}, B^{\chi}]]] + \mathcal{O}(\mu^{-7}). (\star) \end{aligned}$$

Then, the action reduces to

$$\begin{aligned} S &= Tr\left(96\mu B_{\mu}B^{\mu} + 48\mu^{-1}[B_{\mu_1}, B_{\mu_2}][B^{\mu_1}, B^{\mu_2}]\right. \\ &\left. + [\text{higher-order commutators of the order } \mathcal{O}(\mu^{-2k+1}) \text{ with } k = 2, 3, \cdots]\right). \end{aligned}$$

We consider the classical solution of the equation of motion $B_{\mu} = -i\mu^{-1}[B^{\nu}, C_{\mu\nu}]$ with $C_{\mu_1\mu_2}$ substituted for $(\star)$.

Fuzzy-sphere classical solution

1. $SO(3) \times SO(3) \times SO(3)$ fuzzy spheres

   This describes a space formed by the Cartesian product of three fuzzy spheres.

$$[B_i, B_j] = i\mu r\epsilon_{ijk}B_k, \quad B_1^2 + B_2^2 + B_3^2 = \mu^2 r^2 \frac{N^2-1}{4}, (i, j, k = 1, 2, 3)$$
$$[B_l, B_m] = i\mu r\epsilon_{lmn}B_n, \quad B_4^2 + B_5^2 + B_6^2 = \mu^2 r^2 \frac{N^2-1}{4}, (l, m, n = 4, 5, 6)$$
$$[B_p, B_q] = i\mu r\epsilon_{pqr}B_r, \quad B_7^2 + B_8^2 + B_9^2 = \mu^2 r^2 \frac{N^2-1}{4}, (p, q, r = 7, 8, 9)$$
$$B_0 = 0, [B_{\mu}, B_{\nu}] = 0, (\text{otherwise}).$$

   (We consider the Cartesian product of three spheres instead of a single $SO(3)$ fuzzy sphere

$$\begin{aligned} &[B_i, B_j] = i\mu r\epsilon_{ijk}B_k \text{ (for } i, j, k = 1, 2, 3), \\ &B_{\mu} = 0 \text{ (for } \mu = 0, 4, 5, \cdots, 9), \end{aligned}$$

   because the solution $B_4 = \cdots = B_9 = 0$ is trivially unstable.)

2. $SO(9)$ fuzzy sphere

   Generally, the $SO(2k+1)$ fuzzy sphere (S^{2k} fuzzy sphere) is constructed by the n-fold symmetric tensor product of $(2k+1)$-dimensional gamma matrices:

$$B_p = \frac{\mu r}{2}[(\Gamma_p^{(2k)} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}) + \cdots + (\mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes \Gamma_p^{(2k)})]_{\text{sym}}.$$
$$B_pB_p = \frac{\mu^2 r^2}{4}n(n+2k)1_{N_k \times N_k}.$$

We start with the ansatz for the rank-2 fields C_{pq} for the $SO(2k+1)$ fuzzy spheres:

$$C_{pq} = -i\mu^{-1}f(r)B_{pq}.$$

Then, by solving the equations of motion for B_{μ} and $C_{\mu\nu}$ simultaneously, we obtain a constraint on the radius parameter r as

$$r = \frac{1}{2k}.$$

Comparison of the classical energy

- Trivial commutative solution $B_0 = \cdots = B_9 = 0$:

$$E_{B_{\mu}=0} = -S_{B_{\mu}=0} = 0.$$

- $SO(3) \times SO(3) \times SO(3)$ fuzzy spheres ($N_1 = n+1$):

$$\begin{aligned} E_{SO(3)^3} &= -S_{SO(3)^3} = -\frac{16\mu}{r_{SO(3)^3}}Tr(B_{\mu}B^{\mu}) \\ &= -12\mu^3 N_1(N_1-1)(N_1+1) \\ &\sim -\mathcal{O}(\mu^3 n^3) = -\mathcal{O}(\mu^3 N_1^3). \end{aligned}$$

- $SO(9)$ fuzzy sphere:

$$\begin{aligned} E_{SO(9)} &= -S_{SO(9)} = -\frac{5}{8}\mu^3 n(n+8)N_4 \\ &\sim -\mathcal{O}(\mu^3 n^{12}) = -\mathcal{O}(\mu^3 N_4^{\frac{6}{5}}), \end{aligned}$$

  where the size of the matrices B_p is

$$N_4 = \frac{(n+1)(n+2)(n+3)^2(n+4)^2(n+5)^2(n+6)(n+7)}{302400} \sim \mathcal{O}(n^{10}).$$

When the size of the matrices are the same...