Perturbative dynamics of fuzzy spheres at large N

Takehiro Azuma, Keiichi Nagao and Jun Nishimura, hep-th/0410263

1. Introduction

Motivations of the fuzzy sphere studies

- Relation between the noncommutative field theory and the superstring study.
- Novel regularization scheme alternative to the lattice regularization.
- Prototype of the curved-space background in the large-N reduced model.

Matrix models on the homogeneous space G/H:
hep-th/0101102,0103192,0204256,0207115,0209057,0301055,0303120,0307007,0309264,0312241, 0401038,0403242,0405096,0405277

- $G =$ (a Lie group)
- $H =$ (a closed subgroup of G)

$$\mathrm{S}^2 = \mathrm{SU}(2)/\mathrm{U}(1), \quad \mathrm{S}^2 \times \mathrm{S}^2, \quad \mathrm{S}^4 = \mathrm{SO}(5)/\mathrm{U}(2),$$
$$\mathbf{CP}^2 = \mathrm{SU}(3)/\mathrm{U}(2), \cdots.$$

2. 3d bosonic Yang-Mills-Chern-Simons model

Toy model incorporating the fuzzy sphere background:

$$S = N\mathrm{tr}\left(-\frac{1}{4}[A_\mu, A_\nu]^2 + \frac{2i\alpha}{3}\epsilon_{\mu\nu\rho}A_\mu A_\nu A_\rho\right).$$

- Defined in the 3-dimensional Euclidean space: $(\mu, \nu, \cdots = 1, \cdots, 3)$
- A_μ: $N \times N$ hermitian matrices.

Classical equation of motion:

$$[A_\nu, [A_\mu, A_\nu]] - i\alpha\epsilon_{\mu\nu\rho}[A_\nu, A_\rho] = 0.$$

Fuzzy S^2 classical solution:

$$X_\mu = \alpha L_\mu, \text{ where } L_\mu = (N \times N \text{ rep. of } \mathrm{SU}(2)).$$

The Casimir operator:

$$Q = A_1^2 + A_2^2 + A_2^3 = R^2 1_N,$$
$$R = (\text{radius of sphere}) = \frac{\alpha}{2}\sqrt{N^2 - 1}.$$

3. Phase structure

Monte Carlo simulation via the heat bath algorithm:

Observables: $\langle S \rangle$ etc...

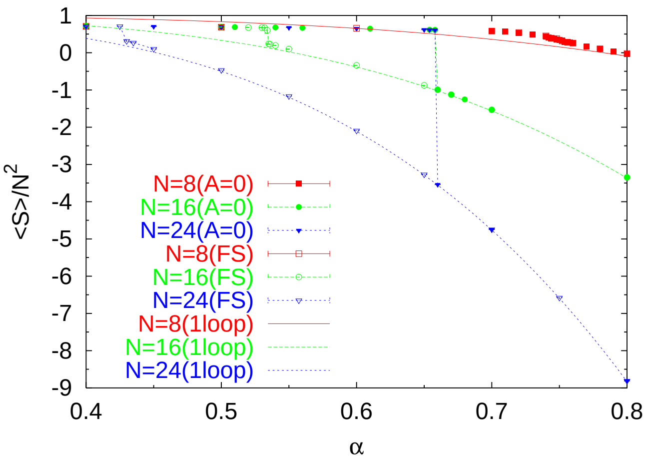
Initial condition:

$$A_\mu = \begin{cases} X_\mu & \text{(FS start)}, \\ 0 & \text{(zero start)}. \end{cases}$$

Discontinuity at

$$\alpha = \begin{cases} \alpha_{\mathrm{cr}}^{(l)} \simeq \frac{2.1}{\sqrt{N}} & \text{(FS start)}, \\ \alpha_{\mathrm{cr}}^{(u)} \simeq 0.66 & \text{(zero start)}. \end{cases}$$

First-order phase transition.



- $\alpha < \alpha_{\mathrm{cr}}$: Yang-Mills phase
- $\alpha > \alpha_{\mathrm{cr}}$: fuzzy sphere phase

One-loop dominance

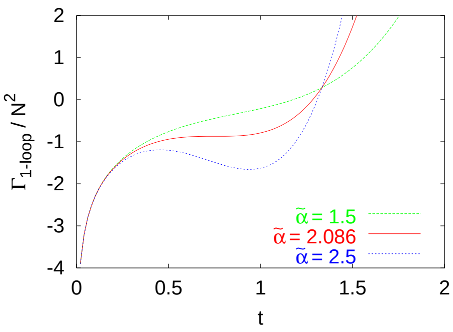
Perturbative calculation ($\tilde{\alpha} = \alpha\sqrt{N}$):

$$\frac{\langle S \rangle_{1-\mathrm{loop}}}{N^2} \overset{\text{large } N}{\to} \underbrace{-\frac{\tilde{\alpha}^4}{24}}_{\text{classical}} \underbrace{+1}_{1-\text{loop}}.$$

Consistent with the numerical results in the fuzzy sphere phase.

Effective action at one loop around $A_\mu = tX_\mu$:

$$\frac{\Gamma_{1-\mathrm{loop}}}{N^2} \overset{\text{large } N}{\to} \tilde{\alpha}^4\left(\frac{t^4}{8} - \frac{t^3}{6}\right) + \log t.$$



The local minimum disappears at

$$\tilde{\alpha} < \tilde{\alpha}_{\mathrm{cr}}^{(l)} = \left(\frac{8}{3}\right)^{\frac{3}{4}} \simeq 2.086\cdots.$$

Consistent with the critical point via the Monte Carlo simulation.

4. Explicit two-loop calculation

Do higher-loop effects survive at large N?

Expansion around $A_\mu = X_\mu + \tilde{A}_\mu$:

$$\begin{aligned} S_{\text{g.f.}} &= -\frac{1}{2} N \,\text{tr}\left([X_\mu, A_\mu]^2\right), \\ S_{\text{gh}} &= -N \,\text{tr}\,([X_\mu, \bar{c}][A_\mu, c]). \end{aligned}$$

$$\begin{aligned} S_{\text{total}} &= S + S_{\text{g.f.}} + S_{\text{gh}} = S[X] + S_{\text{kin}} + S_{\text{int}}, \\ S_{\text{kin}} &= \frac{1}{2} N \,\text{tr}\,(\tilde{A}_\mu [X_\lambda, [X_\lambda, \tilde{A}_\mu]]) + N \,\text{tr}\,(\bar{c}[X_\lambda, [X_\lambda, c]]), \\ S_{\text{int}} &= -\frac{1}{4} N \,\text{tr}\,([\tilde{A}_\mu, \tilde{A}_\nu]^2) - N \,\text{tr}\,([\tilde{A}_\mu, \tilde{A}_\nu][X_\mu, \tilde{A}_\nu]) \\ &\quad + \frac{2}{3} i \alpha N \epsilon_{\mu\nu\rho} \,\text{tr}\,(\tilde{A}_\mu \tilde{A}_\nu \tilde{A}_\rho) - N \,\text{tr}\,([X_\mu, \bar{c}][\tilde{A}_\mu, c]). \end{aligned}$$

Free energy:

$$W = -\log\left(\int d\tilde{A} dc d\bar{c} \exp(-S_{\text{total}})\right).$$

The free energy and the effective action are identical at one loop.

Perturbative expansion:

$$\begin{aligned} W &= \sum_{j=0}^{\infty} \underbrace{W_j}_{\text{j-loop}}, \\ W_0 &= S[X] = -\frac{\tilde{\alpha}^4}{24}(N^2 - 1), \\ W_1 &= \frac{1}{2} \sum_{l=1}^{N-1} (2l+1) \log(\tilde{\alpha}^2 l(l+1)), \\ W_j &= -N^2 \frac{w_j(N)}{j-1} \tilde{\alpha}^{4(1-j)}, \ (\text{for } j \geq 2). \end{aligned}$$

Multi-loop calculation of $\langle S \rangle$:

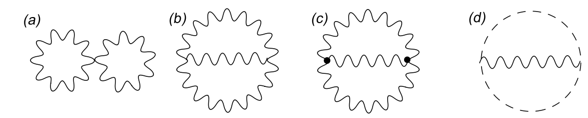
$$\begin{aligned} S(\beta, \alpha) &= \beta S, \\ W(\beta, \alpha) &= -\log\left(\int d\tilde{A} dc d\bar{c} \exp(-S(\beta, \alpha))\right) \\ &= \frac{3}{4}(N^2 - 1)\log\beta + W(1, \alpha\beta^{\frac{1}{4}}), \end{aligned}$$

where $W = W(1, \alpha)$.

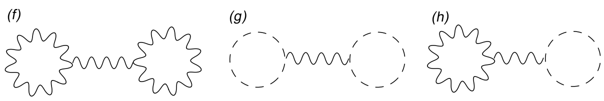
$$\frac{\langle S \rangle}{N^2} = \underbrace{\left(-\frac{\tilde{\alpha}^4}{24} + 1\right)\left(1 - \frac{1}{N^2}\right)}_{= \frac{\langle S \rangle_{1-\text{loop}}}{N^2}} + \sum_{j=2}^{\infty} \tilde{\alpha}^{4(1-j)} w_j(N).$$

W_2 entails the following two-loop diagrams:

1-loop irreducible (1PI)



1-loop reducible (1PR)



$$\begin{aligned} w_2^{(1PI)}(N) &\simeq \underbrace{O\left(\frac{(\log N)^2}{N^2}\right)}_{\text{calculated in hep-th/0303120}}, \\ w_2^{(1PR)}(N) &\simeq 1 - \frac{1}{N^2}. \end{aligned}$$

In total, we obtain

$$w_2(N) = w_2^{(1PI)}(N) + w_2^{(1PR)}(N) \simeq 1 + O\left(\frac{(\log N)^2}{N^2}\right).$$

THE TWO-LOOP EFFECTS SURVIVE AT LARGE N FOR $\frac{\langle S \rangle}{N^2}$.

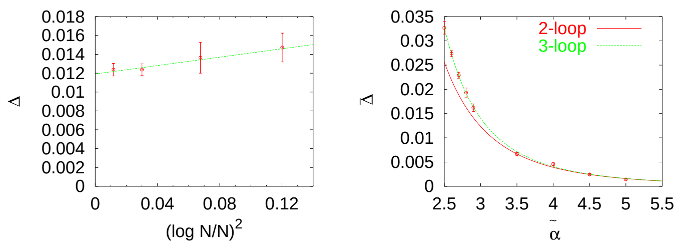
5. Higher-loop contribution

Comparison between the Monte Carlo simulation and the two-loop perturbation.

Discrepancy from the one-loop results:

$$\Delta = \frac{1}{N^2}(\langle S \rangle - \langle S \rangle_{1-\text{loop}}) \simeq O\left(\frac{(\log N)^2}{N^2}\right) + \cdots.$$

Large-N extrapolation of the discrepancy of Δ (left, for $\tilde{\alpha} = 3.0$, $N = 4, 8, 16, 32$)



$$\bar{\Delta} = \lim_{N \to \infty} \Delta \simeq \frac{1}{\tilde{\alpha}^4} + \frac{c}{\tilde{\alpha}^8} + \cdots.$$

c: contribution of the three-loop diagrams

The fitting below gives at large N

$$c = \lim_{N \to \infty} w_3(N) = 11.1(3).$$

THREE-LOOP CONTRIBUTIONS SURVIVE AT LARGE N.