Finite temperature effective action, AdS$_5$ black holes, and $1/N$ expansion (hep-th/0502227)

Luis Alvarez-Gaume, Cesar Gomez, Hong Liu and Spenta Wadia

Journal Club at KEK, presented by Takehiro Azuma,[1] May. 15th 2006, 12:15 ∼ 13:15

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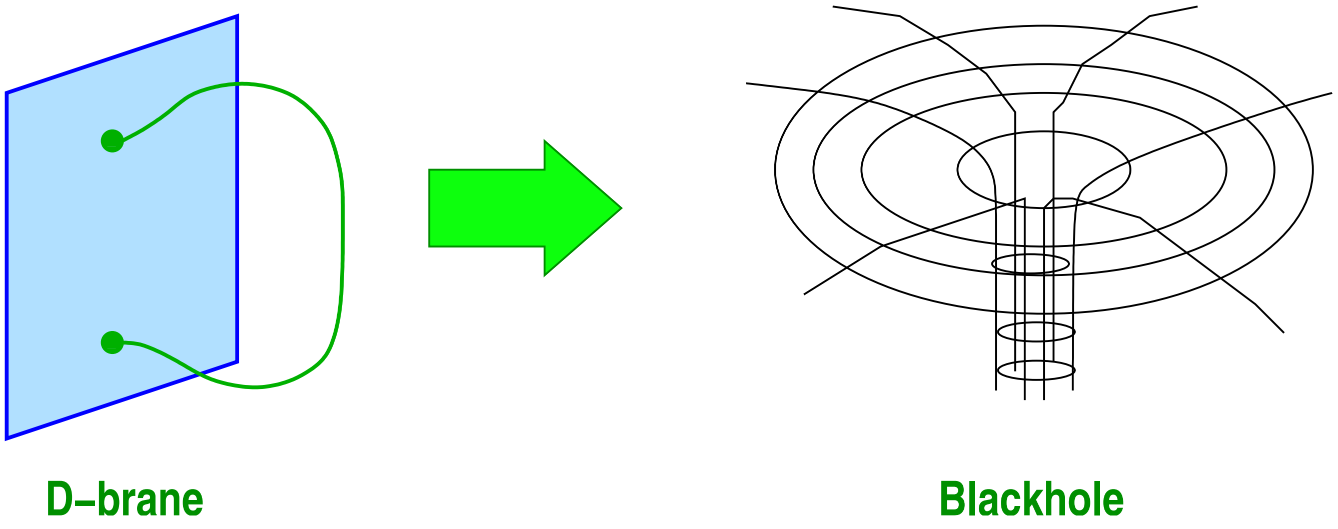
[1]This slide is used for Takehiro Azuma's presentation at KEK. Therefore, it is not the authors *but the presenter Takehiro Azuma* that is responsible for any potential flaws in this slide.

1 Introduction

AdS/CFT correspondence: J. M. Maldacena, hep-th/9711200

duality between type IIB superstring on AdS$_5 \times$ S^5 and $\mathcal{N} = 4$ 4-dim SYM.



We study the non-perturbative aspects of black hole physics in AdS$_5$ using the dual gauge theory at finite temperature.

2 Hawking-Page transition in Euclidean Quantum Gravity

Thermodynamic aspects of quantum gravity in AdS spacetime.

S. W. Hawking and D. N. Page, Commun. Math. Phys. 87, 577 (1983).

AdS spacetime allows two Schwarzschild blackhole solutions.

- **Small black hole (SBH): Horizon radius smaller than AdS.**
  Negative specific heat. Unstable.

- **Big black hole (BBH): horizon radius comparable to AdS. Positive specific heat.**

First-order phase transition between the AdS space and the BBH.

Hawking-Page transition corresponds to a large-N deconfinement transition in the gauge theory at strong coupling. E. Witten, hep-th/9803131.

Metric of the Euclidean AdS_5 and Schwarzschild space:

$$ds^2 = V dt^2 + V^{-1} dr^2 + r^2 d^2\Omega, \text{ where}$$
$$V = \begin{cases} 1 + \frac{r^2}{R^2}, & \text{(AdS metric)}, \\ 1 + \frac{r^2}{R^2} - \frac{w_4 M}{r^2}, & \text{(Schwarzschild (BBH))}. \end{cases}$$

- $R =$ (curvature radius of the AdS_5 space).
- $M =$ (mass of the blackhole)
- $w_4 = \frac{16\pi G_N}{3\text{vol}(S^3)}$, G_N=(3d Newton constant), vol(S^3)=(volume of unit S^3 sphere).

For the Schwarzschild solution, the radius must be $r > r_+$,

where $r_+ =$ [largest solution of $1 + \frac{r^2}{R^2} - \frac{w_4 M}{r^2} = 0$].

The Schwarzschild metric is smooth only if

$$\beta = (\text{period of } t) = \frac{2\pi R^2 r_+}{2r_+^2 + R^2}.$$

Maximum of the period : $\beta_{\max} = \frac{\pi R}{\sqrt{2}}$ at $r_+ = \frac{R}{\sqrt{2}}$.

Schwarzschild blackhole exists only for $T > T_0 = \frac{1}{\beta_{\max}} = \frac{\sqrt{2}}{\pi R}$.

Bulk Euclidean action:

$$I = -\frac{1}{16\pi G_N} \int d^5x \sqrt{g} \left(\mathcal{R} + \frac{6}{R^2} \right).$$

Substituting the classical solution $\mathcal{R} = -\frac{8}{R^2}$, we obtain $I = \frac{1}{2\pi G_N} \int d^5x \sqrt{g}$.

$$\begin{aligned} I_{\text{dif}} &= [\text{difference for (Schwarzschild (BBH))-(AdS}_5\text{ metric)}] \\ &= \frac{1}{2\pi G_N}(V_2 - V_1) = \frac{\text{vol}(S^3)}{8G_N} \frac{R^2 r_+^3 - r_+^5}{2r_+^2 + R^2}. \end{aligned}$$

V_1 (V_2) = [regularized volume of AdS_5 (Schwarzschild) space].

[Hawking-Page transition]

Critical point exists at $r_+ = R$, where $I_{\text{dif}} = 0 \Rightarrow \beta = \frac{2\pi R}{3}$, $T_1 = \frac{3}{2\pi R}$.

- **$T_0 (= \frac{\sqrt{2}}{\pi R}) < T < T_1$: saddle point for AdS dominates.**
- **$T_1 < T$: saddle point for Schwarzschild (BBH) dominates.**

Small black hole (SBH) $\Rightarrow$ instanton for tunneling between AdS to BBH.

3 Effective action at finite temperature

Phenomenological matrix model for string theory in AdS$_5 \times$ S^5 at finite temperature.

$\beta = \frac{1}{T}$: period of the Euclidean time.

Yang-Mills partition function on S^3 at temperature T

$\Rightarrow$ integral over a U(N) matrices U.

$$Z(\lambda, T) = \int dU e^{S_{\rm eff}(U)}, \text{ where } U = \mathrm{P} \exp\left(i \int_0^\beta A dt \right).$$

- $A(t)$: zero mode of the time component of the gauge field in S^3.
- Gauge invariance : U only contributes as $\mathrm{tr}\, U^n$.
- Z_N symmetry: $U \rightarrow U e^{2\pi i/N}$.
- Generic components of the action $S_{\rm eff}$:

$$\mathrm{tr}\, U^{n_1} \mathrm{tr}\, U^{n_2} \cdots \mathrm{tr}\, U^{n_k}, \text{ where } n_1 + n_2 + \cdots + n_k \equiv 0 (\mathrm{mod} N).$$

In general, we consider the action

$$S_{\rm eff} = S(x), \text{ where } x = \frac{1}{N^2} \mathrm{tr}\, U \mathrm{tr}\, U^\dagger, \quad S(x) \text{ is convex (凸)}, S'(x) \text{ is concave (凹)}.$$

For simplicity, we focus on the truncated action

$$Z(a, b) = \int dU \exp\left(a \mathrm{tr}\, U \mathrm{tr}\, U^\dagger + \frac{b}{N^2} (\mathrm{tr}\, U \mathrm{tr}\, U^\dagger)^2 \right), \quad (b > 0).$$

4 Large-N phase structure of the universality class

We introduce new parameters as

$$\begin{aligned} Z(a,b) &= \frac{N}{2\sqrt{\pi b}} \int_{-\infty}^{+\infty} d\mu \exp\left(-\frac{N^2(\mu-a)^2}{4b}\right) \exp(\mu \operatorname{tr} U \operatorname{tr} U^\dagger) \\ &= \frac{N}{2\sqrt{\pi b}} \int_{-\infty}^{+\infty} d\mu \exp\left(-\frac{N^2(\mu-a)^2}{4b}\right) \frac{N^2}{2\mu} \int_0^{+\infty} dg g \exp\left(-\frac{N^2 g^2}{4\mu} + N^2 F(g)\right), \end{aligned}$$

where $\exp(N^2 F(g)) = \int dU \exp(\frac{Ng}{2}(\operatorname{tr} U + \operatorname{tr} U^\dagger))$.

Large-N expansion of the function $F(g)$: D. J. Gross and E. Witten, Phys. Rev. D 21 (1980) 446.

$$F(g) = \begin{cases} \frac{g^2}{4} + \cdots, & (g \leq 1 \text{ or } g \text{ imaginary}) \\ g - \frac{1}{2}\log g - \frac{3}{4} + \cdots, & (g > 1). \end{cases}$$

Vacuum expectation value of the Polyakov loop:

$$\rho_1(g) = \frac{1}{N}\langle \operatorname{tr} U \rangle_g = \frac{1}{N}\langle \operatorname{tr} U^\dagger \rangle_g = \frac{\partial F}{\partial g} = \begin{cases} \frac{g}{2} + \cdots, & (g \leq 1 \text{ or } g \text{ imaginary}) \\ 1 - \frac{1}{2g} + \cdots, & (g > 1). \end{cases}$$

Third-order phase transition at $g = 1$ in the large-N limit.

($F(g), F'(g), F''(g)$ are continuous, but $F'''(g)$ is discontinuous, at $g = 1$).

$$
\begin{aligned}
Z(a,b) &= \frac{N^3}{4\sqrt{\pi b}}\int_{-\infty}^{+\infty}\frac{d\mu}{\mu}\int_0^{+\infty} dgg\exp(-N^2V(\mu,g)) = \frac{N}{2\sqrt{\pi b}}\int_{-\infty}^{+\infty} d\mu\exp(-N^2Q(\mu)), \text{ where} \\
V(\mu,g) &= \begin{cases} (1) \ \frac{(\mu-a)^2}{4b} - \frac{g^2(1-\mu)}{4\mu}, & \mu<0, \\ (2) \ \frac{(\mu-a)^2}{4b} + \frac{g^2(1-\mu)}{4\mu}, & \mu>0, \ 0\le g<1, \\ (3) \ \frac{(\mu-a)^2}{4b} + \frac{g^2}{4\mu} - g + \frac{1}{2}\log g + \frac{3}{4}, & \mu>0, \ g>1. \end{cases} \\
Q(\mu) &= \frac{(\mu-a)^2}{4b} - \mathcal{F}(\mu), \\
\mathcal{F}(\mu) &= \begin{cases} 0 - \frac{1}{N^2}\log(1-\mu)+\cdots, & \mu<1, \\ \frac{w}{2(1-w)} + \frac{1}{2}\log(1-w) + \mathrm{O}(1/N^2), & \mu>1, \end{cases} \quad w = \sqrt{1-\frac{1}{\mu}}.
\end{aligned}
$$

$\mathcal{F}(\mu)$ has a first-order discontinuity at $\mu=1$.

($\mathcal{F}(\mu)$ is continuous, but $\mathcal{F}'(\mu)$ is discontinuous, at $\mu=1$).

Large-N expansion for the general model $S(x)$

For general action $S_{\text{eff}} = S(x)$ (S is convex (凸), $S'(x)$ is concave (凹))

$$
Z = \int dU \int d\mu \exp(N^2(\mu x - \mathcal{S}(\mu))) \overset{\text{large}-\text{N}}{\simeq} \int d\mu \exp(-N^2 Q(\mu)).
$$

$\mathcal{S}(\mu) = \max_x(\mu x - S(x))$: Legendre transformation of $S(x)$, $\quad Q(\mu) = \mathcal{S}(\mu) - \mathcal{F}(\mu)$.

Phase structure of the model at large N

1. From (1)(2) of $V(\mu, g)$ (namely, for $\mu < 0$ or $\mu > 0, 0 \leq g < 1$),

$$(\mu_1, g_1) = (a, 0), \Rightarrow V(\mu_1, g_1) = 0, \quad V'' = \begin{pmatrix} V_{\mu\mu} & V_{\mu g} \\ V_{g\mu} & V_{gg} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \frac{1}{b} & 0 \\ 0 & \frac{1-a}{a} \end{pmatrix}, \quad \rho_1(\mu_1, g_1) = 0.$$

- $a < 1 \Rightarrow \det V'' > 0$, this is a local minimum.
- $a > 1 \Rightarrow \det V'' < 0$, this is tachyonic.

2. From (2) of $V(\mu, g)$: for $a < 1$ and $c = 2(1-a)/b < 1$ (between line III and H),

$$(\mu_2, g_2) = (1, \sqrt{c}), \Rightarrow V(\mu_2, g_2) = \frac{(1-a)^2}{4b}, \quad V'' = \frac{1}{2}\begin{pmatrix} \frac{1}{b} + c & -\sqrt{c} \\ \sqrt{c} & 0 \end{pmatrix}. \quad \rho_1(\mu_2, g_2) = \frac{\sqrt{c}}{2} < \frac{1}{2}.$$

3. From (3) of $V(\mu, g)$: the critical point satisfies

$$\frac{\partial V}{\partial \mu} = \frac{\mu - a}{2b} - \frac{g^2}{4\mu^2} = 0, \quad \frac{\partial V}{\partial g} = \frac{g}{2\mu} - 1 + \frac{1}{2g} = 0.$$

$\frac{\partial V}{\partial g} = 0 \Rightarrow g = \frac{1}{1-w}, \rho_1 = \frac{1+w}{2} > \frac{1}{2}$, $(w = \sqrt{1 - 1/\mu})$.

For $g = \frac{1}{1-w}$ $(w = \sqrt{1 - 1/\mu})$, μ satisfies

$$\frac{\partial V}{\partial \mu} = Q'(\mu) = \frac{\mu - a}{2b} - \frac{(w+1)^2}{4} = 0 \leftarrow (\star).$$

- Below curve I $(Q'(\mu) = Q''(\mu) = 0)$:
  $(a, b) = \left(\frac{1-2w}{(1-w)^2(1+w)}, \frac{2w}{(1-w)^2(1+w)^3}\right)$
  $(\star)$ has no solutions.
  $V(\mu, g)$ has a unique minimum (μ_1, g_1).

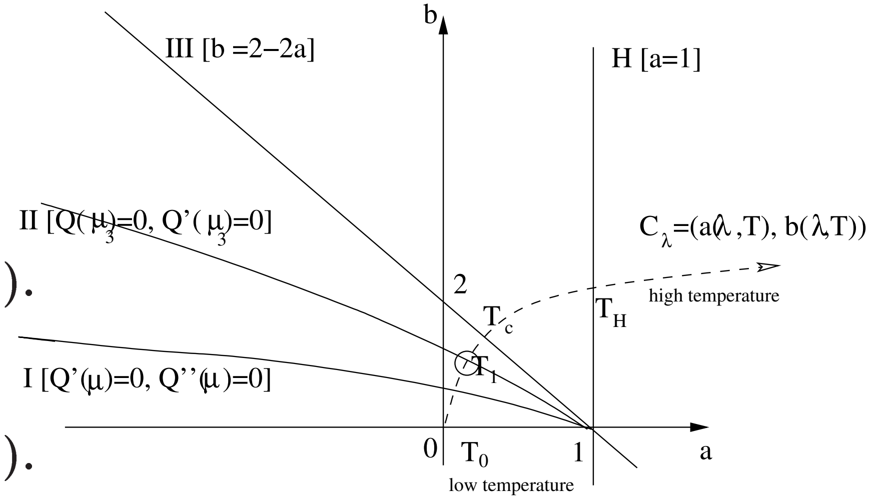
- Between I and line III $(b = 2 - 2a)$:
  $(\star)$ has two solutions $1 < \mu_2 < \mu_3$.
  $V(\mu, g)$ has three extrema μ_1, μ_2, μ_3.
  (μ_1, μ_3: minima, μ_2: saddle point.)

  * Between I and curve II
  $(Q(\mu_3) = Q'(\mu_3) = 0)$,
  $0 = V(\mu_1, g_1) < V(\mu_3, g_3) < V(\mu_2, g_2)$.

  * Between II and III,
  $V(\mu_3, g_3) < 0 = V(\mu_1, g_1) < V(\mu_2, g_2)$.

- Between III and line H $(a = 1)$:
  $V(\mu, g)$ has three extrema
  $\mu_1 (= a) < \mu_2 (= 1) < \mu_3$.
  Below (above) III, $\rho_1(\mu_2) > \frac{1}{2}$ $(< \frac{1}{2})$.
  μ_2 undergoes a Gross-Witten phase transition at large N.

- Right of H $(a > 1)$:
  μ_1 is tachyonic. μ_2 disappears.
  Only μ_3 remains stable.



Critical point for general model and universality class

For the general model $S_{\rm eff} = S(x)$ ($S(x)$ is convex (凸), and $S'(x)$ is concave (凹))

$$
\begin{aligned}
Q(\mu) &= \mathcal{S}(\mu) - \mathcal{F}(\mu), \quad (\mathcal{S}(\mu) \text{ is convex (凸)}), \\
\mathcal{F}(\mu) &= \begin{cases} 0 - \frac{1}{N^2}\log(1-\mu) + \cdots, & \mu < 1, \\ \frac{w}{2(1-w)} + \frac{1}{2}\log(1-w) + O(1/N^2), & \mu > 1, \end{cases} \quad w = \sqrt{1-\frac{1}{\mu}}.
\end{aligned}
$$

- $\mu < 1$: $Q(\mu)(= \mathcal{S}(\mu))$ has only one critical point.
- $\mu > 1$: The critical point satisfies $\mathcal{S}'(\mu) = \frac{(1+w)^2}{4}$.

  Critical points similar to the truncated model.

Universality of the phase structure.

Phenomenological matrix model and blackhole

Relation between the phenomenological matrix model and the AdS$_5$ string theory.

- $\mu_1 = a$: $\rho_1 \simeq 0$ (at large N). Corresponds to thermal AdS.
- μ_2: Negative specific heat $c_v(\mu_c) = -N^2\beta^2 \frac{d^2}{d\beta^2} Q(\beta, \mu_c(\beta))|_{\mu_c=\mu_2} < 0$.
  Corresponds to small black hole (SBH).
- μ_3: Positive specific heat $c_v(\mu_c = \mu_3) > 0$. Corresponds to big black hole (BBH).

($\star$) Varying temperature T, with 't Hooft coupling $\lambda \to C_\lambda =$ (curve of $(a(\lambda, T), b(\lambda, T))$).

- Start from $T = T_0$ (low temperature).
- At $T = T_1$ (where C_λ intersects curve II),
  Exchange of the dominance of μ_1 (AdS) and μ_3 (BBH).
  Hawking-Page transition in the matrix model.
- At $T = T_c$ (where C_λ intersects line III),
  Gross-Witten phase transition of the SBH.
  Horowitz-Polchinski correspondence point (horizon size of SBH is comparable to the string scale)
  G. T. Horowitz and J. Polchinski, hep-th/9612146.
- At $T = T_H$ (where C_λ intersects line H),
  Corresponds to Hagedorn temperature. μ_3 (BBH) is the only stable phase.

5 Small Black Hole and Tunneling

Small black hole (SBH) $\Rightarrow$

Instanton for tunneling between AdS and big black hole (BBH).

($\star$) Tunneling rate for AdS $\to$ BBH.

C.G. Callan and S.R. Coleman, Phys. Rev. D 16, 1762 (1977).

Integral via the contour C_1:

$$Z_1 \simeq e^{-N^2 Q(\mu_1)} K_1 + \frac{i}{2} e^{-N^2 Q(\mu_2)} K_2 + O(N^{-2}).$$

($K_{1,2,3}$: Gaussian factor in saddle point approximation at $\mu = \mu_{1,2,3}$).

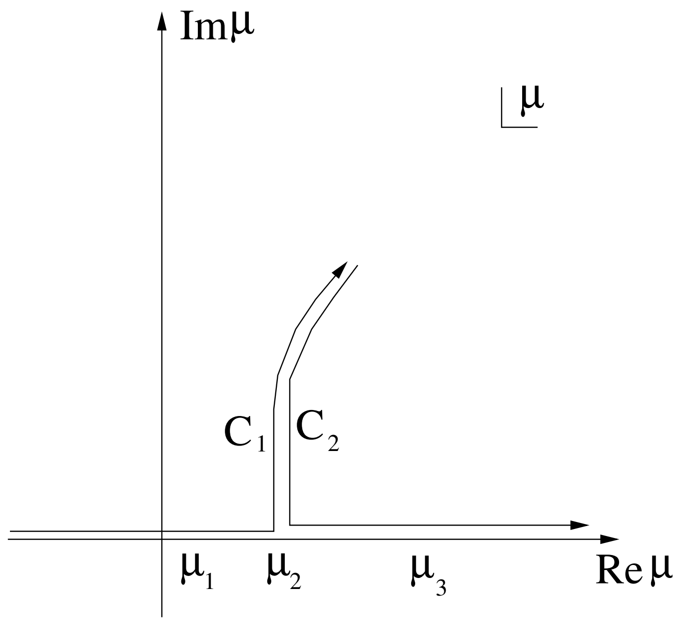
$$\Gamma_1 \simeq \frac{\omega_0 \beta}{\pi} \mathrm{Im}(-\frac{1}{\beta} \log Z_1)$$
$$= \frac{\omega_0}{2\pi} \exp[-N^2(Q(\mu_2) - Q(\mu_1))] \frac{K_2}{K_1} (1 + O(N^{-2})).$$

(ω_0: frequency for the unstable mode around SBH background).

($\star$) Tunneling rate for BBH $\to$ AdS.

Integral via the contour C_2:

$$\Gamma_2 = \frac{\omega_0}{2\pi} \exp[-N^2(Q(\mu_2) - Q(\mu_3))] \frac{K_2}{K_3} (1 + O(N^{-2})).$$



Instanton effect for SBH:

$O(e^{-1/g_s^2})$ for $g_s =$ (string coupling) $= \frac{1}{N}$.

Corresponds to a collective state of N D-instantons.

Gross-Witten transition for small black hole

Behavior of small black hole (SBH) near line III ($b = 2 - 2a$) ($0 < \epsilon \ll 1$).

- $c = 2(1-a)/b = 1+\epsilon$: $\mu_2 = 1 + \frac{\epsilon^2}{4} + \cdots$, $g_2 = 1 + \frac{\epsilon}{2} + \cdots$, $\rho_1(g_2) = \frac{1}{2} + \frac{\epsilon}{4} + \cdots$.
- $c = 2(1-a)/b = 1-\epsilon$: $\mu_2 = 1$, $g_2 = 1 - \frac{\epsilon}{2} + \cdots$, $\rho_1(g_2) = \frac{1}{2} - \frac{\epsilon}{4} + \cdots$.

Smoothening of the discontinuity at $g = 1$ at finite N : Rescale g as $g = 1 - N^{-2/3}y$.

H. Liu, hep-th/0408001, V. Periwal and D. Shevitz, Nucl. Phys. B 344, 731 (1990).

$$V(\mu, g) = \frac{(\mu - a)^2}{4b} + \frac{g_2}{4}\frac{1-\mu}{\mu} - \sum_{n=0}^{\infty} N^{-2n/3} F_n(y),$$

$$F_0(y) = \begin{cases} \frac{y^3}{6} - \frac{1}{8}\log(-y) + \cdots, & -y \gg 1, \\ \frac{1}{2\pi}\exp(-\frac{4\sqrt{2}}{3}y^{3/2})\left(-\frac{1}{8\sqrt{2}y^{3/2}} + \cdots\right), & y \gg 1. \end{cases}$$

This interpolates between $g \le 1$ and $g > 1$.

Physics of the small black hole (SBH) near $c = 1 (T = T_c)$:

$$a(T) = a_0 + a_1 \epsilon q, \;\; b(T) = b_0 + b_1 \epsilon q, \;\; \mu = 1 + \epsilon^2 x, \;\; g = 1 - \epsilon y, \;\; \epsilon = N^{-2/3}, \text{ where}$$
$$a_0 = a(T_c), \;\; b_0 = b(T_c), \;\; a_1 = T_c a'(T_c), \;\; b_1 = T_c b'(T_c), \;\; \epsilon q = \frac{T - T_c}{T_c}, \;\; \frac{2(1-a_0)}{b_0} = 1.$$
$$\Downarrow$$
$$c = 1 - c_1 \epsilon q, \;\; \text{where } c_1 = \frac{1}{b_0}(2a_1 + b_1).$$

The potential is written as

$$N^2 V = \frac{N^2 (1-a)^2}{4b} + \frac{x}{2}\left(y - \frac{c_1 q}{2}\right) - F_0(y) + \mathrm{O}(\epsilon).$$

Partition function around SBH:

$$Z_{\mathrm{SBH}} = iN \sqrt{\frac{\pi}{b}} \exp\left(- \frac{N^2(1-a)^2}{4b} + F_0(c_1 q/2) \right) (1 + \mathrm{O}(\epsilon)).$$

$F_0(t) \to$ full partition function of the type 0B theory in 0 dimension.

I. R. Klebanov, J. M. Maldacena and N. Seiberg, hep-th/0309168

SBH is described by type 0B theory in 0 dimension.

6 Conclusion

Phenomenological matrix model to study the string theory in $AdS_5 \times S^5$ space.

- **The matrix model reproduces the Hawking-Page phase transition.**
- **Gross-Witten-type third-order phase transition of the small black hole (SBH).**

Further development **L. Alvarez-Gaume, P. Basu, M. Marino and S. R. Wadia,** hep-th/0605041

- **Blackhole string transition of the 10-dimensional small black hole.**
- **A key to resolve the information puzzle in blackhole physics?**