Fuzzy CP^2 or S^2 — which is the true vacuum?

Takehiro Azuma, Subrata Bal, Keiichi Nagao and Jun Nishimura, hep-th/0405277

1. Matrix models on the homogeneous space

Large-N reduced models are the most powerful candidates for the constructive definition of superstring theory.

Several alterations of the IIB matrix model have been proposed, to accommodate the curved-space background.

- The matrix model with the Chern-Simons term:
  (hep-th/0101102,0204256,0207115)

These matrix models accommodate the curved-space fuzzy-manifold classical solutions, based on the homogeneous space.

A homogeneous space is realized as G/H:

- $G =$ (a Lie group)
- $H =$ (a closed subgroup of G)

$$\mathrm{S}^2 = \mathrm{SU}(2)/\mathrm{U}(1), \quad \mathrm{S}^2 \times \mathrm{S}^2, \quad \mathrm{S}^4 = \mathrm{SO}(5)/\mathrm{U}(2),$$
$$\mathrm{CP}^2 = \mathrm{SU}(3)/\mathrm{U}(2), \cdots.$$

Such curved-space fuzzy-manifold solutions are interesting in the following senses:

- More manifest realization of the curved-space background: Essential for an eligible framework for gravity.
- We may get insight into the dynamical generation of the gauge group.

2. The model and its classical solutions

Here, we scrutinize the bosonic matrix model that accommodates the four-dimensional fuzzy manifold.
In the following, we focus on the fuzzy CP^2 manifold.

$$S = N\mathrm{tr}\left(-\frac{1}{4}\sum_{\mu,\nu=1}^{8} [A_\mu, A_\nu]^2 + \frac{2i\alpha}{3}\sum_{\mu,\nu,\rho=1}^{8} f_{\mu\nu\rho} A_\mu A_\nu A_\rho \right).$$

- Defined in the 8-dimensional Euclidean space: $(\mu, \nu, \cdots = 1, \cdots, 8)$
- A_μ are promoted to the $N \times N$ hermitian matrices.
- $f_{\mu\nu\rho}$ are the structure constant of the SU(3).

$$f_{123} = 1, \quad f_{458} = f_{678} = \frac{\sqrt{3}}{2},$$
$$f_{147} = f_{246} = f_{257} = f_{345} = f_{516} = f_{637} = \frac{1}{2}.$$

Its equation of motion

$$[A_\nu, [A_\mu, A_\nu]] - i\alpha f_{\mu\nu\rho}[A_\nu, A_\rho] = 0$$

accommodates the following two classical solutions:

(a)fuzzy S^2 sphere

$$A_\mu^{(\mathrm{S}^2)} = \begin{cases} \alpha L_\mu^{(N)}, & (\mu = 1,2,3), \\ 0, & (\text{otherwise}). \end{cases}$$

The Casimir $Q = \sum_{\mu=1}^{8} A_\mu^2$ is given by

$$Q = \rho_{\mathrm{S}^2}^2 \mathbf{1}_N = \alpha^2 \frac{N^2-1}{4}\mathbf{1}_N.$$

(b)fuzzy CP^2 space

The fuzzy CP^2 space is realized by the $(m, 0)$ representation of the SU(3) Lie algebra:

$$A_\mu^{(\mathrm{CP}^2)} = \alpha T_\mu^{(m,0)}.$$

This corresponds to the $\mathrm{SU}(3)/\mathrm{U}(2)$ homogeneous space.

This space is realized by the symmetric tensor product of the fundamental representation of the SU(3) Lie algebra t_μ:

$$T_\mu^{(m,0)} = \underbrace{(t_\mu \otimes \mathbf{1}_3 \otimes \cdots \otimes \mathbf{1}_3)_{\mathrm{sym}}}_{m\text{-fold}} + (\mathbf{1}_3 \otimes t_\mu \otimes \cdots \otimes \mathbf{1}_3)_{\mathrm{sym}} + \cdots$$
$$+ (\mathbf{1}_3 \otimes \cdots \otimes \mathbf{1}_3 \otimes t_\mu)_{\mathrm{sym}}.$$

Here $_{\mathrm{sym}}$ denotes the symmetric tensor product.

The Casimir is given by

$$Q = \rho_{\mathrm{CP}^2}^2 \mathbf{1}_N = \alpha^2 \sum_{\mu=1}^{8} T_\mu^{(m,0)} T_\mu^{(m,0)} = \alpha^2 \frac{m(m+3)}{3}\mathbf{1}_N.$$

The matrix size of this representation is

$$N = \frac{(m+1)(m+2)}{3}, \quad (\text{for } m = 1,2,3,\cdots).$$

Thus, this representation is realized for a limited size of the matrices $N = 3, 6, 10, 15, 21, \cdots$.

We investigate this model via the heat bath algorithm of the Monte Carlo simulation.

In this sense, our analysis is nonperturbative.

3. The fuzzy CP^2 classical solution

We start from the fuzzy CP^2 initial condition:

$$A_\mu^{(0)} = A_\mu^{(\mathrm{CP}^2)}.$$

To see the behavior of this solution, we discuss the following observables:

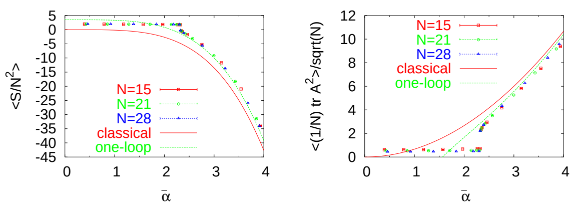
- The action S.
- The spacetime extent $\frac{1}{N}\mathrm{tr}\sum_{\mu=1}^{8} A_\mu^2$.

Here, we introduce the rescaled parameter $\tilde{\alpha} = \alpha N^{\frac{1}{4}}$.

first-order phase transition

We have a first-order phase transition, at the critical point

$$\tilde{\alpha} = \tilde{\alpha}_{\mathrm{cr}}^{(\mathrm{CP}^2)} (= \alpha_{\mathrm{cr}}^{(\mathrm{CP}^2)} N^{\frac{1}{4}} \simeq 2.3).$$



- $\alpha < \alpha_{\rm cr}^{(\mathrm{CP}^2)}$: the effect of the Chern-Simons term is negated, and we see the following behavior typical of the pure Yang-Mills model:

$$\frac{1}{N^2}\langle S\rangle \simeq \mathrm{O}(1), \quad \langle \frac{1}{N}\mathrm{tr} A_\mu^2\rangle \simeq \mathrm{O}(1).$$

- $\alpha > \alpha_{\rm cr}^{(\mathrm{CP}^2)}$: the fuzzy CP^2 is metastable.

one-loop dominance

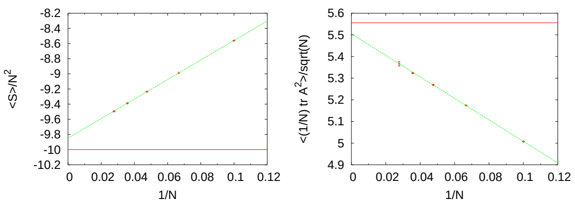
The numerical results are close to the one-loop result at $\alpha > \alpha_{\rm cr}^{(\mathrm{CP}^2)}$:

$$\frac{1}{N^2}\langle S\rangle \simeq -\frac{\tilde{\alpha}^4}{6}+\frac{7}{2}, \quad \frac{1}{\sqrt{N}}\langle \frac{1}{N}\mathrm{tr}\, A_\mu^2\rangle \simeq \frac{2\tilde{\alpha}^2}{3}-\frac{4}{\tilde{\alpha}^2}.$$

finite-N effect

We extrapolate the finite-N effect, by plotting these observables against $\frac{1}{N}$:

- $N = 10, 15, 21, 28, 36$ $(m = 3, 4, 5, 6, 7)$.
- $\tilde{\alpha} = 3.0$ is fixed.



- The finite-N effects are of the order $\mathrm{O}(\frac{1}{N})$.
- We have a deviation from the one-loop calculation at large N.

Since the deviation is rather small, we nevertheless regard this system as retaining the "one-loop dominance".

In fact, the three-dimensional model with fuzzy S^2 classical solution (scrutinized in hep-th/0401038) also has the same deviation.

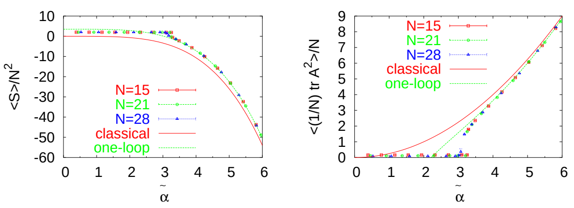
The critical point is consistent with the calculation from the one-loop effective action $\tilde{\alpha}_{\rm cr}^{(\mathrm{CP}^2)} = \frac{4}{\sqrt{3}} \simeq 2.3094011\cdots$.

4. The fuzzy S^2 classical solution

We next start the simulation from the fuzzy S^2 initial condition:

$$A_\mu^{(0)} = A_\mu^{(\mathrm{S}^2)}.$$

We plot the observables against the rescaled parameter $\tilde{\alpha} = \alpha N^{\frac{1}{2}}$.



first-order phase transition

We have a first-order phase transition, at the critical point

$$\tilde{\alpha} = \tilde{\alpha}_{\rm cr}^{(\mathrm{S}^2)} (= \alpha_{\rm cr}^{(\mathrm{S}^2)} N^{\frac{1}{2}} \simeq 3.2).$$

- $\alpha < \alpha_{\rm cr}^{(\mathrm{S}^2)}$: The behavior is similar to the pure Yang-Mills model.
- $\alpha > \alpha_{\rm cr}^{(\mathrm{S}^2)}$: the fuzzy S^2 is stable.

one-loop dominance

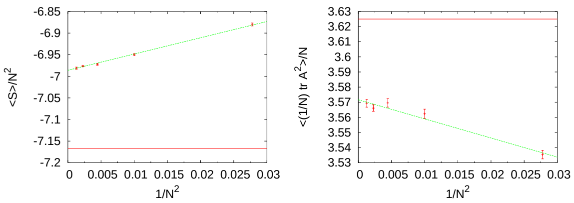
The numerical results are close to the one-loop result at $\alpha > \alpha_{\rm cr}^{(\mathrm{S}^2)}$:

$$\frac{1}{N^2}\langle S\rangle \simeq -\frac{\tilde{\alpha}^4}{24}+\frac{7}{2}, \quad \frac{1}{N}\langle \frac{1}{N}\mathrm{tr}\, A_\mu^2\rangle \simeq \frac{\tilde{\alpha}^2}{4}-\frac{6}{\tilde{\alpha}^2}.$$

finite-N effect

We extrapolate the finite-N effect, by plotting these observables against $\frac{1}{N^2}$:

- $N = 6, 10, 15, 21, 28$.
- $\tilde{\alpha} = 4.0$ is fixed.



For the fuzzy S^2 classical solution, we likewise see the nonperturbative deviation from the one loop at large N.

The critical point is derived from the one-loop effective action as $\tilde{\alpha}_{\rm cr}^{(\mathrm{S}^2)} = \sqrt{\frac{32}{3}} \simeq 3.2659863\cdots$.

5. Fuzzy CP^2 or S^2 — which is the true vacuum?

We determine which is the true vacuum, according to the one-loop dominance.

The one-loop effective action around the fuzzy CP^2 and S^2 is

$$\begin{aligned} W_{\mathrm{CP}^2} &= -\frac{m(m+3)}{12}\alpha^4 N^2 + 3\sum_{c=1}^{m}(c+1)^3 \log[N\alpha^2 c(c+2)] \\ &\simeq N^2\left(-\frac{\alpha^4 N}{6} + 6\log\alpha + 6\log N\right), \\ W_{\mathrm{S}^2} &= -\frac{1}{24}\alpha^4 N^2(N^2-1) + 3\sum_{l=1}^{N-1}(2l+1)\log[N\alpha^2 l(l+1)] \\ &\simeq N^2\left(-\frac{\alpha^4 N^2}{24} + 6\log\alpha + 9\log N\right). \end{aligned}$$

The difference is calculated (at large N) as

$$\Delta = W_{\mathrm{S}^2} - W_{\mathrm{CP}^2} = N^2\left\{\alpha^4\left(-\frac{N^2}{24}+\frac{N}{6}\right) + 3\log N\right\}.$$

- The classical effect is $\mathrm{O}(N^4)$.
- Whereas, the one-loop quantum effect is $\mathrm{O}(N^2\log N)$.

Therefore, $\Delta < 0$, namely $W_{\mathrm{S}^2} < W_{\mathrm{CP}^2}$.

The fuzzy S^2 is the true vacuum, and the fuzzy CP^2 is a metastable state.

Nevertheless, the fuzzy CP^2 state retains a very strong metastability.