Spontaneous breakdown of Lorentz symmetry in a simplified IKKT matrix model

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1. Introduction

Matrix models as a constructive definition of superstring theory

iKKT model (IIB matrix model)
$\Rightarrow$ Promising candidate for constructive definition of superstring theory.
N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, hep-th/9612115.

$$S = N\left(-\frac{1}{4}\mathrm{tr}\,[A_\mu, A_\nu]^2 + \frac{1}{2}\mathrm{tr}\,\bar{\psi}\Gamma^\mu[A_\mu,\psi]\right).$$

- Dimensional reduction of $\mathcal{N} = 1$ 10d Super-Yang-Mills (SYM) theory to 0d.
  A_μ (10d vector) and ψ (10d Majorana-Weyl spinor)
  $\Rightarrow$ $N \times N$ matrices .
- Evidences for spontaneous breakdown of SO(10) $\to$ SO(4).
  J. Nishimura and F. Sugino, hep-th/0111102,
  H. Kawai, et. al. hep-th/0204240,0211272,0602044,0603146.
- Complex determinant (from integrating out fermions) :
  - * Crucial for breakdown of rotational symmetry.
    J. Nishimura and G. Vernizzi, hep-th/0003223.
  - * Difficulty of Monte Carlo simulation.

2. Simplified IKKT matrix model

Simplified model with spontaneous rotational symmetry breakdown,
J. Nishimura, hep-th/0108070.

$$S = \underbrace{\frac{N}{2}\mathrm{tr}\,A_\mu^2}_{=S_b} \underbrace{-\bar{\psi}^f_\alpha(\Gamma_\mu)_{\alpha\beta}A_\mu\psi^f_\beta}_{=S_f}$$

- A_μ: $N \times N$ hermitian matrices ($\mu = 1, \cdots, 4$)
  $\bar{\psi}^f_\alpha, \psi^f_\alpha$: N-dim vector ($\alpha = 1, 2, f = 1, \cdots, N_f$),
  N_f = (number of flavors).

$$\Gamma_1 = i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \Gamma_2 = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$
$$\Gamma_3 = i\sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \Gamma_4 = \sigma_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

- SU(N) symmetry and SO(4) rotational symmetry.
- Partition function:

$$Z = \int dA e^{-S_B}(\det\mathcal{D})^{N_f} = \int dA e^{-S_0}e^{i\Gamma}, \text{ where}$$
$$\mathcal{D} = \Gamma_\mu A_\mu = (2N \times 2N \text{ matrices}),$$
$$e^{-S_0} = e^{-S_B}|\det\mathcal{D}|^{N_f}.$$

Analytical studies of the model

Solvable at $N \to \infty$ using random matrix theory (RMT) technique.

$$\langle\frac{1}{N}\mathrm{tr}\,A_\mu^2\rangle = \begin{cases} 1 + r + o(r), & (\mu = 1,2,3) \\ 1 - r + o(r), & (\mu = 4), \end{cases}$$

for small $r = N_f/N$.

Spontaneous breakdown of SO(4) symmetry to SO(3).

For the phase-quenched partition function $Z_0 = \int dA e^{-S_0}$,

$\langle\frac{1}{N}\mathrm{tr}\,A_\mu^2\rangle = 1 + r/2$ for $\mu = 1,2,3,4$.

Phase $\Rightarrow$ crucial in rotational symmetry breakdown.

Gaussian expansion analysis up to 9th order:
T. Okubo, J. Nishimura and F. Sugino, hep-th/0412194.
Spontaneous breakdown of SO(4) to SO(2) at finite r.

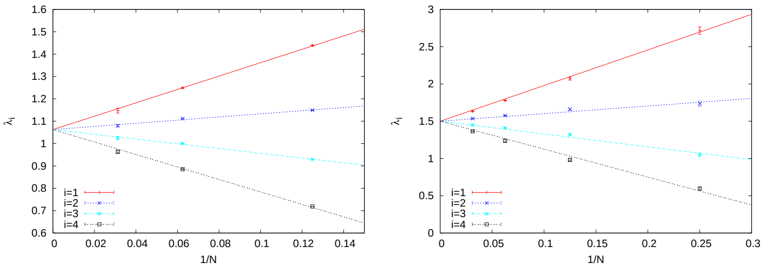
3. Monte Carlo studies of the model

Hybrid Monte Carlo (HMC) simulation of the phase-quenched model

HMC simulation of partition function Z_0 (with the phase omitted).

Observable for probing dimensionality : $T_{\mu\nu} = \frac{1}{N}\mathrm{tr}\,(A_\mu A_\nu)$.

λ_i ($i = 1,2,3,4$) : eigenvalues of $T_{\mu\nu}$ ($\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$)



Results for $r = \frac{1}{8}$ (left) and $r = 1$ (right).

$$\lambda_1, \cdots, \lambda_4 \to 1 + \frac{r}{2} \text{ (as } N \to \infty).$$

Factorization method

Numerical approach to the complex action problem.
K. N. Anagnostopoulos and J. Nishimura, hep-th/0108041,
J.Ambjorn, K.N.Anagnostopoulos, J.Nishimura and J.J.M.Verbaarschot, hep-lat/0208025.

Overlap problem: Discrepancy of a distribution function between the phase-quenched model Z_0 and the full model Z.

Force the simulation to sample the important region for the full model.

Standard reweighting method:

$$\langle\lambda_i\rangle = \frac{\langle\lambda_i\cos\Gamma\rangle_0}{\langle\cos\Gamma\rangle_0},$$

where $\langle * \rangle_0$ = (V.E.V. for the phase-quenched model Z_0).

(♯ of configurations required) $\simeq e^{O(N^2)}$. $\Rightarrow$ complex-action problem.
$\tilde{\lambda}_i \stackrel{\text{def}}{=} \lambda_i/\langle\lambda_i\rangle_0$: deviation from 1 $\Rightarrow$ effect of the phase.

Distribution function

$$\rho_i(x) \stackrel{\text{def}}{=} \langle\delta(x - \tilde{\lambda}_i)\rangle = \frac{1}{C}\rho_i^{(0)}(x)w_i(x),$$

where

$$C = \langle\cos\Gamma\rangle_0, \quad \rho_i^{(0)}(x) = \langle\delta(x - \tilde{\lambda}_i)\rangle_0, \quad w_i(x) = \langle\cos\Gamma\rangle_{i,x},$$

$$\langle * \rangle_{i,x} = [\text{V.E.V. for the partition function } Z_{i,x} = \int dA e^{-S_0}\delta(x - \tilde{\lambda}_i)].$$

Resolution of the overlap problem:
$\Rightarrow$ Visit the configurations where $\rho_i(x)$ is important.

Monte Carlo evaluation of $\rho_i^{(0)}(x)$ and $w_i(x)$

Approximation of the partition function $Z_{i,x}$:

$$Z_{i,V} = \int dA e^{-S_0} \underbrace{e^{-V(\lambda_i)}}_{\simeq \delta(x-\tilde{\lambda}_i)}, \quad \text{where}$$

$$V(x) = \frac{\gamma}{2}(x-\xi)^2, \quad \gamma, \xi = \text{(parameters)}.$$

Monte Carlo evaluation of $\rho_i^{(0)}(x)$ and $w_i(x)$:

$$\rho_{i,V}(x) \stackrel{\text{def}}{=} \langle \delta(x - \tilde{\lambda}_i) \rangle_{i,V} \propto \rho_i^{(0)}(x) \exp(-V(\langle \lambda_i \rangle_0 x)).$$

The position x_p of the peak for $\rho_{i,V}(x)$:

$$0 = \frac{\partial}{\partial x} \log \rho_{i,V}(x) = f_i^{(0)}(x) - \langle \lambda_i \rangle_0 V'(\langle \lambda_i \rangle_0 x),$$

$$f_i^{(0)}(x) \stackrel{\text{def}}{=} \frac{\partial}{\partial x} \log \rho_i^{(0)}(x).$$

- Determination of x_p: Approximated as $x_p \simeq \langle \tilde{\lambda}_i \rangle_{i,V}$.
- Determination of $\rho_i^{(0)}(x)$:
  1. Vary ξ.
  2. Calculate $f_i^{(0)}(x_p)$ for different x_p (and ξ).
  3. Evaluate $\rho_i^{(0)}(x) = \exp[\int_0^x dz f_i^{(0)}(z) + \text{const.}]$.

Why such a roundabout way?
$\Rightarrow$ To capture the skirt of $\rho_i^{(0)}(x)$.

Monte Carlo evaluation of $\langle \tilde{\lambda}_i \rangle$

$\tilde{\lambda}_i = \lambda_i / \langle \lambda_i \rangle_0$: deviation from phase-quenched model.

Direct evaluation:

$$\langle \tilde{\lambda}_i \rangle = \int_0^\infty dx x \rho_i(x) = \frac{\int_0^\infty dx x \rho_i^{(0)}(x) w_i(x)}{\int_0^\infty dx \rho_i^{(0)}(x) w_i(x)}.$$

Difficult because $w_i(x) \simeq 0$ at large N.

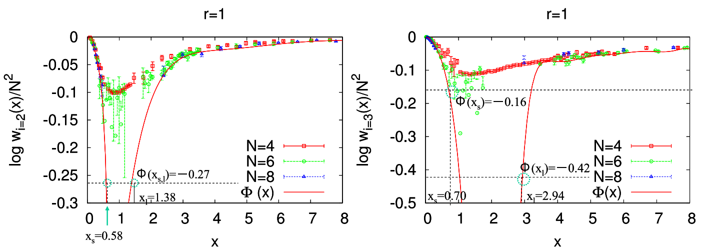
$w_i(x) > 0 \Rightarrow \langle \tilde{\lambda}_i \rangle$ is the minimum of $\mathcal{F}_i(x)$:

$$\mathcal{F}_i(x) = \text{(free energy density)} = -\frac{1}{N^2} \log \rho_i(x).$$

We solve $\mathcal{F}_i'(x) = 0$, namely

$$\frac{1}{N^2} f_i^{(0)}(x) = -\frac{d}{dx}\left(\frac{1}{N^2} \log w_i(x)\right).$$

Analysis for $r = N_f/N = 1$.



$i = 2, 3$ cases

Both $\frac{1}{N^2} f_i^{(0)}(x)$ and $\frac{1}{N^2} \log w_i(x)$ scales at large N.

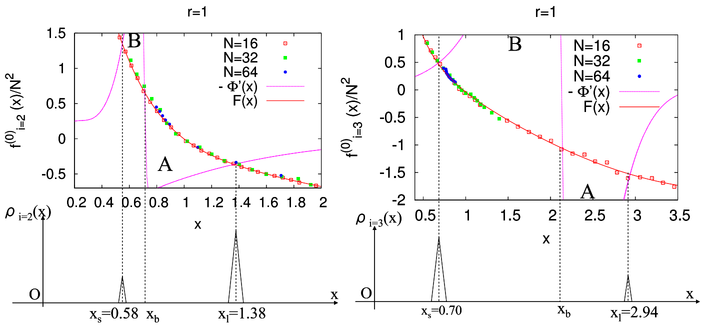
$$\frac{1}{N^2} f_i^{(0)}(x) \to F_i(x), \quad \frac{1}{N^2} \log w_i(x) \to \Phi_i(x).$$

Extrapolation of $F_i(x)$ and $\Phi_i(x)$:

$$F_i(x) \simeq a_{i,0} + (a_{i,1} x + \frac{b_{i,1}}{x}) + \cdots + (a_{i,4} x^4 + \frac{b_{i,4}}{x^4}),$$

$$\Phi_i(x) \simeq \begin{cases} \phi_{i,s}(x) = c_{i,0} + c_{i,1} x + \cdots + c_{i,4} x^4, & (x < x_s), \\ \phi_{i,l}(x) = d_{i,0} + d_{i,1} x + \cdots + d_{i,8} x^8, & (x > x_l), \\ \frac{\phi_{i,s}(x) e^{-C(x-\alpha)} + \phi_{i,l}(x) e^{C(x-\alpha)}}{e^{-C(x-\alpha)} + e^{C(x-\alpha)}}, \\ (x_s < x < x_l). \end{cases}$$

At $x = \alpha$, $\phi_{i,s}(x) = \phi_{i,l}(x)$.



Three solutions of $\mathcal{F}_i'(x) = 0$ ($x_s < x_b < x_l$).

Double-peak structure of $\rho_i(x)$.

Which peak is higher?

- $$\frac{1}{N^2}(\log \rho_i(x_l) - \log \rho_i(x_b)) = \int_{x_b}^{x_l} dx (F_i(x) + \Phi_i'(x)) = \text{(A's area)}.$$
- $$\frac{1}{N^2}(\log \rho_i(x_s) - \log \rho_i(x_b)) = -\int_{x_s}^{x_b} dx (F_i(x) + \Phi_i'(x)) = \text{(B's area)}.$$

Difference of the height:

$$\begin{aligned} \Delta_i &= \frac{1}{N^2}(\log \rho_i(x_l) - \log \rho_i(x_s)) \\ &= (\Phi_i(x_l) - \Phi_i(x_s)) + \int_{x_s}^{x_l} dx F_i(x) \\ &= \text{(A's area)-(B's area)} \\ &\simeq \begin{cases} +0.12 \cdots > 0, & (i=2), \\ -1.93 \cdots < 0, & (i=3). \end{cases} \end{aligned}$$

For this extrapolation, $\rho_i(x)$ is dominant at

$$x = \begin{cases} x_l \simeq 1.38, & (i=2), \\ x_s \simeq 0.70, & (i=3). \end{cases}$$

Result of the 9th order Gaussian expansion:
T. Okubo, J. Nishimura and F. Sugino, hep-th/0412194.

$$\tilde{\lambda}_{i=1} \simeq 1.4, \quad \tilde{\lambda}_{i=2} \simeq 1.4, \quad \tilde{\lambda}_{i=3} \simeq 0.7, \quad \tilde{\lambda}_{i=4} \simeq 0.5.$$

Scenario for Lorentz symmetry breakdown SO(4) $\to$ SO(2).

Ambiguity of the extrapolation of $\Phi_i(x)$:

- Position of the peaks $x_{s,l}$.
- Value (even the sign) of Δ_i (which peak is higher?).

More solid analysis in the future:
$\Rightarrow$ Analysis of the region at which $w_i(x) \simeq 0$.