Exact fuzzy sphere thermodynamics in matrix quantum mechanics

(arXiv:0704.3183)

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Tea-duality Seminar at Tata Institute of Fundamental Research (TIFR),

Takehiro Azuma,[1] May. 4th 2007, 16:00 ∼ 17:00

Contents

[1] This slide is used for Takehiro Azuma's presentation in the Tea-duality Seminar at TIFR. It is not the authors but *the speaker Takehiro Azuma* that is responsible for any flaw in this slide.

1 Introduction

Large-N reduced models $\Rightarrow$ promising candidates for the constructive definition of superstring theory.

The IIB matrix model

N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, hep-th/9612115

$$\boxed{S = -\frac{1}{g^2}\mathrm{tr}\left(\frac{1}{4}[A_\mu, A_\nu]^2 + \frac{1}{2}\bar{\psi}\Gamma^\mu[A_\mu, \psi]\right).}$$

Relation with the type IIB superstring theory:

- **Matrix regularization of the Green-Schwarz action of type IIB superstring theory.**
- **D-brane interaction.**
- **Derivation of the string field theory.**

Matrix models on a homogeneous space

Motivations of fuzzy manifold studies:

- **Relation between the non-commutative field theory and the superstring.**
- **Novel regularization scheme alternative to lattice regularization.**
- **Prototype of the curved-space background in the large-N reduced models.**

Fuzzy spheres are compact, and thus realized by finite matrices.

The Chern-Simons term is added to accommodate the classical solution of the fuzzy manifolds.

2 Review of the (0+0)-dimensional matrix model

3d Yang-Mills-Chern-Simons (YMCS) model

$\Rightarrow$ **a toy model with fuzzy sphere solutions:**

S. Iso, Y. Kimura, K. Tanaka and K. Wakatsuki, hep-th/0101102.

$$\boxed{S = N\mathrm{tr}\left(-\frac{1}{4}[A_\mu, A_\nu]^2 + \frac{2i\alpha}{3}\epsilon_{ijk}A_iA_jA_k\right).}$$

- **Defined in the D-dimensional Euclidean space ($D = 3$)**
  $(\mu, \nu, \rho = 1, 2, \cdots, D, \quad i, j, k = 1, 2, 3)$.
- **Classical equation of motion:** $[A_j, [A_i, A_j]] - i\alpha\epsilon_{ijk}[A_j, A_k] = 0$.
- **fuzzy S^2 classical solutions:** $A_i = Y_i = \oplus_{I=1}^{s}(\alpha L_i^{(n_I)} \otimes 1_{k_I})$,
  (where $[L_i^{(n_I)}, L_j^{(n_I)}] = i\epsilon_{ijk}L_k^{(n_I)}, \quad \Sigma_{I=1}^{s} n_I k_I = N$).
  $L_i^{(n_I)} =$ ($n_I \times n_I$ **representation of the SU(2) Lie algebra).**

First-order phase transition

Monte Carlo simulation launched from single fuzzy sphere classical solution:

$$A_i = \alpha L_i^{(N)} \ (s = 1, n_1 = N, k_1 = 1).$$

T. Azuma, S. Bal, K. Nagao and J. Nishimura, hep-th/0401038.

Critical point at $\alpha_{cr} \simeq \frac{2.1}{\sqrt{N}}$.

- $\alpha < \alpha_{cr}$: Yang-Mills phase

  Strong quantum effects.

  behavior like the $\alpha = 0$ case.

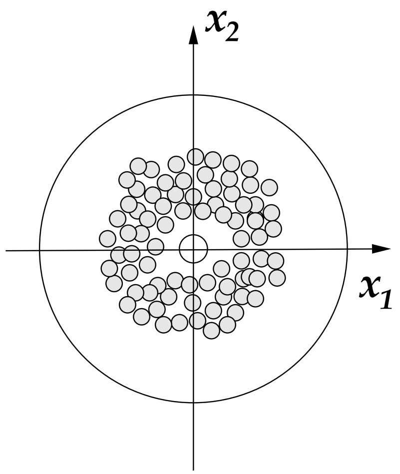
  T. Hotta, J. Nishimura and A. Tsuchiya, hep-th/9811220,

$$\langle \frac{S}{N^2} \rangle \simeq \mathrm{O}(1), \ \langle \frac{1}{N} \mathrm{tr}\, A_i^2 \rangle \simeq \mathrm{O}(1).$$

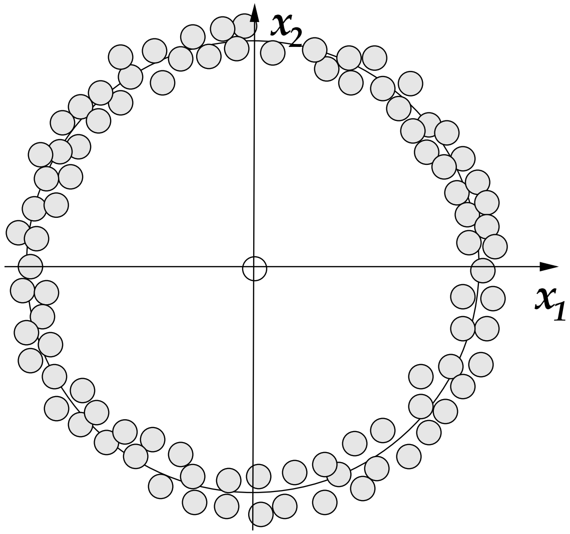
- $\alpha > \alpha_{cr}$: fuzzy sphere phase.

  Fuzzy sphere configuration is stable.

Yang–Mills phase



Fuzzy sphere phase



Calculation of the observables

Expansion around the single fuzzy sphere $B_i = \kappa L_i^{(N)}$: $A_i = B_i + \tilde{A}_i$.

Gauge fixing term and ghost term:

$$S_{\text{total}} = S + S_{\text{g.f.}} + S_{\text{total}} = S_0 + S_2 + S_3 + \cdots, \text{ where}$$
$$S_{\text{g.f.}} = -\frac{N}{2}\text{tr}\,[B_i, A_i]^2 \ , \quad S_{\text{ghost}} = -N\text{tr}\,\left([B_i, \bar{c}][A_i, c]\right) \ ,$$
$$S_0 = \frac{1}{4}N^2(N^2-1)\left(\frac{\kappa^4}{2} - \frac{2\alpha\kappa^3}{3}\right),$$
$$S_2 = \frac{N}{2}\text{tr}\,(\tilde{A}_i \mathcal{L}_j^2 \tilde{A}_i) + N\text{tr}\,(\bar{c}\mathcal{L}_j^2 c), \text{ where } \mathcal{L}_j Z = [B_j, Z]$$

One-loop effective action:

$$\Gamma(\kappa) = \underbrace{\Gamma^{(0)}(\kappa)}_{\text{tree}} + \underbrace{\Gamma^{(1)}(\kappa)}_{\text{one-loop}} + \cdots, \text{ where } \Gamma^{(0)}(\kappa) = S_0,$$
$$\Gamma^{(1)}(\kappa) = \frac{D-2}{2}\text{Tr}\log(N\mathcal{L}_j^2) = \frac{D-2}{2}\sum_{l=1}^{N-1}(2l+1)\log[N\kappa^2 l(l+1)].$$

Free energy at one-loop level:

$$\begin{aligned} W(\beta_1, \beta_2, \alpha) &= -\log \int dA \exp(-S(\beta_1, \beta_2, \alpha)) \\ &= \frac{3}{4}(N^2 - 1) \log \beta_1 + W(1, 1, \alpha\beta_1^{-\frac{3}{4}}\beta_2), \text{ where} \\ S(\beta_1, \beta_2, \alpha) &= N\text{tr} \left(-\frac{\beta_1}{4}[A_i, A_j]^2 + \frac{2i\alpha\beta_2}{3}\epsilon_{ijk}A_iA_jA_k \right). \end{aligned}$$

At one-loop level, $W_{\text{one-loop}}(1, 1, \alpha)$ is equal to free energy $\Gamma(\alpha)_{\text{one-loop}}$.

Observables obtained from the derivative of the free energy (where $\bar{\alpha} = \alpha\sqrt{N}$)..

$$\begin{aligned} \frac{1}{\sqrt{N}}\langle M \rangle &= \frac{1}{\sqrt{N}}\langle \frac{2i}{3N}\text{tr}\, \epsilon_{ijk}A_iA_jA_k \rangle = \frac{1}{\alpha N^{\frac{5}{2}}}\frac{\partial W}{\partial \beta_2}|_{\beta_1=\beta_2=1} \simeq -\frac{\bar{\alpha}^3}{6} + \frac{1}{\bar{\alpha}}, \\ \langle \frac{1}{N}\text{tr}\, F_{ij}^2 \rangle &= \langle \frac{-1}{N}\text{tr}\,[A_i, A_j]^2 \rangle = \frac{4}{N^2}\frac{\partial W}{\partial \beta_1}|_{\beta_1=\beta_2=1} \simeq \frac{\bar{\alpha}^4}{2}. \end{aligned}$$

Space-time extent is calculated by evaluating the tadpole.

$$\frac{1}{N}\langle \frac{1}{N}\text{tr}\, A_i^2 \rangle = \frac{1}{N^2}\left(\text{tr}\,(\alpha L_i^{(N)})^2 + 2\text{tr}\,((\alpha L_i^{(N)})\langle \tilde{A}_i \rangle) + \langle \text{tr}\, \tilde{A}_i^2 \rangle\right) \simeq \frac{\bar{\alpha}^2}{4} - \frac{1}{\bar{\alpha}^2}.$$

Phase transition from the one-loop effective action

The effective action Γ is saturated at the one-loop level at large N.

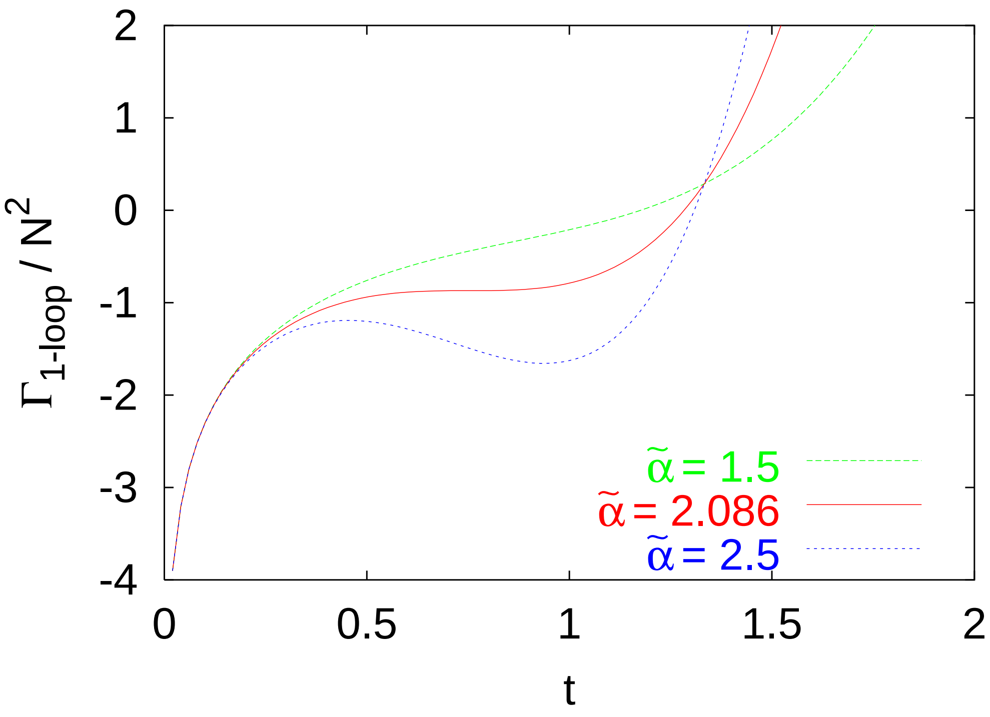
T. Imai, Y. Kitazawa, Y. Takayama and D. Tomino, hep-th/0307007.

Effective action at one-loop around $A_i = t\alpha L_i$.

$$\frac{\Gamma_{1-\mathrm{loop}}}{N^2} \simeq \bar{\alpha}^4 \left(\frac{t^4}{8} - \frac{t^3}{6} \right) + \log t.$$

The local minimum disappears at $\bar{\alpha} < \bar{\alpha}_{\mathrm{cr}} = \frac{4}{3}(6(D-2))^{\frac{1}{4}} \simeq 2.086\cdots$ (for $D = 3$).

Consistent with the Monte Carlo simulation.



All order calculation from one-loop effective action

The effective action Γ is saturated at one loop at large N

Y. Kitazawa, Y. Takayama and D. Tomino, hep-th/0403242

The free energy W can be obtained by the extremum of the effective action.

Expansion around $A_i = B_i$: ($\bar{\kappa} = \kappa\sqrt{N}$)

$$\lim_{N\to\infty} \frac{1}{N^2}\Gamma(\bar{\kappa}) = \left(\frac{\bar{\kappa}^4}{8} - \frac{1}{6}\bar{\alpha}\bar{\kappa}^3\right) + \log \bar{\kappa}.$$

Local minimum for $\bar{\alpha} > \bar{\alpha}_{\rm cr} = \sqrt[4]{\frac{512}{27}}$:

$$\begin{aligned}
\bar{\kappa} &= f(\bar{\alpha}) = \frac{\bar{\alpha}}{4}\left(1 + \sqrt{1+\delta} + \sqrt{2 - \delta + \frac{2}{\sqrt{1+\delta}}}\right) \\
&= \bar{\alpha}\left(1 - \frac{2}{\bar{\alpha}^4} - \frac{12}{\bar{\alpha}^8} - \frac{120}{\bar{\alpha}^{12}} - \frac{1456}{\bar{\alpha}^{16}} - \cdots\right), \text{ where} \\
\delta &= 4\bar{\alpha}^{-\frac{4}{3}}\left[\left(1 + \sqrt{1 - \frac{512}{27\bar{\alpha}^4}}\right)^{\frac{1}{3}} + \left(1 - \sqrt{1 - \frac{512}{27\bar{\alpha}^4}}\right)^{\frac{1}{3}}\right].
\end{aligned}$$

Free energy and observables:

$$\begin{aligned}
\lim_{N\to\infty} \frac{1}{N^2} W &= \left(\frac{1}{8}f(\bar{\alpha})^4 - \frac{1}{6}\bar{\alpha} f(\bar{\alpha})^3\right) + \log f(\bar{\alpha}) \\
&= -\frac{\bar{\alpha}^4}{24} + \log \bar{\alpha} - \frac{1}{\bar{\alpha}^4} - \frac{14}{3\bar{\alpha}^8} - \frac{110}{3\bar{\alpha}^{12}} - \frac{364}{\bar{\alpha}^{16}} - \cdots,
\end{aligned}$$

All order calculation of generic observables $\mathcal{O}$

Consider the action $S_\epsilon = S + \epsilon \mathcal{O}$.

Corresponding free energy:

$$\begin{aligned} W_\epsilon &= -\log\left(\int d\tilde{A} e^{-(S+\epsilon\mathcal{O})}\right) = -\log\left(\int d\tilde{A} e^{-S}\right) + \epsilon \frac{\int d\tilde{A}\mathcal{O}e^{-S}}{\int d\tilde{A} e^{-S}} + \mathrm{O}(\epsilon^2) \\ &= W + \epsilon\langle\mathcal{O}\rangle + \mathrm{O}(\epsilon^2). \end{aligned}$$

One-loop effective action (take only 1PI diagrams into account)

$$\Gamma_\epsilon(\bar{\kappa}) = \Gamma(\bar{\kappa}) + \epsilon\Gamma_1(\bar{\kappa}) + \mathrm{O}(\epsilon^2).$$

Its saddle point:

$$\frac{\partial}{\partial\bar{\kappa}}\Gamma_\epsilon(\bar{\kappa}) = 0, \quad \Rightarrow \bar{\kappa} = f(\bar{\alpha}) + \epsilon g(\bar{\alpha}) + \mathrm{O}(\epsilon^2).$$

Plugging this solution, we obtain the free energy as

$$W_\epsilon = \Gamma_\epsilon(f(\bar{\alpha}) + \epsilon g(\bar{\alpha}) + \cdots) = \Gamma(f(\bar{\alpha})) + \epsilon\left(\Gamma_1(f(\bar{\alpha})) + g(\bar{\alpha})\underbrace{\left(\frac{\partial\Gamma}{\partial\bar{\kappa}}\right)\Big|_{\bar{\kappa}=f(\bar{\alpha})}}_{=0}\right) + \mathrm{O}(\epsilon^2).$$

We thus obtain $\langle\mathcal{O}\rangle = \Gamma_1(f(\bar{\alpha}))$.

All order calculation of the spacetime content:

$$\lim_{N\to\infty}\frac{1}{N}\langle\frac{1}{N}\mathrm{tr}\,A_i^2\rangle = \frac{\bar{\alpha}^2}{4} \underbrace{-\frac{1}{\bar{\alpha}^2}}_{\text{one-loop}} .$$

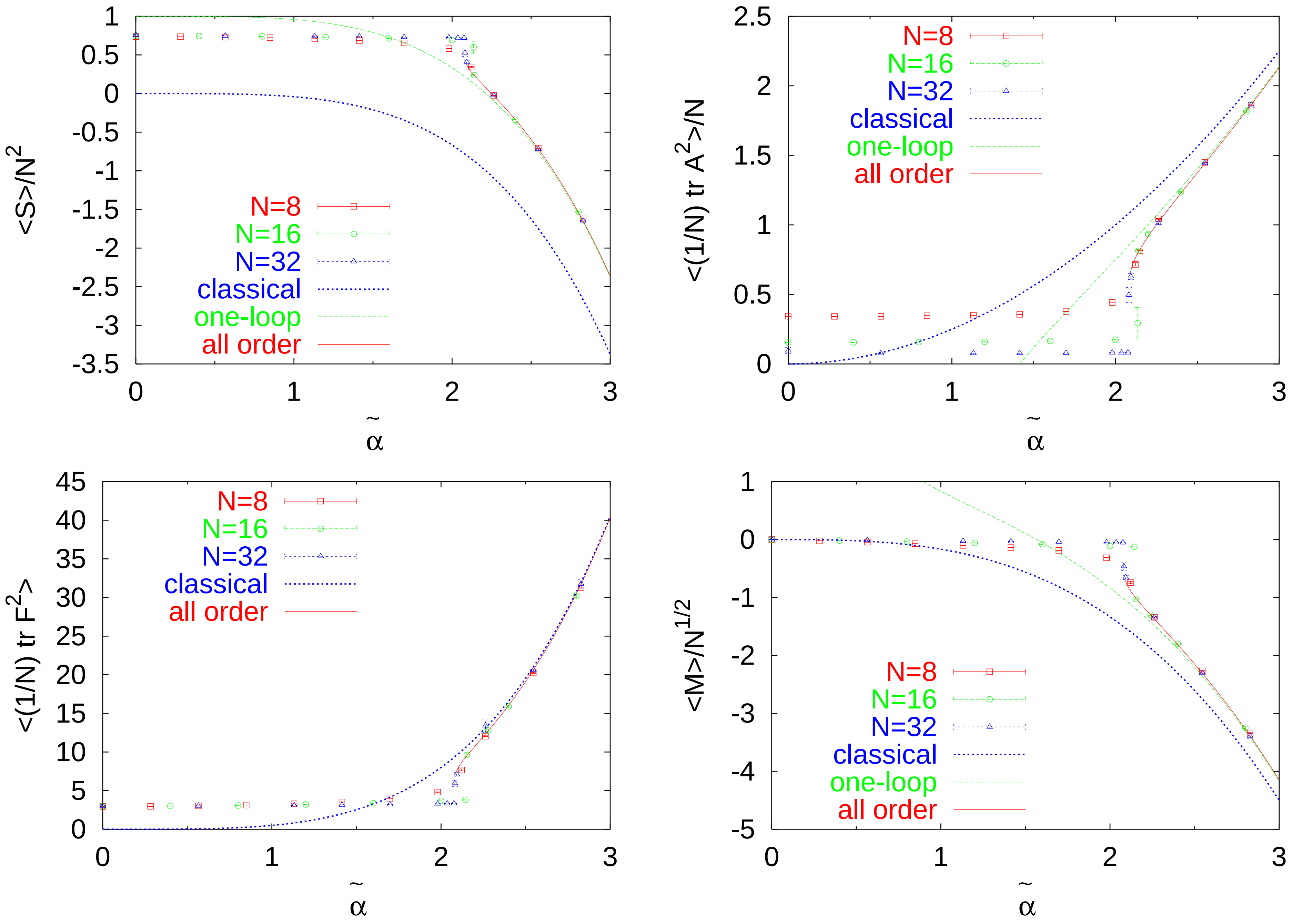
The one-loop effect comes from tadpole diagrams.

$$\lim_{N\to\infty}\frac{1}{N}\langle\frac{1}{N}\mathrm{tr}\,A_i^2\rangle = \frac{1}{4}f(\bar{\alpha})^2 = \frac{1}{4}\bar{\alpha}^2 - \frac{1}{\bar{\alpha}^2} - \frac{5}{\bar{\alpha}^6} - \frac{48}{\bar{\alpha}^{10}} - \frac{572}{\bar{\alpha}^{14}} - \cdots ,$$

$$\lim_{N\to\infty}\frac{1}{\sqrt{N}}\langle M\rangle = -\frac{1}{6}f(\bar{\alpha})^3 = -\frac{1}{6}\bar{\alpha}^3 + \frac{1}{\bar{\alpha}} + \frac{4}{\bar{\alpha}^5} + \frac{112}{3\bar{\alpha}^9} + \frac{440}{\bar{\alpha}^{13}} + \cdots ,$$

$\langle\frac{1}{N}\mathrm{tr}\,F_{ij}^2\rangle$ **is derived from the Schwinger-Dyson equation (SDE).**

$$\lim_{N\to\infty}\left\langle\frac{1}{N}\mathrm{tr}\,(F_{ij})^2\right\rangle = \underbrace{3 - 3\alpha\langle M\rangle}_{\text{SDE}} = 3 + \frac{1}{2}\bar{\alpha}f(\bar{\alpha})^3 = \frac{1}{2}\bar{\alpha}^4 - \frac{12}{\bar{\alpha}^4} - \frac{112}{\bar{\alpha}^8} - \frac{1320}{\bar{\alpha}^{12}} - \cdots .$$



<S>/N^2
N=8
N=16
N=32
classical
one-loop
all order
α̃
<(1/N) tr A^2>/N
N=8
N=16
N=32
classical
one-loop
all order
α̃
<(1/N) tr F^2>
N=8
N=16
N=32
classical
all order
α̃
<M>/N^1/2
N=8
N=16
N=32
classical
one-loop
all order
α̃

3 Finite-temperature (0+1)-dimensional matrix model

N. Kawahara, J. Nishimura and S. Takeuchi, arXiv:0704.3183.[2]

$$\boxed{S = N\int_0^\beta dt \mathrm{tr}\left(\frac{1}{2}(D_t X_i(t))^2 - \frac{1}{4}[X_i(t), X_j(t)]^2 + \frac{2i\alpha}{3}\epsilon_{ijk} X_i(t) X_j(t) X_k(t)\right).}$$

- **Covariant derivative :** $D_t X_i(t) = \partial_t X_i(t) - i[A(t), X_i(t)]$.
- **1-dim. gauge symmetry :** $X_i(t) \to g(t) X_i(t) g^\dagger(t)$, $A(t) \to g(t) A(t) g^\dagger(t) + i g(t) \frac{dg^\dagger(t)}{dt}$.
- $\beta = 1/T$=(period). **Periodicity** $X_i(t+\beta) = X_i(t)$, $A(t+\beta) = A(t)$.
- **Equation of motion:**

$$(D_t)^2 X_i(t) = [X_j(t), [X_j(t), X_i(t)]] + i\alpha\epsilon_{ijk}[X_j(t), X_k(t)], \quad [X_i(t), D_t X_i(t)] = 0.$$

Fuzzy S^2 sphere solution:

$$X_i(t) = \oplus_{I=1}^{s}(\alpha L_i^{(n_I)} \otimes 1_{k_I}), \quad A(t) = \oplus_{I=1}^{s}(1_{n_I} \otimes \bar{A}^{(I)}).$$

[2]The figures in this section are quoted from arXiv:0704.3183.

Perturbative calculation around the fuzzy sphere

Expansion around the single fuzzy sphere $B_i = \kappa L_i^{(N)}$:

$$X_i(t) = B_i + \tilde{X}_i(t), \quad A(t) = 0 + \tilde{A}(t).$$

Gauge fixing term and ghost term:

$$S_{\text{total}} = S_{\text{g.f.}} + S_{\text{gh}} = S_0 + S_2 + S_3 + \cdots, \text{ where}$$
$$S_{\text{g.f.}} = \frac{N}{2}\int_0^\beta dt \text{tr}\,(\partial_t A(t) - i[B_i, \tilde{X}_i(t)])^2,$$
$$S_{\text{gh}} = N\int_0^\beta dt \text{tr}\,(\partial_t \bar{c}(t) D_t c(t) - [B_i, \bar{c}(t)][X_i(t), \bar{c}(t)]),$$
$$S_0 = \frac{\beta}{4} N^2(N^2-1)\left(\frac{\kappa^4}{2} - \frac{2\alpha\kappa^3}{3}\right),$$
$$S_2 = N\int_0^\beta dt \text{tr}\left(\frac{1}{2}\tilde{X}_i(t)\mathcal{P}\tilde{X}_i(t) + \frac{1}{2}\tilde{A}(t)\mathcal{P}\tilde{A}(t) + \bar{c}(t)\mathcal{P}c(t)\right),$$
$$(\mathcal{P} = -\partial_t^2 + \kappa^2 \mathcal{L}_i^2, \quad \mathcal{L}_i Z = [B_i, Z]).$$

One-loop effective action:

$$\Gamma(\kappa) = \underbrace{\Gamma^{(0)}(\kappa)}_{\text{tree}} + \underbrace{\Gamma^{(1)}(\kappa)}_{\text{one-loop}} + \cdots, \text{ where } \Gamma^{(0)}(\kappa) = S_0,$$
$$\Gamma_1(\kappa) = \log\det\mathcal{P} = 2\sum_{l=1}^{N-1}(2l+1)\log\left(\sinh\left(\frac{\beta\kappa}{2}\sqrt{l(l+1)}\right)\right).$$

Effective action at large N:

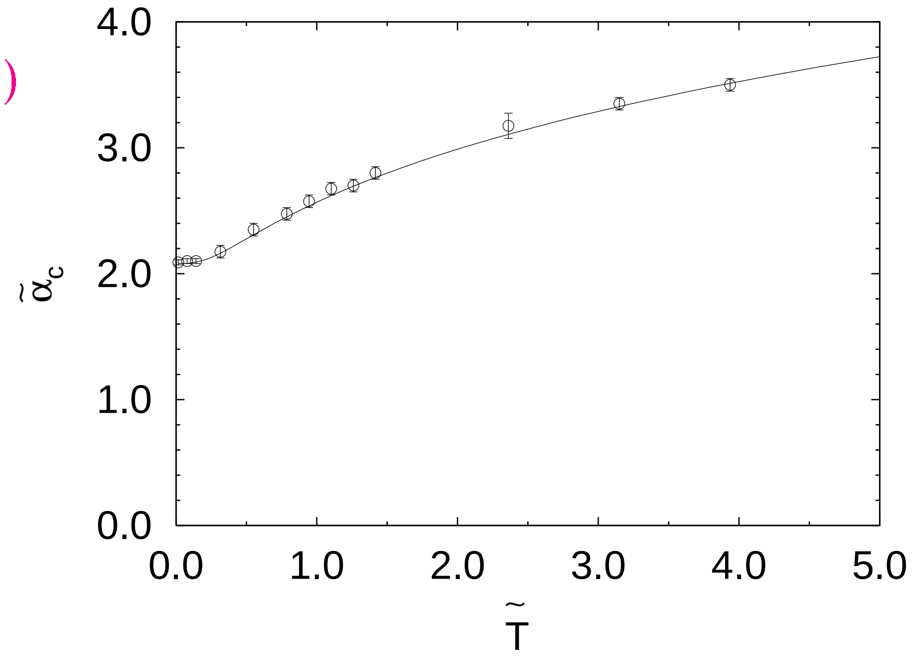
$$\frac{\Gamma(\kappa)}{N^2} \overset{N\to\infty}{\simeq} \frac{\tilde{\beta}}{4}\left(\frac{\tilde{\kappa}^4}{2} - \frac{2}{3}\tilde{\alpha}\tilde{\kappa}^3\right) + \Phi(\tilde{\beta}\tilde{\kappa})$$
$$= f(\tilde{\kappa};\tilde{\alpha},\tilde{\beta}), \text{ where}$$
$$\tilde{\alpha} = N^{\frac{1}{3}}\alpha, \ \tilde{\beta} = N^{\frac{2}{3}}\beta, \ \tilde{\kappa} = N^{\frac{1}{3}}\kappa,$$
$$\Phi(x) = \lim_{N\to\infty}\frac{2}{N}\int_0^N d\xi 2\xi \log(\sinh(\frac{x\xi}{2N}))$$
$$= \frac{x}{3} - 2\log(1-e^x) + 2\log(\sinh\frac{x}{2})$$
$$- \frac{4}{x}\mathrm{Li}_2(e^x) + \frac{4}{x^2}\mathrm{Li}_3(e^x) - \frac{4}{x^2}\zeta(3).$$

Local minimum κ_0

$\Rightarrow$ **Solution of $\frac{\partial}{\partial\kappa}f(\tilde{\kappa};\tilde{\alpha},\tilde{\beta}) = 0$.**

Local minimum disappears at $\tilde{\alpha} < \tilde{\alpha}_c$, where ($\tilde{T} = 1/\tilde{\beta}$)

$$\tilde{\alpha}_c = \begin{cases} 9^{\frac{1}{3}} \simeq 2.08 & (\tilde{T} = 0), \\ \left(\frac{1024\tilde{T}}{27}\right)^{\frac{1}{4}} \simeq 2.48\tilde{T}^{\frac{1}{4}} & (\tilde{T} \gg 1). \end{cases}$$



Calculation of the observables

Large-N limit of the free energy around the single fuzzy sphere classical solution $X_i(t) = \alpha L_i^{(N)}$.

$$W(\alpha,\beta) = -\log\left(\int dX dA e^{-S}\right) \Rightarrow \lim_{N\to\infty} \frac{W_{\text{one-loop}}(\alpha,\beta)}{N^2} = -\frac{1}{24}\tilde{\alpha}^4\tilde{\beta} + \Phi(\tilde{\beta}\tilde{\alpha}).$$

Observables obtained from the derivative of the free energy.

$$\begin{aligned}
\frac{1}{N}\langle M\rangle &= \frac{1}{N}\langle \frac{2i}{3N\beta}\int_0^\beta dt \epsilon_{ijk}\text{tr}\,(X_i(t)X_j(t)X_k(t))\rangle = \frac{1}{N^3\beta}\frac{\partial}{\partial\alpha}W(\alpha,\beta) \\
&= -\frac{1}{6}\tilde{\alpha}^3 + \Phi'(\tilde{\beta}\tilde{\alpha}), \\
\frac{1}{N^{\frac{2}{3}}}\langle F^2\rangle &= \frac{1}{N^{\frac{2}{3}}}\langle \frac{-1}{N\beta}\int_0^\beta dt \text{tr}[X_i(t),X_j(t)]^2\rangle = \frac{4}{N^{\frac{8}{3}}\beta}\left[-\frac{5\alpha}{6}\frac{\partial}{\partial\alpha}W(\alpha,\beta) + \frac{\beta}{3}\frac{\partial}{\partial\beta}W(\alpha,\beta)\right] \\
&= \frac{\tilde{\alpha}^4}{2} - 2\tilde{\alpha}\Phi'(\tilde{\beta}\tilde{\alpha}).
\end{aligned}$$

Space-time extent is calculated by evaluating the tadpole.

$$\begin{aligned}
\frac{1}{N^{\frac{4}{3}}}\langle R^2\rangle &= \frac{1}{N^{\frac{4}{3}}}\langle \frac{1}{N}\int_0^\beta dt \text{tr}\, X_i^2(t)\rangle = \frac{1}{N^{\frac{7}{3}}\beta}\int_0^\beta dt\left\{\text{tr}\, B_i^2 + 2\text{tr}\, B_i\langle X_i(t)\rangle + \langle \text{tr}\, X_i^2(t)\rangle\right\} \\
&= \frac{\tilde{\alpha}^2}{4} - \frac{1}{\tilde{\alpha}}\Phi'(\tilde{\beta}\tilde{\alpha}).
\end{aligned}$$

All order calculation of the observables

Free energy W $\to$ extremum of the effective action $\Gamma(\kappa)$.

$$\lim_{N\to\infty} \frac{W_{\text{all-order}}(\alpha,\beta)}{N^2} = f(\tilde{\kappa}_0;\tilde{\alpha},\tilde{\beta}), \text{ where } \tilde{\kappa}_0 = (\text{solution of } \frac{\partial}{\partial\tilde{\kappa}} f(\tilde{\kappa};\tilde{\alpha},\tilde{\beta}) = 0).$$

One-loop contribution of M and R^2 comes from 1PR (one particle reducible) diagrams.

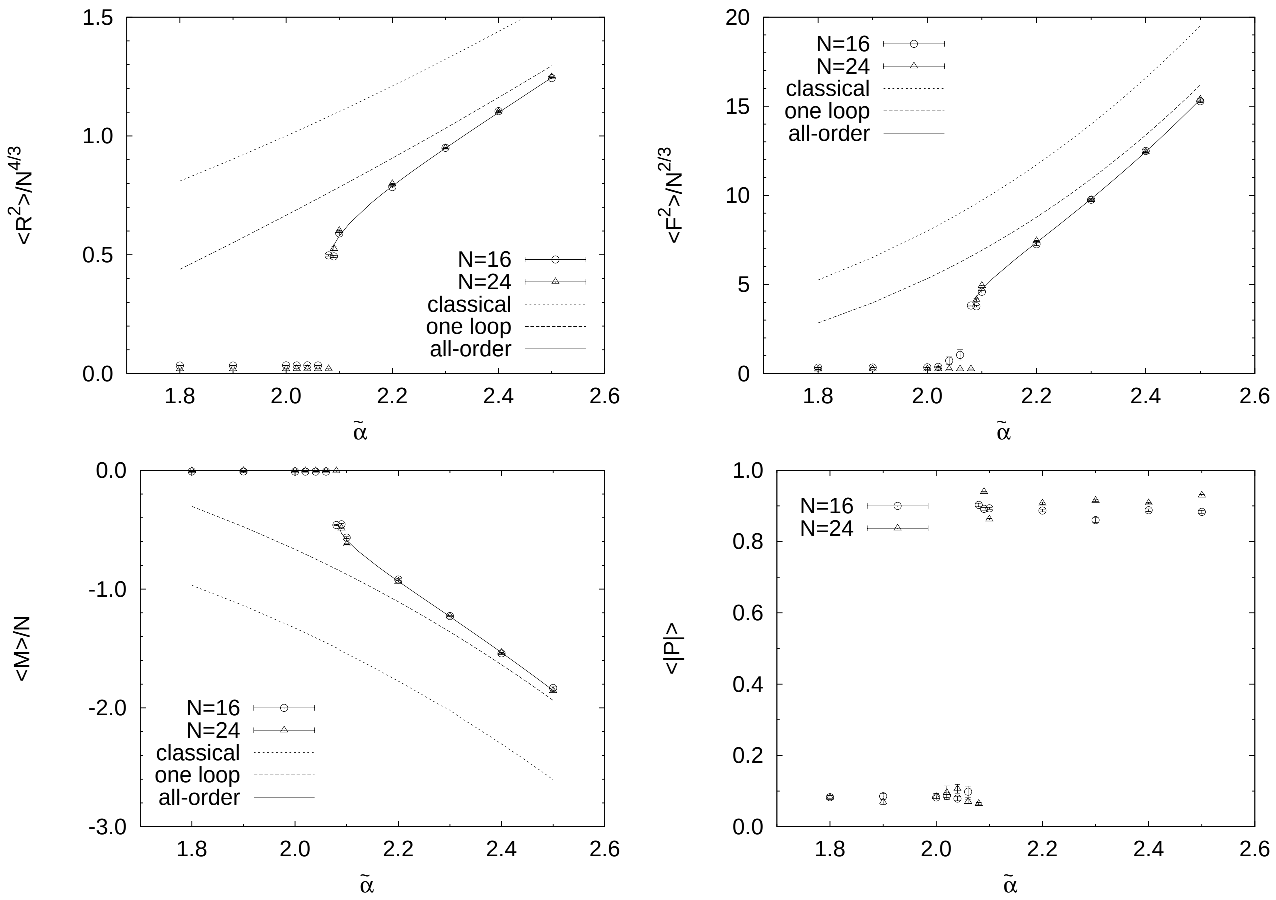
$$\frac{1}{N^{\frac{4}{3}}}\langle R^2\rangle_{\text{all-order}} = \frac{1}{4}(\tilde{\kappa}_0)^2, \quad \frac{1}{N}\langle M\rangle_{\text{all-order}} = -\frac{1}{6}(\tilde{\kappa}_0)^3.$$

The observable $\langle F^2\rangle$ comes from the derivative of the free energy.

$$\lim_{N\to\infty} \frac{\langle F^2\rangle_{\text{all-order}}}{N^{\frac{2}{3}}} = \frac{4}{\tilde{\beta}}\left(-\frac{5}{6}\tilde{\alpha}\frac{d}{d\tilde{\alpha}} + \frac{1}{3}\tilde{\beta}\frac{d}{d\tilde{\beta}}\right) f(\tilde{\kappa}_0;\tilde{\alpha},\tilde{\beta}).$$

Comparison with Monte Carlo simulations

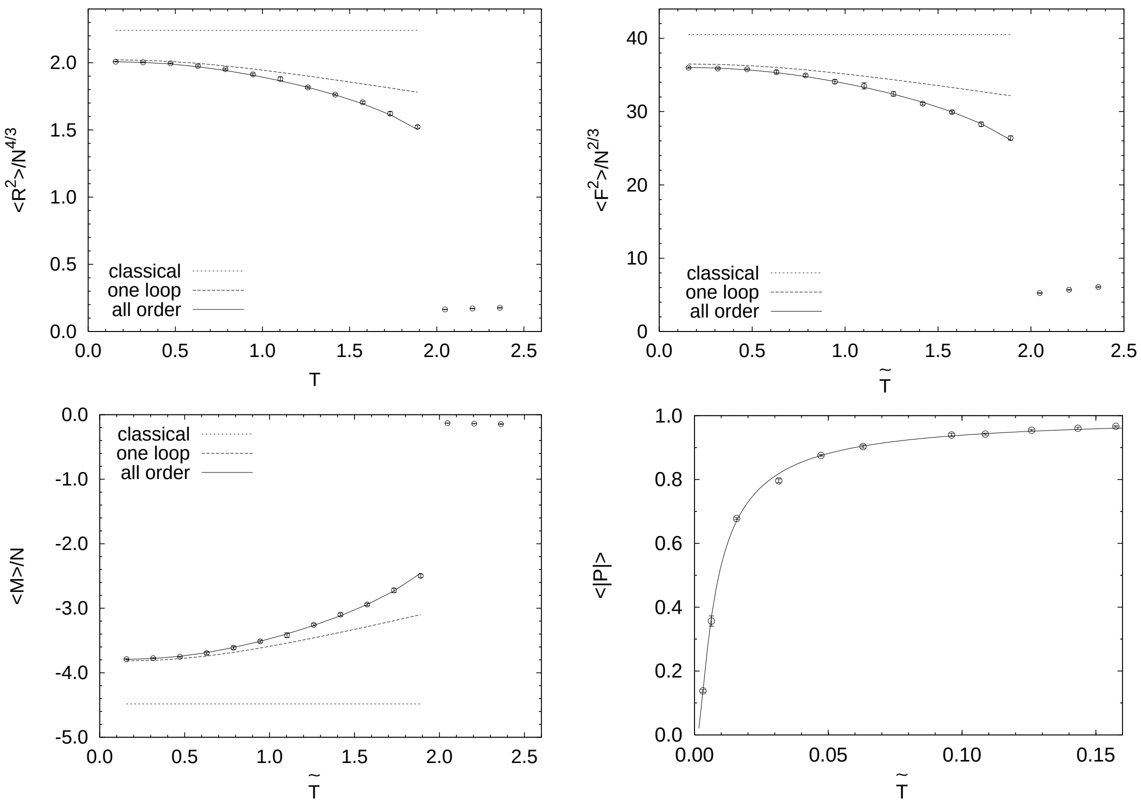
Simulation for low temperature $N = 16, 24, \tilde{T} = 0.1$, where $P = \frac{1}{N}\mathrm{tr}\,\mathcal{P}\exp\langle(i\int_0^\beta dt\,A(t)\rangle$.



Discontinuity at $\tilde{\alpha} \simeq 2.1$.

Temperature dependence of the observables in fuzzy sphere phase

Simulation for $N = 16, \tilde{\alpha} = 3.0 (> \tilde{\alpha}_{\rm cr} \simeq 2.1)$.

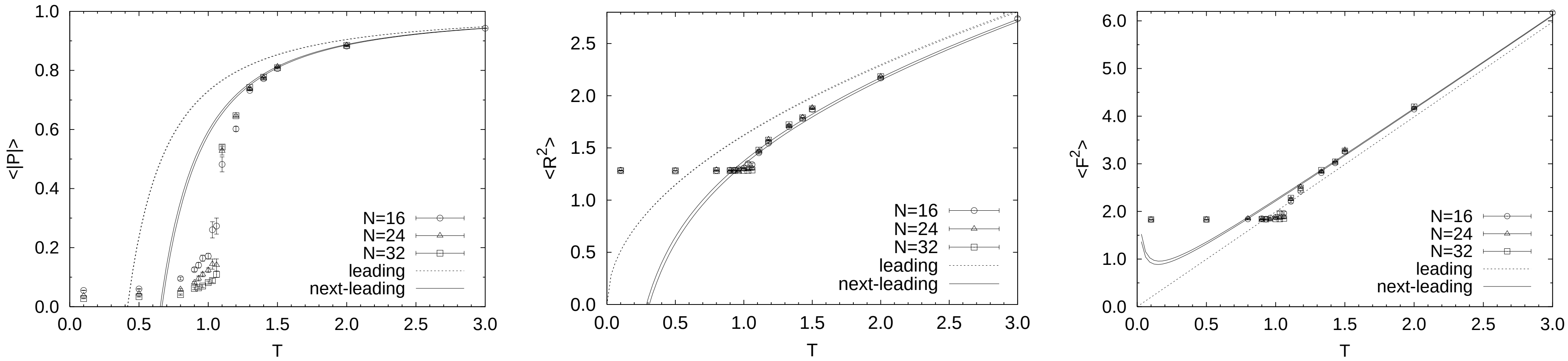


Discontinuity at $\tilde{T} = 2.0$.

$\langle |P| \rangle = \exp(-c/\tilde{T})$ $\Rightarrow$ **deconfined phase.**

Hagedorn transition in Yang-Mills phase

Monte Carlo simulation at $\tilde{\alpha} = 0$.



Discontinuity at $T = T_H$(Hagedorn temperature) $\simeq 1.1$.

- **Center symmetry is broken at $T > T_H$.**
- **$T < T_H \rightarrow$ confined phase. Observables are independent of T.**
  Eguchi-Kawai equivalence
  ($U(1)^D$ symmetry is unbroken $\rightarrow$ single-trace operators are volume independent).

Dimensionally reduced model at high temperature

Dimension of time is reduced at high temperature (small periodicity $\beta = 1/T$).

Integral $\int_0^\beta \to$ Multiplication of $\beta = \frac{1}{T}$ at $T \gg 1$.

$$\begin{aligned} S_{\rm DR} &= \frac{N}{T}\left\{-\frac{1}{2}[A, X_i]^2 - \frac{1}{4}[X_i, X_j]^2 + \frac{2i\alpha}{3}\epsilon_{ijk}X_iX_jX_k\right\} \\ &= N\mathrm{tr}\left(-\frac{1}{4}[A_\mu, A_\nu]^2 + \frac{2i\gamma}{3}\epsilon_{ijk}A_iA_jA_k\right). \end{aligned}$$

- Defined in the D-dimensional Euclidean space ($D = 4$) ($\mu, \nu, \rho = 1, 2, 3, 4, \quad i, j, k = 1, 2, 3$).
- $A_i = T^{-\frac{1}{4}}X_i$, $A_4 = T^{-\frac{1}{4}}A$, $\gamma = T^{-\frac{1}{4}}\alpha$.

Observables at high temperature.

$$\langle R^2\rangle \simeq T^{\frac{1}{2}}\langle\frac{1}{N}\mathrm{tr}\, A_i^2\rangle_{\mathrm{DR},\gamma}, \qquad \langle M\rangle \simeq T^{\frac{3}{4}}\langle\frac{2i}{3N}\epsilon_{ijk}\mathrm{tr}\, A_iA_jA_k\rangle_{\mathrm{DR},\gamma},$$

$$\langle F^2\rangle \simeq -T\langle\mathrm{tr}\,[A_i, A_j]^2\rangle_{\mathrm{DR},\gamma}, \quad \langle|P|\rangle \simeq \langle\frac{1}{N}\mathrm{tr}\,\exp\left(T^{-\frac{3}{4}}A_4\right)\rangle_{\mathrm{DR},\gamma}.$$

Observables at Yang-Mills phase

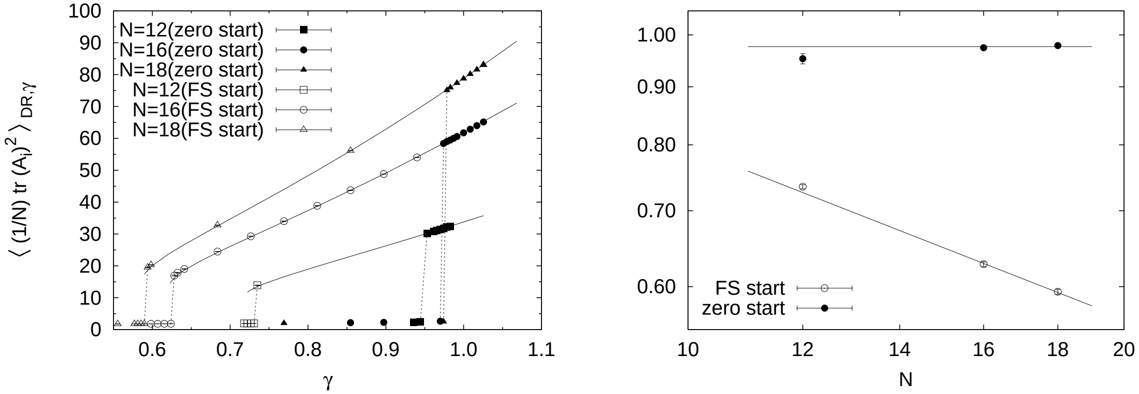
Simulation for $\alpha = 0$ at high T:

$$\lim_{N\to\infty} \langle R^2 \rangle \simeq 1.62\cdots\sqrt{T}, \quad \langle F^2 \rangle \simeq 2T\left(1 - \frac{1}{N^2}\right).$$

Observables at fuzzy sphere phase

Simulation from different initial configuration:

$$A_4 = 0, \quad A_i = \begin{cases} \gamma L_i^{(N)} & \text{(single fuzzy sphere start)}, \\ 0 & \text{(zero start)} \end{cases}$$



Discontinuity at $\alpha = \begin{cases} \alpha_c^{(l)} \simeq \frac{4T^{\frac{1}{4}}}{3\sqrt{N}}(6(D-2))^{\frac{1}{4}} \simeq \frac{2.5}{\sqrt{N}}T^{\frac{1}{4}} & \text{(single fuzzy sphere start)}, \\ \alpha_c^{(u)} \simeq 0.98T^{\frac{1}{4}}, & \text{(zero start)}. \end{cases}$

N dependence is the same as $D = 3$ case.

4 Conclusion

Numerical simulation and all order calculation of the (0+1)-dimensional (BFSS-type) matrix model.

- **All order calculation of the observables' VEV at large N.**
- **Phase transitions for varying α and temperature T.**