Perturbative dynamics of fuzzy spheres at large N

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1. Introduction

Motivations of the fuzzy sphere studies

- Relation between the noncommutative field theory and the superstring study.
- Novel regularization scheme alternative to the lattice regularization.
- Prototype of the curved-space background in the large-N reduced model.

Matrix models on the homogeneous space G/H:

hep-th/0101102,0103192,0204256,0207115,0209057,0301055,0303120,0307007,0309264,0312241,
0401038,0403242,0405096,0405277

- G = (a Lie group)
- H = (a closed subgroup of G)

$$S^2 = \mathrm{SU}(2)/\mathrm{U}(1), \quad S^2 \times S^2, \quad S^4 = \mathrm{SO}(5)/\mathrm{U}(2),$$
$$CP^2 = \mathrm{SU}(3)/\mathrm{U}(2), \cdots.$$

2. 3d bosonic Yang-Mills-Chern-Simons model

Toy model incorporating the fuzzy sphere background:

$$S = N\mathrm{tr}\left(-\frac{1}{4}[A_\mu, A_\nu]^2 + \frac{2i\alpha}{3}\epsilon_{\mu\nu\rho}A_\mu A_\nu A_\rho\right).$$

- Defined in the 3-dimensional Euclidean space: $(\mu, \nu, \cdots = 1, \cdots, 3)$
- A_μ: $N \times N$ hermitian matrices.

Classical equation of motion:

$$[A_\nu, [A_\mu, A_\nu]] - i\alpha\epsilon_{\mu\nu\rho}[A_\nu, A_\rho] = 0.$$

Fuzzy S^2 classical solution:

$$X_\mu = \alpha L_\mu, \text{ where } L_\mu = (N \times N \text{ rep. of SU}(2)).$$

The Casimir operator:

$$Q = A_1^2 + A_2^2 + A_3^2 = R^2 1_N,$$
$$R = (\text{radius of sphere}) = \frac{\alpha}{2}\sqrt{N^2 - 1}.$$

3. Phase structure

Monte Carlo simulation via the heat bath algorithm:

Observables: $\langle S \rangle$ etc...

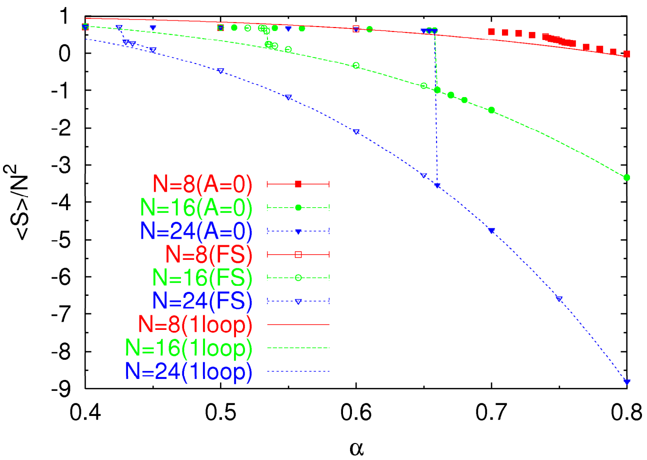
Initial condition:

$$A_\mu = \begin{cases} X_\mu & \text{(FS start)}, \\ 0 & \text{(zero start)}. \end{cases}$$

Discontinuity at

$$\alpha = \begin{cases} \alpha_{cr}^{(l)} \simeq \frac{2.1}{\sqrt{N}} & \text{(FS start)}, \\ \alpha_{cr}^{(u)} \simeq 0.66 & \text{(zero start)}. \end{cases}$$

FIRST-ORDER PHASE TRANSITION.



- $\alpha < \alpha_{cr}$: Yang-Mills phase
- $\alpha > \alpha_{cr}$: fuzzy sphere phase

One-loop dominance

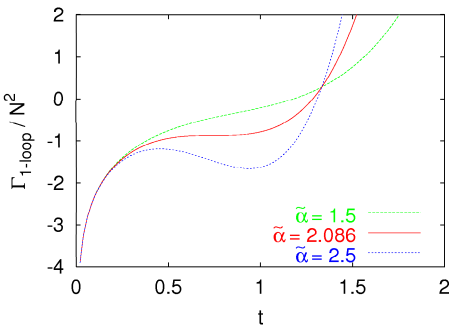
Perturbative calculation ($\tilde{\alpha} = \alpha\sqrt{N}$):

$$\frac{\langle S \rangle_{1-\text{loop}}}{N^2} \overset{\text{large } N}{\to} \underbrace{-\frac{\tilde{\alpha}^4}{24}}_{\text{classical}} \underbrace{+1}_{1-\text{loop}}.$$

Consistent with the numerical results in the fuzzy sphere phase.

Effective action at one loop around $A_\mu = tX_\mu$:

$$\frac{\Gamma_{1-\text{loop}}}{N^2} \overset{\text{large } N}{\to} \tilde{\alpha}^4\left(\frac{t^4}{8} - \frac{t^3}{6}\right) + \log t.$$



The local minimum disappears at

$$\tilde{\alpha} < \tilde{\alpha}_{cr}^{(l)} = \left(\frac{8}{3}\right)^{\frac{3}{4}} \simeq 2.086\cdots.$$

Consistent with the critical point via the Monte Carlo simulation.

4. Explicit two-loop calculation

Do higher-loop effects survive at large N?

Expansion around $A_\mu = X_\mu + \tilde{A}_\mu$:

$$S_{\rm g.f.} = -\frac{1}{2} N \,{\rm tr} \left([X_\mu, A_\mu]^2 \right) ,$$
$$S_{\rm gh} = -N \,{\rm tr}\, ([X_\mu, \bar{c}][A_\mu, c]) .$$

$$S_{\rm total} = S + S_{\rm g.f.} + S_{\rm gh} = S[X] + S_{\rm kin} + S_{\rm int} ,$$
$$S_{\rm kin} = \frac{1}{2} N \,{\rm tr}\, (\tilde{A}_\mu [X_\lambda, [X_\lambda, \tilde{A}_\mu]]) + N \,{\rm tr}\, (\bar{c}[X_\lambda, [X_\lambda, c]]) ,$$
$$S_{\rm int} = -\frac{1}{4} N \,{\rm tr}\, ([\tilde{A}_\mu, \tilde{A}_\nu]^2) - N \,{\rm tr}\, ([\tilde{A}_\mu, \tilde{A}_\nu][X_\mu, \tilde{A}_\nu]) + \frac{2}{3} i \alpha N \epsilon_{\mu\nu\rho} \,{\rm tr}\, (\tilde{A}_\mu \tilde{A}_\nu \tilde{A}_\rho) - N \,{\rm tr}\, ([X_\mu, \bar{c}][\tilde{A}_\mu, c]) .$$

Free energy:

$$W = -\log \left(\int d\tilde{A} dc d\bar{c} \exp(-S_{\rm total}) \right) .$$

The free energy and the effective action are identical at one loop.

Perturbative expansion:

$$W = \sum_{j=0}^{\infty} \underbrace{W_j}_{j-{\rm loop}} ,$$
$$W_0 = S[X] = -\frac{\tilde{\alpha}^4}{24}(N^2 - 1),$$
$$W_1 = \frac{1}{2} \sum_{l=1}^{N-1} (2l+1) \log(\tilde{\alpha}^2 l(l+1)),$$
$$W_j = -N^2 \frac{w_j(N)}{j-1} \tilde{\alpha}^{4(1-j)}, \quad (\text{for } j \geq 2).$$

Multi-loop calculation of $\langle S \rangle$:

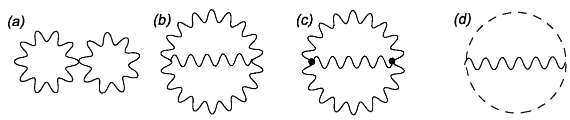
$$S(\beta, \alpha) = \beta S,$$
$$W(\beta, \alpha) = -\log \left(\int d\tilde{A} dc d\bar{c} \exp(-S(\beta, \alpha)) \right)$$
$$= \frac{3}{4}(N^2 - 1) \log \beta + W(1, \alpha \beta^{\frac{1}{4}}),$$

where $W = W(1, \alpha)$.

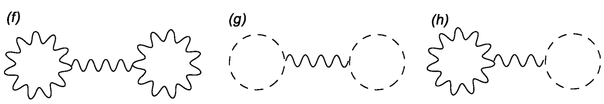
$$\frac{\langle S \rangle}{N^2} = \underbrace{(-\frac{\tilde{\alpha}^4}{24} + 1)(1 - \frac{1}{N^2})}_{= \frac{\langle S \rangle_{1-{\rm loop}}}{N^2}} + \sum_{j=2}^{\infty} \tilde{\alpha}^{4(1-j)} w_j(N).$$

W_2 entails the following two-loop diagrams:

1-loop irreducible (1PI)



1-loop reducible (1PR)



$$w_2^{(1PI)}(N) \simeq \underbrace{O(\frac{(\log N)^2}{N^2})}_{\text{calculated in hep-th/0303120}} ,$$
$$w_2^{(1PR)}(N) \simeq 1 - \frac{1}{N^2}.$$

In total, we obtain

$$w_2(N) = w_2^{(1PI)}(N) + w_2^{(1PR)}(N) \simeq 1 + O(\frac{(\log N)^2}{N^2}).$$

THE TWO-LOOP EFFECTS SURVIVE AT LARGE N FOR $\frac{\langle S \rangle}{N^2}$.

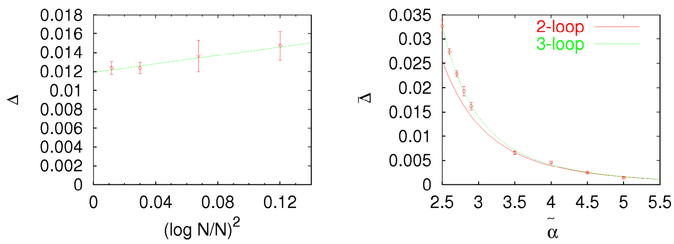
5. Higher-loop contribution

Comparison between the Monte Carlo simulation and the two-loop perturbation.

Discrepancy from the one-loop results:

$$\Delta = \frac{1}{N^2}(\langle S \rangle - \langle S \rangle_{1-{\rm loop}}) \simeq O(\frac{(\log N)^2}{N^2}) + \cdots .$$

Large-N extrapolation of the discrepancy of Δ (left, for $\tilde{\alpha} = 3.0$, $N = 4, 8, 16, 32$)



$$\bar{\Delta} = \lim_{N \to \infty} \Delta \simeq \frac{1}{\tilde{\alpha}^4} + \frac{c}{\tilde{\alpha}^8} + \cdots .$$

c: contribution of the three-loop diagrams

The fitting below gives at large N

$$c = \lim_{N \to \infty} w_3(N) = 11.1(3).$$

THREE-LOOP CONTRIBUTIONS SURVIVE AT LARGE N.