Monte Carlo simulation of the large-N reduced models

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D3 presentation at Kyoto University
Feb. 4th 2004 10:20 $\sim$ 10:40

References: hep-th/9811220,0401038

Contents

1 Introduction

Large-N reduced models are the most powerful candidates for the constructive definition of superstring theory.

$\boxed{\text{The IIB matrix model}}$

$$\boxed{S = -\frac{1}{g^2} Tr_{N\times N}\left(\frac{1}{4}\sum_{\mu,\nu=1}^{10}[A_\mu, A_\nu]^2 + \frac{1}{2}\bar{\psi}\sum_{\mu=1}^{10}\Gamma^\mu[A_\mu, \psi]\right),}$$

- Dimensional reduction of $\mathcal{N} = 1$ 10-dimensional SYM theory to 0 dimension.
  A_μ and ψ are $N \times N$ Hermitian matrices.
  - ⋆ A_μ: 10-dimensional vectors
  - ⋆ ψ: 10-dimensional Majorana-Weyl (i.e. 16-component) spinors
- Matrix regularization of the Schild form of the Green-Schwarz action of the type IIB superstring theory.

The action of the IIB matrix model does not include the integral.

The numerical treatment is easier than that for the quantum field theory!

2 Heat bath algorithm of the matrix model

In this talk, we focus on the heat bath algorithm of the bosonic part of the matrix model.

The treatment of the fermions entails the hybrid Monte Carlo simulation, and we do not delve into the fermions' analysis here.

(a) Quadratic $U(N)$ one-matrix model

We start with the simplest case – quadratic $U(N)$ one-matrix model:

$$S = \frac{N}{2} Tr \phi^2.$$

To analyze this model via the heat bath algorithm, we rewrite the $U(N)$ matrix ϕ as

$$\phi_{ii} = \frac{a_i}{\sqrt{N}}, \quad \begin{cases} \phi_{ij} = \frac{x_{ij}+iy_{ij}}{\sqrt{2N}} \\ \phi_{ji} = \frac{x_{ij}-iy_{ij}}{\sqrt{2N}}, \end{cases} \quad (\text{for } i < j).$$

The N^2 real quantities a_i, x_{ij}, y_{ij} comply with the independent normal Gaussian distribution.

$$S = \frac{1}{2}\sum_{i=1}^{N} a_i^2 + \frac{1}{2}\sum_{i<j}((x_{ij})^2 + (y_{ij})^2).$$

$$Z = \int \prod_{i=1}^{N} da_i \prod_{1 \le i < j \le N} dx_{ij} dy_{ij} \exp\left(-\frac{1}{2}\sum_{i=1}^{N} a_i^2 - \frac{1}{2}\sum_{1 \le i < j \le N}((x_{ij})^2 + (y_{ij})^2)\right).$$

a_i, x_{ij}, y_{ij} are updated by the Gaussian random number.

Generation of the uniform random number

We use the congruence method to generate the uniform random number.

- We give the random seed z_1, such as $a_1 = \mathsf{time}()$.
- We solve the recursion formula

$$z_{k+1} = az_k + c \pmod{2^{31} - 1}.$$

  The choice $(a, c) = (5^{11}, 0)$ is known to give a good pseudo-random number.

- The sequence $\{\frac{z_k}{2^{31}-1}\}$ gives a uniform pseudo-random number $[0{:}1]$.

Generation of the Gaussian random number

- We take two uniform random numbers $x, y \in [0 : 1]$.
- We introduce the quantity $r = \sqrt{-a^2 \log x^2}$. This complies with the probability distribution

$$P(r)dr = P(x)\frac{dx}{dr}dr = \frac{2r}{a^2} \exp\left(-\frac{r^2}{a^2}\right).$$

- We next introduce the quantities

$$X = r\cos(2\pi y), \quad Y = r\sin(2\pi y).$$

  They comply with the probability distribution

$$P(r)drdy \propto \exp\left(-\frac{1}{a^2}(X^2 + Y^2)\right).$$

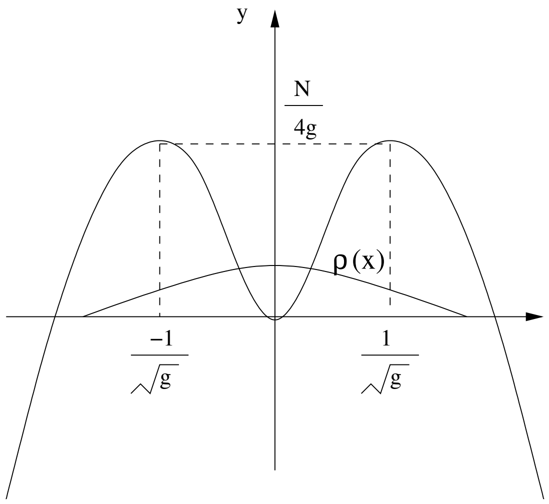
(b) Quartic $U(N)$ one-matrix model

We next analyze the quartic one-matrix model via the heat-bath algorithm:

$$S = \frac{N}{2} Tr\phi^2 - \frac{gN}{4} Tr\phi^4.$$

This action is unbounded below.

However, we can avoid the divergence in the large-N limit, due to the metastability of the origin.



We introduce the auxiliary fields Q (where $\alpha = \sqrt{\frac{g}{2}}$) in order to render the action quadratic:

$$\tilde{S} = \frac{N}{2} Tr\phi^2 + \frac{N}{2} TrQ^2 - \alpha N TrQ\phi^2 = \frac{N}{2} Tr(Q - \alpha\phi^2)^2 + S.$$

We update Q as

$$Q_{ii} = \frac{a_i}{\sqrt{N}} + \alpha(\phi^2)_{ii}, \; Q_{ij} = \frac{x_{ij} + iy_{ij}}{\sqrt{2N}} + \alpha(\phi^2)_{ij},$$

where a_i, x_{ij}, y_{ij} comply with the normal Gaussian distribution.

In updating the diagonal part ϕ_{ii}, we extract the dependence of ϕ_{ii}:

$$\tilde{S} = \frac{N}{2}(\phi_{ii})^2 \underbrace{(1 - 2\alpha Q_{ii})}_{=c_i} - N\phi_{ii} \underbrace{(\alpha \sum_{j \neq i} (\phi_{ji} Q_{ij} + Q_{ji}\phi_{ij}))}_{=h_i}.$$

Then, ϕ_{ii} is updated as

$$\phi_{ii} = \frac{a_i}{\sqrt{N c_i}} + \frac{h_i}{c_i}.$$

We likewise extract the ϕ_{ij} dependence:

$$\tilde{S} = N \underbrace{(1 - \alpha(Q_{ii} + Q_{jj}))}_{=c_{ij}} |\phi_{ij}|^2 - N(\phi_{ij} h_{ji} + \phi_{ji} h_{ij}), \text{ where}$$

$$h_{ij} = \alpha(\sum_{k \neq j} (\phi_{ik} Q_{kj} + \sum_{k \neq i} Q_{ik}\phi_{kj})).$$

Then, ϕ_{ij} is updated as follows:

$$\phi_{ij} = \frac{x_{ij} + i y_{ij}}{\sqrt{2N c_{ij}}} + \frac{h_{ij}}{c_{ij}}.$$

The legitimacy of the algorithm is ascertained by checking the following results (as $N \to \infty$):

E.Brezin, C.Itzykson, G.Parisi and J.Zuber, Comm. Math. Phys. 59, 35 (1978).

$$\langle \frac{1}{N} Tr\phi^2 \rangle = \frac{1}{3} a^2 (4 - a^2), \text{ where } a^2 = \frac{2}{1 + \sqrt{1 - 12g}}.$$

The eigenvalue distribution is given by

$$\rho(x) = \frac{1}{2\pi}(-gx^2 - 2ga^2 + 1)\sqrt{4a^2 - x}.$$

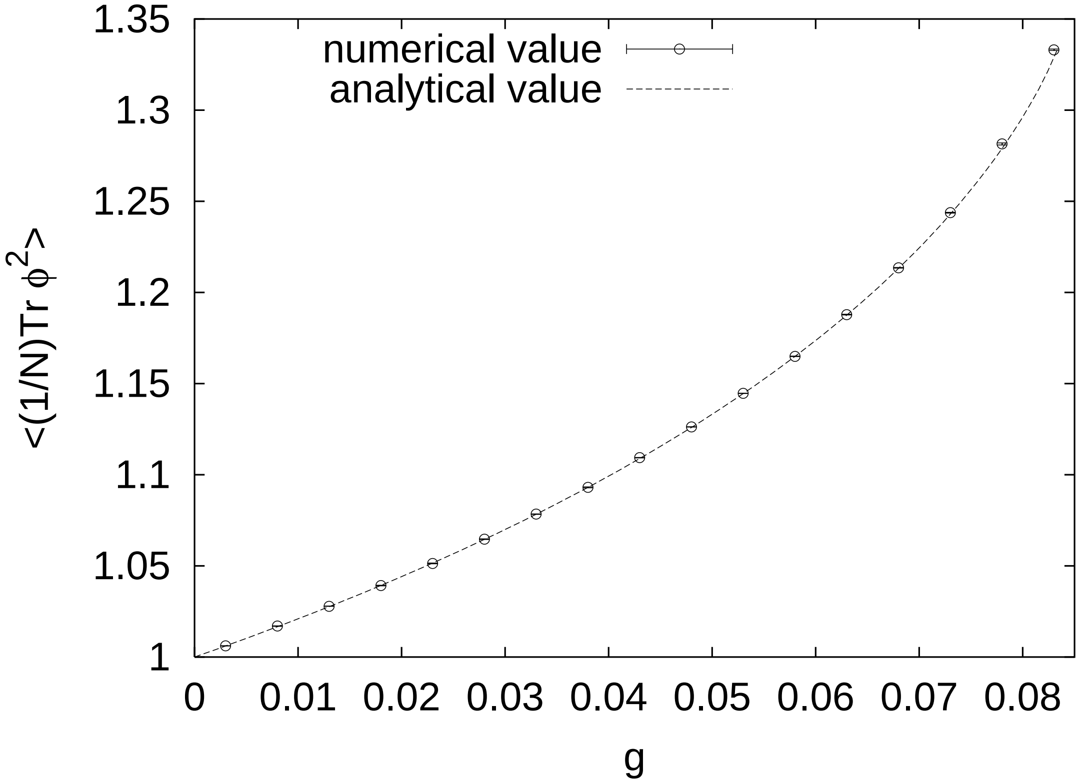


Figure 1: The plot of $\langle \frac{1}{N} Tr\phi^2 \rangle$ for $N = 32$.

(c) The bosonic IIB matrix model

T. Hotta, J. Nishimura and A. Tsuchiya hep-th/9811220.

We investigate the bosonic IIB matrix model via the the heat-bath algorithm:

$$S = -\frac{N}{4}\sum_{\mu,\nu=1}^{d} Tr[A_\mu, A_\nu]^2$$
$$= -\frac{N}{2}\sum_{1\le\mu<\nu\le d} Tr\{A_\mu, A_\nu\}^2 + 2N\sum_{1\le\mu<\nu\le d} Tr(A_\mu^2 A_\nu^2),$$

defined in the d-dimensional Euclidean space.

This action is equivalent to $\tilde{S}$, after integrating out $Q_{\mu\nu}$ (where $G_{\mu\nu} = \{A_\mu, A_\nu\}$):

$$\tilde{S} = \sum_{1\le\mu<\nu\le d}\left(\frac{N}{2} TrQ_{\mu\nu}^2 - NTr(Q_{\mu\nu}G_{\mu\nu}) + 2NTr(A_\mu^2 A_\nu^2)\right)$$
$$= \frac{N}{2}\sum_{1\le\mu<\nu\le d} Tr(Q_{\mu\nu} - G_{\mu\nu})^2 + S.$$

Then, $Q_{\mu\nu}$ is updated as

$$(Q_{\mu\nu})_{ii} = \frac{a_i}{\sqrt{N}} + (G_{\mu\nu})_{ii},\ (Q_{\mu\nu})_{ij} = \frac{x_{ij} + iy_{ij}}{\sqrt{2N}} + (G_{\mu\nu})_{ij},$$

We next update A_μ. We extract the dependence of A_λ.

$$\tilde{S} = -NTr(T_\lambda A_\lambda) + 2NTr(S_\lambda A_\lambda^2) + \cdots, \text{ where}$$
$$S_\lambda = \sum_{\mu \neq \lambda} (A_\mu^2), \; T_\lambda = \sum_{\mu \neq \lambda} (A_\mu Q_{\lambda\mu} + Q_{\lambda\mu} A_\mu).$$

- The diagonal part A_μ is updated by extracting the dependence of $(A_\mu)_{ii}$:

  $$\tilde{S} = 2N(S_\lambda)_{ii}(A_\mu)_{ii}^2 - 4Nh_i(A_\mu)_{ii}, \text{ where}$$
  $$h_i = \frac{N}{4}[(T_\lambda)_{ii} - 2\sum_{j \neq i}((S_\lambda)_{ji}(A_\lambda)_{ij} + (S_\lambda)_{ij}(A_\lambda)_{ji})].$$

  Then, $(A_\lambda)_{ii}$ is updated as

  $$(A_\lambda)_{ii} = \frac{a_i}{\sqrt{4N(S_\lambda)_{ii}}} + \frac{h_i}{(S_\lambda)_{ii}}.$$

- The other components $(A_\mu)_{ij}$ are updated likewise by extracting their dependence:

  $$\tilde{S} = 2Nc_{ij}|(A_\lambda)_{ij}|^2 - 2Nh_{ji}(A_\lambda)_{ij}, \text{ where}$$
  $$c_{ij} = (S_\lambda)_{ii} + (S_\lambda)_{jj},$$
  $$h_{ij} = \frac{1}{2}(T_\lambda)_{ij} - \sum_{k \neq i}(S_\lambda)_{ik}(A_\lambda)_{kj} - \sum_{k \neq j}(S_\lambda)_{kj}(A_\lambda)_{ik}.$$

  Then, $(A_\mu)_{ij}$ are updated as

  $$(A_\lambda)_{ij} = \frac{x_{ij} + iy_{ij}}{\sqrt{4Nh_{ij}}} + \frac{h_{ij}}{c_{ij}}.$$

The following Schwinger-Dyson equation serves as the consistency check of the algorithm.

$$-\langle \frac{1}{N} Tr[A_\mu, A_\nu]^2 \rangle = d(1 - \frac{1}{N^2}).$$

3 Matrix model with the Chern-Simons term

The generalizations of the IIB matrix model have been hitherto considered in order to accommodate the curved-space background.

The most typical alteration is the addition of the Chern-Simons term.

$$S = NTr\left(-\frac{1}{4}\sum_{\mu,\nu=1}^{3}[A_\mu, A_\nu]^2 + \frac{2i\alpha}{3}\sum_{\mu,\nu,\rho=1}^{3}\epsilon_{\mu\nu\rho}A_\mu A_\nu A_\rho\right),$$

defined in the three-dimensional spacetime.

Its equation of motion

$$[A_\mu, [A_\mu, A_\nu]] + i\alpha\epsilon_{\nu\rho\chi}[A_\rho, A_\chi] = 0$$

accommodates the S^2 fuzzy sphere classical solution

$$A_\mu = \alpha J_\mu, \text{ where } [J_\mu, J_\nu] = i\epsilon_{\mu\nu\rho}J_\rho.$$

The algorithm is similar, since the Chern-Simons term is **linear with respect to each A_1, A_2 and A_3.**

T. Azuma, S. Bal, K. Nagao and J. Nishimura, hep-th/0401038

We likewise introduce the auxiliary field $Q_{\mu\nu}$ as

$$\begin{aligned}\tilde{S} &= \sum_{1\leq\mu<\nu\leq 3}\left(\frac{N}{2}TrQ_{\mu\nu}^2 - NTr(Q_{\mu\nu}G_{\mu\nu}) + 2NTr(A_\mu^2 A_\nu^2)\right) \\ &+\frac{2i\alpha N}{3}\epsilon_{\mu\nu\rho}TrA_\mu A_\nu A_\rho.\end{aligned}$$

$Q_{\mu\nu}$ can be updated via the Gaussian variables:

In updating A_λ, its dependence is extracted as

$$\tilde{S} = -NTr(T_\lambda A_\lambda) + 2NTr(S_\lambda A_\lambda^2) + \cdots, \text{ where}$$

$$S_\lambda = \sum_{\mu\neq\lambda}(A_\mu^2),$$

$$T_\lambda = \sum_{\mu\neq\lambda}(A_\mu Q_{\lambda\mu} + Q_{\lambda\mu}A_\mu) \quad \underbrace{-2i\alpha\epsilon_{\lambda\mu\nu}A_\mu A_\nu}_{\text{The only difference!!}} \quad .$$

Other than the alteration of T_λ, the algorithm is **totally parallel to the bosonic IIB matrix model**!

Main results

- The **first-order phase transition**, as we vary the parameter α.

  Lower critical point $(A_\mu^{(0)} = \alpha J_\mu$ start$)$ $\alpha_{cr}^{(l)} \sim \frac{2.1}{\sqrt{N}}$,

  Higher critical point $(A_\mu^{(0)} = 0$ start$)$ $\alpha_{cr}^{(u)} \sim 0.66$.

  ⋆ **The Yang-Mills phase** $(\alpha < \alpha_{cr})$

  The fuzzy sphere is unstable.

  ⋆ **The fuzzy sphere phase** $(\alpha > \alpha_{cr})$

  The fuzzy sphere is stable.

- **One-loop dominance**:

  The higher-loop effect is suppressed for $N \to \infty$.

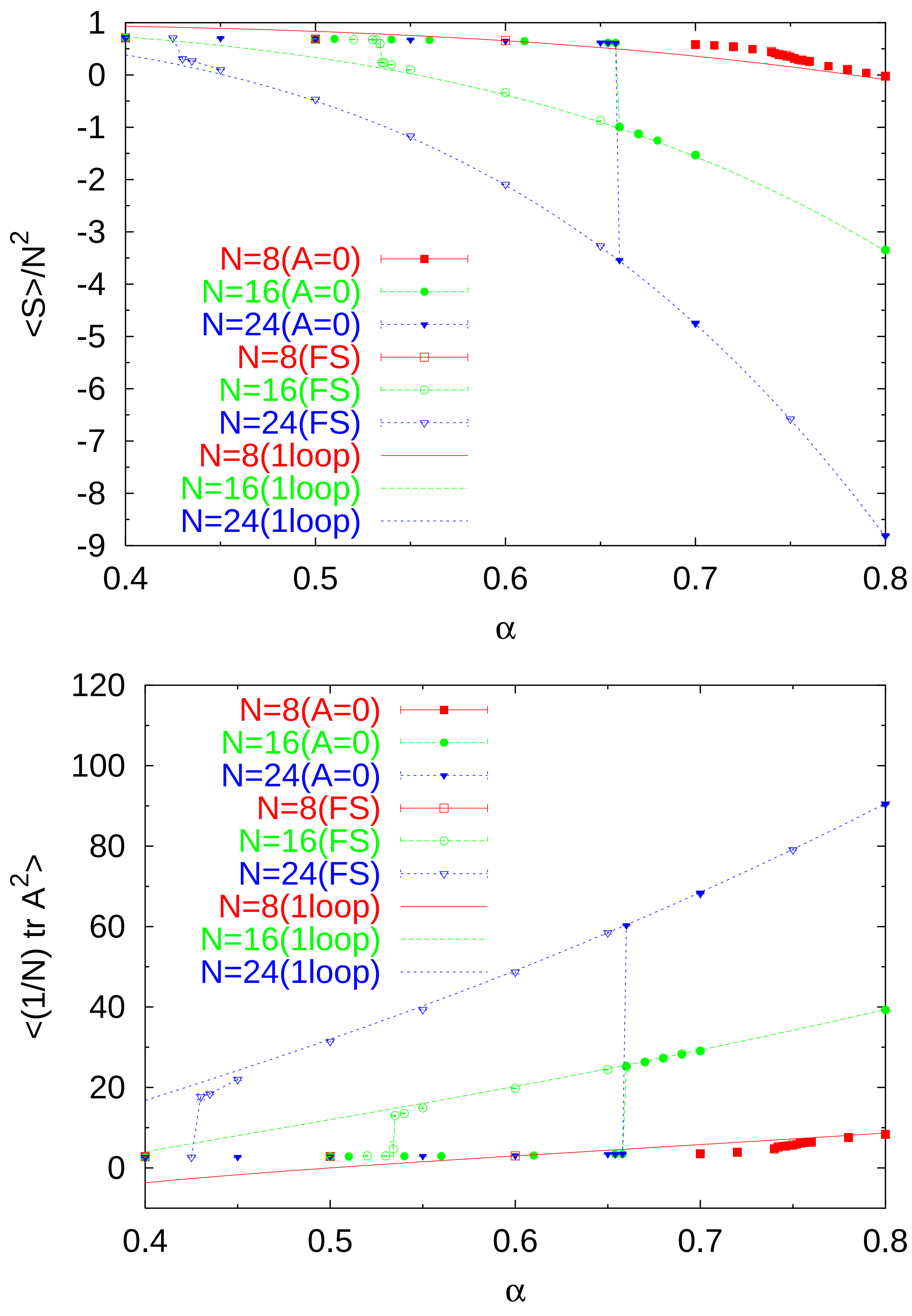


Figure 2: The hysteresis cycle of (Upper) $\langle \boldsymbol{S} \rangle / \boldsymbol{N^2}$ and (Lower) $\langle \frac{1}{N} \boldsymbol{Tr A_\mu^2} \rangle$.

4 Conclusion

In this talk, we have reviewed the basic technicality of the heat bath algorithm of the large-N reduced model.

The simulation of the IIB matrix model is much easier than the quantum field theory, since the IIB matrix model is the totally reduced model.

In hep-th/0401038, we investigated the matrix model with the Chern-Simons term, to deepen the understanding of the fuzzy-sphere background.

Miscellaneous future directions (of hep-th/0401038)

- Understanding of the dynamical generation of the gauge group.
- The numerical treatment of the supersymmetric case via the hybrid Monte Carlo simulation.
- Extension to the higher-dimensional case:
  S^{2k} fuzzy sphere, $S^2 \times S^2$ fuzzy sphere $\cdots$.