Matrix Configurations for Spherical 4-branes and Non-Commutative Structures on S^4

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Journal Club at KEK presented by Takehiro Azuma[1]

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Contents

[1]This slide is used for Takehiro Azuma's presentation at KEK. Therefore, it is not the authors but *the presenter Takehiro Azuma* that is responsible for any potential flaws in this slide.

1 Introduction

The studies of the fuzzy sphere are interesting from the following points of view:

Matrix-model side

The fuzzy spheres play an important role as the prototype of the curved-space background.

The IIB matrix model is the most promising candidate of the constructive definition of the superstring theory.

On the other hand, this has only a classical background, which makes the perturbation around the curved space impossible.

In order to evade this difficulty, the matrix models with the Chern-Simons term have been extensively studied.

hep-th/0101102,0103192,0108002, 0204256,0207115,0209057,0301055,0303120,0307007, 0309264,0312241,0401038, 0403242,0405096

String-theory side

The fuzzy spheres appear as the classical solutions in the presence of an external RR field (Myers effect):

R.Myers, hep-th/9910053

The interests of the higher-dimensional fuzzy S^4 spheres:

- **Longitudinal 5-branes in the Matrix theory:**
  J.Castelino, S.Lee and W.Taylor, hep-th/9712105
- **Four-dimensional noncommutative theory in the large-N limit of the large-N reduced model:**
  Y.Kimura, hep-th/0204256,
  T. Azuma, S. Bal, K.Nagao and J. Nishimura, hep-th/0405096
- **Application to the quantum Hall effect.**
  K.Hasebe and Y. Kimura, hep-th/0310274

The purpose of this paper:

- **The authors attempt to build the algebra of the fuzzy S^4 spheres, retaining the $SO(3) \otimes SO(2)$ symmetry.**
- **They build the associative star product for the fuzzy S^4 spheres.**

2 The algebra on the fuzzy S^4 sphere

To retain the $SO(5)$ symmetry of the fuzzy S^4 sphere, one has to introduce the extra degrees of freedom.

The algebra is realized by the 6-dimensional homogeneous space $SO(5)/U(2)$.

Namely, the fuzzy S^2 sphere is attached on every point of the S^4 sphere.

$$\begin{aligned} SO(5)/U(2) &\sim (SO(5)/SO(4)) \times (SO(4)/U(2)) \\ &\sim \underbrace{(SO(5)/SO(4))}_{S^4 \text{ sphere}} \times \underbrace{(SO(3)/U(1))}_{S^2 \text{ sphere}}. \end{aligned}$$

Such an algebra is constructed through the n-fold symmetric tensor product of the 5-dimensional gamma matrices:

$$G_A = \underbrace{(\Gamma_A \otimes 1 \otimes \cdots \otimes 1)_{\text{sym}}}_{n\text{-fold product}} + \cdots + (1 \otimes \cdots \otimes 1 \otimes \Gamma_A)_{\text{sym}},$$

where Γ_A satisfies the following Clifford algebra:

$$\{\Gamma_A, \Gamma_B\} = 2\delta_{AB}, \ (A, B, \cdots = 1, 2, 3, 4, 5).$$

More explicitly, they are constructed as

$$\Gamma_a = \sigma_3 \otimes \sigma_a, \ \Gamma_4 = \sigma_2 \otimes 1_{2\times 2}, \ \Gamma_5 = \sigma_1 \otimes 1_{2\times 2}.$$

- G_A is realized by the $N = \frac{1}{6}(n+1)(n+2)(n+3)$-dimensional matrices:
  Unlike the S^2 fuzzy sphere, the S^4 fuzzy sphere is realized by a limited size of the matrices.

- G_A does not close with respect to the commutator.

  G_A and the commutator $G_{AB} = [G_A, G_B]$ satisfy the following algebras:

  (1) $G_A^2 = n(n+4)1_{N\times N}$,
  (2) $G_{AB}G_{AB} = -16n(n+4)1_{N\times N}$,
  (3) $[G_{AB}, G_C] = 4(-\delta_{AC}G_B + \delta_{BC}G_A)$,
  (4) $[G_{AB}, G_{CD}] = 4(\delta_{BC}G_{AD} + \delta_{AD}G_{BC} - \delta_{AC}G_{BD} - \delta_{BD}G_{AC})$,
  (5) $\epsilon_{ABCDE}G_BG_CG_DG_E = 8(n+2)G_A$.
  $\Rightarrow G_{AB} = \frac{1}{n+2}\epsilon_{ABCDE}G_CG_DG_E$.

Here, we break the $SO(5)$ rotational symmetry as

$$SO(5) \to SO(3) \otimes SO(2),$$

and try to build the $N = 2(2j+1)$-dimensional representation.

Namely, we take the following tensor product:

(Pauli matrices) $\otimes$ (spin-j rep. of $SO(3)$).

Then, the representation is constructed as

$$\boxed{\begin{array}{l}
\hat{X}^a = \frac{2}{3}\sigma_3 \otimes T^a_{(j)}, \ \hat{X}^4 = \frac{1}{3}\sigma_1 \otimes 1_{2j+1}, \\
\hat{X}^5 = \frac{1}{3}\sigma_2 \otimes 1_{2j+1}, \ \text{where} \\
a, b, c, \cdots = 1, 2, 3, \ i, j, \cdots = 4, 5, \\
A, B, C, \cdots = 1, 2, 3, 4, 5, \\
[T^a_{(j)}, T^b_{(j)}] = i\epsilon^{abc}T^c_{(j)}, \ (T^a_{(j)})^2 = j(j+1).
\end{array}}$$

This representation satisfies the following relations:

$$[\hat{X}^4, \hat{X}^5] = \frac{3}{4j(j+1)}\epsilon_{45abc}\hat{X}^a\hat{X}^b\hat{X}^c,$$
$$[\hat{X}^A, \hat{X}^B] = \epsilon_{ABCDE}\hat{X}^C\hat{X}^D\hat{X}^E, \ \text{(otherwise)}.$$

In general, it is impossible to resurrect the $SO(5)$ rotational symmetry via the following rescaling:

$$\hat{X}^a = \frac{2\alpha}{3}\sigma_3 \otimes T^a_{(j)}, \ \ \hat{X}^i = \frac{\beta}{3}\sigma_{i-3} \otimes 1_{2j+1},$$

which satisfies the following commutation relations:

$$[\hat{X}^a, \hat{X}^b] = \frac{\alpha}{\beta^2}\epsilon^{abcij}(\hat{X}^c\hat{X}^i\hat{X}^j - \hat{X}^i\hat{X}^c\hat{X}^j + \hat{X}^i\hat{X}^j\hat{X}^c),$$
$$[\hat{X}^a, \hat{X}^i] = \frac{1}{\alpha}\epsilon^{aibcj}(\hat{X}^b\hat{X}^c\hat{X}^j - \hat{X}^b\hat{X}^j\hat{X}^c + \hat{X}^j\hat{X}^b\hat{X}^c),$$
$$[\hat{X}^4, \hat{X}^5] = \frac{3\beta^2}{4j(j+1)\alpha^3}\epsilon^{45abc}\hat{X}^a\hat{X}^b\hat{X}^c.$$

The $SO(5)$ symmetry is resurrected only if $\frac{\alpha}{\beta^2} = \frac{1}{\alpha} = \frac{3\beta^2}{4j(j+1)\alpha^3}$, which is realized only for $j = \frac{1}{2}$.

The matrix theory action for this fuzzy-4 sphere is

$$\begin{aligned} S \ = \ \int dt \mathrm{tr} \Bigg\{ & \frac{1}{2R_M}(D_0\hat{X}^A)^2 + \frac{1}{4}\left([\hat{X}^a, \hat{X}^b] - \frac{\alpha}{\beta^2}\epsilon^{abCDE}\hat{X}^C\hat{X}^D\hat{X}^E\right)^2 \\ & + \frac{1}{2}\left([\hat{X}^a, \hat{X}^i] - \frac{1}{\alpha}\epsilon^{aiCDE}\hat{X}^C\hat{X}^D\hat{X}^E\right)^2 \\ & + \frac{1}{2}\left([\hat{X}^4, \hat{X}^5] - \frac{3\beta^2}{4j(j+1)\alpha^3}\epsilon^{45CDE}\hat{X}^C\hat{X}^D\hat{X}^E\right)^2 \Bigg\}. \end{aligned}$$

3 Associative product of the fuzzy S^2 sphere

The construction of the associative product for the fuzzy S^2 sphere plays a pivotal role in the following S^4 case.

Here, we review the work for S^2

K. Hayasaka, R. Nakayama and Y. Takaya, hep-th/0209240

We start with the star product of the arbitrary representation:

$$f(x)\star_\lambda g(x) = f(x)g(x) + \lambda J^{ab}(\partial_a f(x))(\partial_b g(x)) + \sum_{n=2}^{\infty}\lambda^n\left(\sum_{m=2}^{n}\chi^{(n)}_{m,m}J^{a_1b_1}\cdots J^{a_mb_m}(\partial_{a_1}\cdots\partial_{a_m}f(x))(\partial_{b_1}\cdots\partial_{b_m}g(x))\right).$$

Here, J^{ab} is defined as

$$J^{ab} = r^2\delta^{ab} - x^a x^b + ir\epsilon^{abc}x^c,$$

($r = \sqrt{(x^1)^2+(x^2)^2+(x^3)^2}$) and has the following properties:

$$\begin{aligned}
&(1)\quad x^aJ^{ab} = x^bJ^{ab} = 0,\\
&(2)\quad J^{ab}J^{ac} = J^{ba}J^{ca} = 0,\\
&(3)\quad J^{ab}(\partial_aJ^{cb}) = J^{ca}(\partial_aJ^{db}),\\
&(4)\quad J^{ba}J^{dc}(\partial_a\partial_cJ^{fe}) = -J^{bf}J^{de} - J^{be}J^{df},\\
&(5)\quad J^{a_1b_1}J^{a_2b_2}\cdots J^{a_nb_n}(\partial_{b_1}\cdots\partial_{b_n}J^{cd}) = 0,\ \ (\text{for } n\geq 3).
\end{aligned}$$

From (1) and (2), r is a constant for $\star_\lambda$ product. This is a necessary condition for $\star_\lambda$ to be a product on the S^2 sphere.

$$f(r)\star_\lambda g(x^a) = g(x^a)\star_\lambda f(r) = f(r)g(x^a).$$

We determine $\chi^{(n)}_{m,m}$, to satisfy the associativity:

$$\boxed{(f(x) \star_\lambda g(x)) \star_\lambda h(x) = f(x) \star_\lambda (g(x) \star_\lambda h(x)).}$$

- Up to the first order: for both $(f(x) \star_\lambda g(x)) \star_\lambda h(x)$ and $f(x) \star_\lambda (g(x) \star_\lambda h(x))$,

  $$fgh + \lambda J^{ab}\{(\partial_a f)(\partial_b g)h + (\partial_a f)g(\partial_b h) + f(\partial_a g)(\partial_b h)\}.$$

- To the second order, the difference $(f(x) \star_\lambda g(x)) \star_\lambda h(x) - f(x) \star_\lambda (g(x) \star_\lambda h(x))$ is

  $$\lambda^2[\underbrace{(J^{ab}\partial_a J^{cd} - J^{ca}\partial_a J^{db})}_{\text{vanishes due to (3).}}(\partial_c f)(\partial_d g)(\partial_b h)$$
  $$+ \quad (1 - 2\chi^{(2)}_{2,2})J^{ac}J^{bd}(\partial_a\partial_b f\partial_c g\partial_d h - \partial_a f\partial_b g\partial_c\partial_d h)].$$

  This gives the constraint $\chi^{(2)}_{2,2} = \frac{1}{2}$.

- By the same token, we derive the relation

  $$\chi^{(n)}_{m,m} = \frac{1}{m!}\sum_P (m-1)^{P_1}(m-2)^{P_2}\cdots 2^{P_{m-2}}1^{P_{m-1}}.$$

  $\{P_i\}$ is the partition of $n - m$ into $m - 1$ nonnegative integers.

  $$\chi^{(n)}_{3,3} = \frac{1}{3!}\left(2^0 1^{n-3} + 2^1 1^{n-4} + \cdots + 2^{n-4}1^1 + 2^{n-3}1^0\right) = \frac{2^{n-2}-1}{6}.$$

We thus obtain the following product:

$$\boxed{\begin{aligned} f(x) \star_\lambda g(x) &= fg + \sum_{m=1}^{\infty} C_m(\lambda) J^{a_1 b_1}\cdots J^{a_m b_m} \\ &\qquad \times(\partial_{a_1}\cdots\partial_{a_m} f)(\partial_{b_1}\cdots\partial_{b_m} g), \ \text{where} \\ C_m(\lambda) &= \frac{\lambda^m}{m!(1-\lambda)(1-2\lambda)\cdots(1-(m-1)\lambda)}. \end{aligned}}$$

For the unitary representation, this should reproduce the ordinary $SU(2)$ algebra and its Casimir:

$$[x^a, x^b]_\star = 2i\lambda r \epsilon^{abc} x^c.$$

Then the rescaling $y^a = \frac{x^a}{2\lambda r}$ gives

$$[y^a, y^b]_\star = i\epsilon^{abc} y^c, \; y^a \star_\lambda y^a = \frac{1+2\lambda}{(2\lambda)^2} = j(j+1).$$

Then, λ is determined as $\lambda = \frac{1}{2j}, -\frac{1}{2j+2}$.

$$\boxed{\lambda = \lambda_j^{(A)} = \frac{1}{2j}}$$

In this case, the coefficient $C_m(\lambda)$ is given by

$$C_m(\lambda_j^{(A)}) = \begin{cases} \frac{(2j-m)!}{m!(2j)!} & (\text{for } m \leq 2j), \\ \infty & (\text{for } m > 2j). \end{cases}$$

This product is limited to the finite representation! We denote this product as $\star$.

Especially for $j = \frac{1}{2}$, this gives

$$x^a \star x^b = r^2 \delta_{ab} + ir\epsilon^{abc} x^c, \; 1 \star x^a = x^a \star 1 = x^a.$$

This corresponds to the algebra of the Pauli matrices σ_a and the unit matrices $1_{2\times 2}$.

$$\boxed{\lambda = \lambda_j^{(B)} = -\frac{1}{2j+2}}$$

In this case, the coefficient is given by

$$C_m(\lambda_j^{(B)}) = \frac{(-1)^m}{m!}\frac{(2j+1)!}{(2j+1+m)!}, \text{ (for any } m).$$

This has no divergence for any m. Therefore, this product is applicable to an arbitrary size of representation. We denote this product as $\bullet$.

Especially for $j = \frac{1}{2}$, this gives

$$\begin{aligned} x^a \bullet x^b &= \frac{4}{3}(x^a x^b - \frac{1}{3}r^2\delta^{ab}) + \frac{1}{9}r^2\delta^{ab} - \frac{i}{3}r\epsilon^{abc}x^c, \\ x^a \bullet x^b \bullet x^c &= \frac{20}{9}x^a x^b x^c - \frac{i}{27}r^3\epsilon^{abc} + \cdots. \end{aligned}$$

The algebra of the Pauli matrices is not realized.

On the other hand, the integration on the sphere gives

$$\underbrace{\int d\Omega x^a \bullet x^b = \frac{4\pi}{9}r^2\delta^{ab}}_{=\mathrm{tr}\sigma^a\sigma^b}, \ \underbrace{\int d\Omega x^a \bullet x^b \bullet x^c = -\frac{4\pi i}{27}r^3\epsilon^{abc}}_{=\mathrm{tr}\sigma^a\sigma^b\sigma^c}.$$

The trace corresponds to the product of the Pauli matrices.

4 Noncommutative product on $(S^4)_j$

We construct the star product on the fuzzy S^4 sphere.

Here, we focus on the finite set of the function, and focus on the product $\star$ for $\lambda = \lambda_j^{(A)}$.

$\boxed{S^2 \otimes S^2 \text{ parameterization of } S^4}$

S^4 representation is expressed by the tensor product

$$\begin{aligned} & (2 \times 2 \text{ matrices spanned by } 1_{2\times 2}, \sigma_a) \\ \otimes \;\; & ((2j+1) \times (2j+1) \text{ matrices spanned by } 1_{2j+1}, T^a_{(j)}). \end{aligned}$$

We assign the S^2 coordinate for each representation as

$$\begin{aligned} & x^1, x^2, x^3 : \quad (\text{for the } 2 \times 2 \text{ rep. of } S^2), \\ & y^1, y^2, y^3 : \quad (\text{for the } (2j+1) \times (2j+1) \text{ rep. of } S^2). \end{aligned}$$

We define the radii of the S^2 spheres as

$$r = \sqrt{(x^a)^2}, \;\; \rho = \sqrt{(y^a)^2}.$$

The correspondence with the matrices is

$$\frac{\sigma^a}{2} \Leftrightarrow \frac{x^a}{2r}, \;\; T^a_{(j)} \Leftrightarrow \frac{j y^a}{\rho}.$$

We then find the following correspondence (for $\alpha = \frac{3R}{2j}$, $\beta = 3R$):

$$\begin{aligned} & \hat{X}^a = \tfrac{R}{j}\sigma_3 \otimes T^a_{(j)} \Leftrightarrow X^a = \tfrac{Rx^3 y^a}{r\rho}, && (\text{for } a = 1,2,3), \\ & \hat{X}^i = R\sigma_{i-3} \otimes 1_{2j+1} \Leftrightarrow X^i = \tfrac{Rx^{i-3}}{r}, && (\text{for } i = 4,5). \end{aligned}$$

This mapping $S^2 \otimes S^2 \to S^4$ gives a double cover of the S^4 sphere:

The following two points on $S^2 \otimes S^2$ fall onto the same point on S^4:

$$\begin{aligned} P &: \left(\frac{(x^1,x^2,x^3)}{r}, \frac{(y^1,y^2,y^3)}{\rho}\right) \in S^2_+ \ (x^3 \geq 0), \\ P' &: \left(\frac{(x^1,x^2,-x^3)}{r}, \frac{(-y^1,-y^2,-y^3)}{\rho}\right) \in S^2_- \ (x^3 \leq 0). \end{aligned}$$

The overlapping point of S^2 (namely, the equator $x^3 = 0$) constitutes $S^1 \otimes S^2$.

This $S^1 \otimes S^2$ is mapped on the S^1 circle $\mathcal{C}$:

$$\mathcal{C} = \{(X^1, \cdots, X^5) | X^1 = X^2 = X^3 = 0, X^4 + X^5 = R^2\}.$$

Therefore, the inverse map $S^4 \to S^2_+ \otimes S^2$ is multi-valued (indeterminate) on the circle:

$$\begin{aligned} & x^1(X) = \frac{rX^4}{R}, x^2(X) = \frac{rX^5}{R}, x^3(X) = \frac{rD(X)}{R}, \\ & y^a(X) = \frac{\rho X^a}{D(X)}, \ \text{where} \\ & D(X) = \sqrt{(X^1)^2 + (X^2)^2 + (X^3)^2}. \end{aligned}$$

The mapping $S^4 \to S^2_- \otimes S^2$ is likewise obtained by the replacement $D(X) \Rightarrow -D(X)$. It is also indeterminate on $\mathcal{C}$.

It is straightforward to build the noncommutative product on $(S^4)_j$, from the star product on fuzzy S^2 sphere:

To this end, we combine the noncommutative product for each S^2:

$$
\begin{aligned}
& F(X) \star G(X) \\
= \; & F(X)G(X) + (r^2\delta^{ab} - x^a x^b + ir\epsilon^{abc}x^c)\frac{\partial F(X)}{\partial x^a}\frac{\partial G(X)}{\partial x^b} \\
& +\lambda(\rho\delta^{ab} - y^a y^b + i\rho\epsilon^{abc}y^c)\frac{\partial F(X)}{\partial y^a}\frac{\partial G(X)}{\partial y^b} \\
& +\lambda(r^2\delta^{ab} - x^a x^b + ir\epsilon^{abc}x^c) \\
& \times(\rho^2\delta^{de} - y^d y^e + i\rho\epsilon^{def}y^f)\frac{\partial^2 F(X)}{\partial x^a \partial y^d}\frac{\partial^2 G(X)}{\partial x^b \partial y^e} + \cdots
\end{aligned}
$$

In this way, we can easily build the product for the S^4 sphere, by reducing the symmetry as
$SO(5) \rightarrow SO(3) \otimes SO(2)$.

$\boxed{\text{Functions on the noncommutative } (S^4)_j}$

The functions of the fuzzy $(S^4)_j$ are given by the product of the functions on the two fuzzy S^2:

- First S^2: spanned by $1, \frac{x^a}{r}$.
- Second S^2: spanned by $1, \frac{y^a}{\rho}, \cdots, \frac{y^{a_1}y^{a_2}\cdots y^{a_{2j}}}{\rho^{2j}}$.

Then, the functions of the fuzzy $(S^4)_j$ are given by

$$1, \frac{x^a}{r}, \frac{y^{a_1}}{\rho}, \frac{x^a y^{a_1}}{r\rho}, \cdots, \frac{y^{a_1}\cdots y^{a_{2j}}}{\rho^{2j}}, \frac{x^a y^{a_1}\cdots y^{a_{2j}}}{r\rho^{2j}}.$$

The functions on $(S^4)_j$ are expressed via X^A as

$$\boxed{\begin{gathered} 1, \underbrace{\frac{X^i}{R}, \frac{D(X)}{R}}_{=\frac{x^a}{r}}, \cdots, \\ \underbrace{\frac{X^{a_1}\cdots X^{a_{2j}}}{D(X)^n}}_{=\frac{y^{a_1}\cdots y^{a_{2j}}}{\rho^{2j}}}, \underbrace{\frac{X^i X^{a_1}\cdots X^{a_{2j}}}{RD(X)^n}}_{=\frac{x^{i-3}y^{a_1}\cdots y^{a_{2j}}}{r\rho^{2j}}}, \underbrace{\frac{X^3 X^{a_1}\cdots X^{a_{2j}}}{RD(X)^n}}_{=\frac{x^3 y^{a_1}\cdots y^{a_{2j}}}{r\rho^{2j}}}. \end{gathered}}$$

The number of the independent function is

$$4 \times (2j+1)^2 = N^2.$$

Thus, the function of $(S^4)_j$ corresponds to the $2(2j+1) \times 2(2j+1)$ hermitian matrices.

The spherical harmonics of fuzzy S^4 is given by $Y_{l_1 l_2 l_3 m}$.

The degree of freedom is (up to order l)

$$\sum_{l_1=0}^{l} \underbrace{\sum_{l_2=0}^{l_1} \sum_{l_3=0}^{l_2} \sum_{m=-l_3}^{l_3} 1}_{\text{degree of freedom for the } l_1\text{-order polynomials}} = \frac{(l+1)(l+2)^2(l+3)}{12}.$$

This is not a square number. This makes it difficult to establish the isomorphism

$$(\text{matrices}) \Leftrightarrow (\text{polynomial on } S^4).$$

This construction avoids this problem, since they introduce a non-polynomial function.

These functions are not well-defined on the equator $\mathcal{C}$, since these functions are indeterminate on $\mathcal{C}$.

However, these singularities are not serious in the noncommutative field theory.

The derivative is given by the commutator $\nabla_A F(X) = \frac{i}{R}[X^A, F(X)]_\star$.

We define the star product on the equator $\mathcal{C}$ by the limit outside $\mathcal{C}$, such that the singularities are canceled on $\mathcal{C}$.

In this sense, the products and derivatives are "well-defined".

This is evident from the matrix $\Leftrightarrow$ function correspondence, because the matrix configuration does not have any singularity.

5 Noncommutative product on $(S^4)_{j=\frac{1}{2}}$

In this section, we scrutinize the simplest case $j = \frac{1}{2}$ in more detail.

In principle, we can obtain the explicit form of the star product by replacing (x^a, y^a) with X^A.

On the other hand, it is easier to determine the product for $j = \frac{1}{2}$, such that it reproduces the multiplication rule.

For $j = \frac{1}{2}$, we have the following functions:

$$1, \ \underbrace{\frac{X^i}{R} = \frac{x^{i-3}}{r}, \ \frac{D(X)}{R} = \frac{x^3}{r}}_{=\frac{x^a}{r}}, \ \underbrace{\frac{X^a}{D} = \frac{y^a}{\rho}}_{=\frac{y^a}{\rho}},$$

$$\underbrace{\frac{X^a}{R} = \frac{x^3 y^a}{r\rho}, \ \frac{X^i X^a}{RD(X)} = \frac{x^{i-3} y^a}{r\rho}}_{\frac{x^b y^a}{r\rho}}.$$

The star products of these functions can be built from the S^2 algebra (and likewise for $\frac{y^a}{\rho}$):

$$1 \star 1 = 1, \ 1 \star \frac{x^a}{r} = \frac{x^a}{r} \star 1 = \frac{x^a}{r}, \ \frac{x^a}{r} \star \frac{x^b}{r} = \delta^{ab} + i\epsilon^{abc}\frac{x^c}{r}.$$

This gives the following multiplication laws:

$$
\begin{aligned}
\frac{X^a}{R}\star\frac{X^b}{R} &= \delta_{ab}+i\epsilon_{abc}\frac{X^c}{D(X)}, \qquad \frac{X^i}{R}\star\frac{X^j}{R}=\delta_{ij}+i\epsilon_{ij}\frac{D(X)}{R},\\
\frac{X^a}{R}\star\frac{X^i}{R} &= i\epsilon_{ij}\frac{X^jX^a}{RD(X)}, \qquad \frac{X^i}{R}\star\frac{X^a}{R}=-i\epsilon_{ij}\frac{X^jX^a}{RD(X)},\\
\frac{X^a}{D(X)}\star\frac{D(X)}{R} &= \frac{X^a}{R}, \qquad \frac{D(X)}{R}\star\frac{X^a}{D(X)}=\frac{X^a}{R},\\
\frac{X^i}{R}\star\frac{D(X)}{R} &= -i\epsilon_{ij}\frac{X^j}{R}, \qquad \frac{D(X)}{R}\star\frac{X^i}{R}=i\epsilon_{ij}\frac{X^j}{R},\\
\frac{X^i}{R}\star\frac{X^jX^a}{RD(X)} &= \delta_{ij}\frac{X^a}{D(X)}+i\epsilon_{ij}\frac{X^a}{R},\\
\frac{X^iX^a}{RD(X)}\star\frac{X^j}{R} &= \delta_{ij}\frac{X^a}{D(X)}+i\epsilon_{ij}\frac{X^a}{R},\\
\frac{X^a}{D(X)}\star\frac{X^b}{D(X)} &= \delta_{ab}+i\epsilon_{abc}\frac{X^c}{D(X)},\\
\frac{X^iX^a}{RD(X)}\star\frac{X^b}{D(X)} &= \delta_{ab}\frac{X^i}{R}+i\epsilon_{abc}\frac{X^iX^c}{RD(X)},\\
\frac{X^a}{D(X)}\star\frac{X^iX^b}{RD(X)} &= \delta_{ab}\frac{X^i}{R}+i\epsilon_{abc}\frac{X^iX^c}{RD(X)},\\
\frac{X^a}{R}\star\frac{X^b}{D(X)} &= \delta_{ab}\frac{D(X)}{R}+i\epsilon_{abc}\frac{X^c}{R},\\
\frac{X^a}{D(X)}\star\frac{X^b}{R} &= \delta_{ab}\frac{D(X)}{R}+i\epsilon_{abc}\frac{X^c}{R},\\
\frac{X^iX^a}{RD(X)}\star\frac{D(X)}{R} &= -i\epsilon_{ij}\frac{X^jX^a}{RD(X)},\\
\frac{D(X)}{R}\star\frac{X^iX^a}{RD(X)} &= i\epsilon_{ij}\frac{X^jX^a}{RD(X)},\\
\frac{X^a}{R}\star\frac{D(X)}{R} &= \frac{X^a}{D(X)}, \qquad \frac{D(X)}{R}\star\frac{X^a}{R}=\frac{X^a}{D(X)},\\
\frac{X^a}{R}\star\frac{X^iX^b}{RD(X)} &= i\epsilon_{ij}\delta_{ab}\frac{X^j}{R}-\epsilon_{ij}\epsilon_{abc}\frac{X^jX^c}{RD(X)},\\
\frac{X^iX^a}{RD(X)}\star\frac{X^b}{R} &= -i\epsilon_{ij}\delta_{ab}\frac{X^j}{R}+\epsilon_{ij}\epsilon_{abc}\frac{X^jX^c}{RD(X)},\\
\frac{X^iX^a}{RD(X)}\star\frac{X^jX^b}{RD(X)} &= \delta_{ij}\delta_{ab}+i\delta_{ij}\epsilon_{abc}\frac{X^c}{D(X)}\\
&\quad +i\epsilon_{ij}\delta_{ab}\frac{D(X)}{R}-\epsilon_{ij}\epsilon_{abc}\frac{X^c}{R},\\
\frac{D(X)}{R}\star\frac{D(X)}{R} &= 1.
\end{aligned}
$$

The product of $j=\frac{1}{2}$ is at most of the second order of the derivative.

Then, the star product is constrained as

$$
\begin{aligned}
F(X)\star G(X) = \quad & FG \\
+ \; & L^{(1)}_{a,b}\frac{\partial F}{\partial X^a}\frac{\partial G}{\partial X^b} + L^{(2)}_{i,j}\frac{\partial F}{\partial X^i}\frac{\partial G}{\partial X^j} \\
+ \; & L^{(3)}_{a,i}\frac{\partial F}{\partial X^a}\frac{\partial G}{\partial X^i} + L^{(4)}_{i,a}\frac{\partial F}{\partial X^i}\frac{\partial G}{\partial X^a} \\
+ \; & L^{(5)}_{ab,c}\frac{\partial^2 F}{\partial X^a \partial X^b}\frac{\partial G}{\partial X^c} + L^{(6)}_{a,bc}\frac{\partial F}{\partial X^a}\frac{\partial^2 G}{\partial X^b \partial X^c} \\
+ \; & L^{(7)}_{ab,i}\frac{\partial^2 F}{\partial X^a \partial X^b}\frac{\partial G}{\partial X^i} + L^{(8)}_{a,bi}\frac{\partial F}{\partial X^a}\frac{\partial^2 G}{\partial X^b \partial X^i} \\
+ \; & L^{(9)}_{ai,b}\frac{\partial^2 F}{\partial X^a \partial X^i}\frac{\partial G}{\partial X^b} + L^{(10)}_{i,ab}\frac{\partial F}{\partial X^i}\frac{\partial^2 G}{\partial X^a \partial X^b} \\
+ \; & L^{(11)}_{ai,j}\frac{\partial^2 F}{\partial X^a \partial X^i}\frac{\partial G}{\partial X^j} + L^{(12)}_{a,ij}\frac{\partial F}{\partial X^a}\frac{\partial^2 G}{\partial X^i \partial X^j} \\
+ \; & L^{(13)}_{i,aj}\frac{\partial F}{\partial X^i}\frac{\partial G}{\partial X^a \partial X^j} + L^{(14)}_{ij,a}\frac{\partial^2 F}{\partial X^i \partial X^j}\frac{\partial G}{\partial X^a} \\
+ \; & L^{(15)}_{ij,k}\frac{\partial^2 F}{\partial X^i \partial X^j}\frac{\partial G}{\partial X^k} + L^{(16)}_{i,jk}\frac{\partial F}{\partial X^i}\frac{\partial^2 G}{\partial X^j \partial X^k} \\
+ \; & L^{(17)}_{ab,cd}\frac{\partial^2 F}{\partial X^a \partial X^b}\frac{\partial^2 G}{\partial X^c \partial X^d} + L^{(18)}_{ij,kl}\frac{\partial^2 F}{\partial X^i \partial X^j}\frac{\partial^2 G}{\partial X^k \partial X^l} \\
+ \; & L^{(19)}_{ab,ci}\frac{\partial^2 F}{\partial X^a \partial X^b}\frac{\partial^2 G}{\partial X^c \partial X^i} + L^{(20)}_{ai,bc}\frac{\partial^2 F}{\partial X^a \partial X^i}\frac{\partial^2 G}{\partial X^b \partial X^c} \\
+ \; & L^{(21)}_{ab,ij}\frac{\partial^2 F}{\partial X^a \partial X^b}\frac{\partial^2 G}{\partial X^i \partial X^j} + L^{(22)}_{ij,ab}\frac{\partial^2 F}{\partial X^i \partial X^j}\frac{\partial^2 G}{\partial X^a \partial X^b} \\
+ \; & L^{(23)}_{ai,bj}\frac{\partial^2 F}{\partial X^a \partial X^i}\frac{\partial^2 G}{\partial X^b \partial X^j} \\
+ \; & L^{(24)}_{ai,jk}\frac{\partial^2 F}{\partial X^a \partial X^i}\frac{\partial^2 G}{\partial X^j \partial X^k} + L^{(25)}_{ij,ak}\frac{\partial^2 F}{\partial X^i \partial X^j}\frac{\partial^2 G}{\partial X^a \partial X^k}.
\end{aligned}
$$

The coefficients $L^{(1)}_{a,b} \sim L^{(25)}_{ij,ak}$ are determined such that

- The product should agree with the multiplication laws of each function.
- $R=\sqrt{(X^A)^2}$ should be constant; namely

$$f(R)\star G(X) = G(X)\star f(R) = f(R)G(X).$$

One of such solutions is given by

$$
\begin{aligned}
L^{(1)}_{a,b} &= R^2\delta_{ab} - X^aX^b + iR^2\epsilon_{abc}\frac{X^c}{D(X)}, \\
L^{(2)}_{i,j} &= R^2\delta_{ij} - X^iX^j + iR^2\epsilon_{ij}\frac{D(X)}{R}, \\
L^{(3)}_{a,i} &= -X^aX^i + iR^2\epsilon_{ij}\frac{X^jX^a}{RD(X)}, \\
L^{(4)}_{i,a} &= -X^aX^i - iR^2\epsilon_{ij}\frac{X^jX^a}{RD(X)}, \\
L^{(5)}_{ab,c} &= \frac{R^2-D(X)^2}{2}(X^a\delta_{bc}+X^b\delta_{ac}) - \frac{R^2-D(X)^2}{D(X)^2}X^aX^bX^c \\
&\quad + i\frac{R^2-D(X)^2}{2}(X^b\epsilon_{acd}+X^a\epsilon_{bcd})\frac{X^d}{D(X)}, \\
L^{(6)}_{a,bc} &= \frac{R^2-D(X)^2}{2}(X^c\delta_{ab}+X^b\delta_{ac}) - \frac{R^2-D(X)^2}{D(X)^2}X^aX^bX^c \\
&\quad + i\frac{R^2-D(X)^2}{2}(X^c\epsilon_{abd}+X^b\epsilon_{acd})\frac{X^d}{D(X)}, \\
L^{(7)}_{ab,i} &= -\frac{1}{2}X^aX^bX^i + i\frac{R}{D(X)}\epsilon_{ij}X^jX^aX^b, \\
L^{(8)}_{a,bi} &= -D(X)^2X^i\delta_{ab} + X^aX^bX^i - iD(X)\epsilon_{abc}X^cX^i + iRD(X)\delta_{ab}\epsilon_{ij}X^j \\
&\quad - i\frac{R}{D(X)}\epsilon_{ij}X^jX^aX^b - R\epsilon_{ij}\epsilon_{abc}X^jX^c, \\
L^{(9)}_{ai,b} &= -D(X)^2X^i\delta_{ab} + X^aX^bX^i - iD(X)\epsilon_{abc}X^cX^i - iRD(X)\delta_{ab}\epsilon_{ij}X^j \\
&\quad + i\frac{R}{D(X)}\epsilon_{ij}X^jX^aX^b + R\epsilon_{ij}\epsilon_{abc}X^jX^c, \\
L^{(10)}_{i,ab} &= -\frac{1}{2}X^aX^bX^i - i\frac{R}{D(X)}\epsilon_{ij}X^jX^aX^b, \\
L^{(11)}_{ai,j} &= \frac{D(X)^2}{2}X^a\delta_{ij} + iR^2X^a\epsilon_{ij}\frac{D(X)}{R}, \\
L^{(12)}_{a,ij} &= -i\frac{R}{2}X^a(\epsilon_{ik}X^j+\epsilon_{jk}X^i)\frac{X^k}{D(X)}, \\
L^{(13)}_{i,aj} &= \frac{D(X)^2}{2}X^a\delta_{ij} + iR^2X^a\epsilon_{ij}\frac{D(X)}{R}, \\
L^{(14)}_{ij,a} &= i\frac{R}{2}X^a(\epsilon_{ik}X^j+\epsilon_{jk}X^i)\frac{X^k}{D(X)}, \\
L^{(15)}_{ij,k} &= -i\frac{RD(X)}{2}(\delta_{ik}\epsilon_{jl}+\delta_{jk}\epsilon_{il})X^l + \frac{D(X)^2}{4}(\epsilon_{ik}\epsilon_{jl}+\epsilon_{il}\epsilon_{jk})X^l, \\
L^{(16)}_{i,jk} &= i\frac{RD(X)}{2}(\delta_{ij}\epsilon_{kl}+\delta_{ik}\epsilon_{jl})X^l - \frac{D(X)^2}{4}(\epsilon_{ij}\epsilon_{kl}+\epsilon_{ik}\epsilon_{jl})X^l.
\end{aligned}
$$

$$
\begin{aligned}
L^{(17)}_{ab,cd} &= \frac{D(X)^2}{4}(R^2-D(X)^2)(\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc})+\frac{R^2-D(X)^2}{2D(X)^2}X^aX^bX^cX^d \\
&\quad +i\frac{R^2-D(X)^2}{4}(X^aX^c\epsilon_{bde}+X^aX^d\epsilon_{bce}+X^bX^d\epsilon_{ace}+X^bX^c\epsilon_{ade})\frac{X^e}{D(X)} \\
&\quad -\frac{R^2-D(X)^2}{4}(\epsilon_{ace}\epsilon_{bdf}+\epsilon_{ade}\epsilon_{bcf})X^eX^f, \\
L^{(18)}_{ij,kl} &= -\frac{D(X)^4}{4}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})+\frac{D(X)^4}{4}(\epsilon_{ik}\epsilon_{jl}+\epsilon_{il}\epsilon_{jk}), \\
L^{(19)}_{ab,ci} &= -\frac{D(X)^2}{2}X^i(X^b\delta_{ac}+X^a\delta_{bc})-i\frac{D(X)}{2}X^i(X^b\epsilon_{acd}+X^a\epsilon_{bcd})X^d \\
&\quad +i\frac{RD(X)}{2}(X^b\delta_{ac}+X^a\delta_{bc})\epsilon_{ij}X^j-\frac{R}{2}(X^b\epsilon_{acd}+X^a\epsilon_{bcd})\epsilon_{ij}X^jX^d, \\
L^{(20)}_{ai,bc} &= -\frac{D(X)^2}{2}X^i(X^c\delta_{ab}+X^b\delta_{ac})-i\frac{D(X)}{2}X^i(X^c\epsilon_{abd}+X^b\epsilon_{acd})X^d \\
&\quad -i\frac{RD(X)}{2}(X^c\delta_{ab}+X^b\delta_{ac})\epsilon_{ij}X^j+\frac{R}{2}(X^c\epsilon_{abd}+X^b\epsilon_{acd})\epsilon_{ij}X^jX^d, \\
L^{(21)}_{ab,ij} &= -\frac{1}{2}\epsilon_{ik}\epsilon_{jl}X^kX^lX^aX^b, \\
L^{(22)}_{ij,ab} &= -\frac{1}{2}\epsilon_{ik}\epsilon_{jl}X^kX^lX^aX^b, \\
L^{(23)}_{ai,bj} &= R^2D(X)^2\delta_{ij}\delta_{ab}-D(X)^2X^iX^j\delta_{ab}-\frac{R^2-D(X)^2}{2}X^aX^b\delta_{ij} \\
&\quad +iR^2D(X)\delta_{ij}\epsilon_{abc}X^c-iD(X)X^iX^j\epsilon_{abc}X^c \\
&\quad +iR^2D(X)^2\delta_{ab}\epsilon_{ij}\frac{D(X)}{R}-RD(X)^2\varepsilon_{ij}\epsilon_{abc}X^c, \\
L^{(24)}_{ai,jk} &= \frac{D(X)^2}{4}X^a(X^k\delta_{ij}+X^j\delta_{ik}), \\
L^{(25)}_{ij,ak} &= \frac{D(X)^2}{4}X^a(X^j\delta_{ik}+X^i\delta_{jk}).
\end{aligned}
$$

The covariant derivatives are defined (and given) as

$$
\begin{aligned}
\nabla_a G(X) &= \frac{i}{R}[X_a, G(X)]_\star \\
&= \frac{i}{R}\Bigg((L^{(1)}_{a,b} - L^{(1)}_{b,a})\frac{\partial G}{\partial X^b} + (L^{(3)}_{a,i} - L^{(4)}_{i,a})\frac{\partial G}{\partial X^i} + (L^{(6)}_{a,bc} - L^{(5)}_{bc,a})\frac{\partial^2 G}{\partial X^b \partial X^c} \\
&\quad +(L^{(8)}_{a,bi} - L^{(9)}_{bi,a})\frac{\partial^2 G}{\partial X^b \partial X^i} + (L^{(12)}_{a,ij} - L^{(11)}_{ij,a})\frac{\partial^2 G}{\partial X^i \partial X^j}\Bigg) \\
&\equiv -2\frac{R}{D(X)}\epsilon_{abc}X^c\frac{\partial G}{\partial X^b} - 2\frac{1}{D(X)}\epsilon_{ij}X^j X^a\frac{\partial G}{\partial X^i} \\
&\quad -\frac{R^2 - D(X)^2}{RD(X)}(\epsilon_{acd}X^b + \epsilon_{bcd}X^a)X^d\frac{\partial^2 G}{\partial X^b \partial X^c} \\
&\quad +\frac{2}{R}D(X)\epsilon_{abc}X^c X^i\frac{\partial^2 G}{\partial X^b \partial X^i} - 2i\epsilon_{ij}\epsilon_{abc}X^j X^c\frac{\partial^2 G}{\partial X^b \partial X^i} \\
&\quad +\frac{1}{D(X)}(\epsilon_{ik}X^j + \epsilon_{jk}X^i)X^k X^a\frac{\partial^2 G}{\partial X^i \partial X^j},
\end{aligned}
$$

$$
\begin{aligned}
\nabla_i G(X) &= \frac{i}{R}[X_i, G(X)]_\star \\
&= \frac{i}{R}\Bigg((L^{(2)}_{i,j} - L^{(2)}_{j,i})\frac{\partial G}{\partial X^j} + (L^{(4)}_{i,a} - L^{(3)}_{a,i})\frac{\partial G}{\partial X^a} + (L^{(10)}_{i,ab} - L^{(7)}_{ab,i})\frac{\partial^2 G}{\partial X^a \partial X^b} \\
&\quad +(L^{(13)}_{i,aj} - L^{(11)}_{aj,i})\frac{\partial^2 G}{\partial X^a \partial X^j} + (L^{(16)}_{i,jk} - L^{(15)}_{jk,i})\frac{\partial^2 G}{\partial X^j \partial X^k}\Bigg) \\
&\equiv -2D(X)\epsilon_{ij}\frac{\partial G}{\partial X^j} - 2\frac{1}{D(X)}\epsilon_{ij}X^j X^a\frac{\partial G}{\partial X^a} \\
&\quad -2\frac{1}{D(X)}\epsilon_{ij}X^j X^a X^b\frac{\partial^2 G}{\partial X^a \partial X^b} - 2D(X)X^a\epsilon_{ij}\frac{\partial^2 G}{\partial X^a \partial X^j} \\
&\quad -i\frac{D(X)^2}{2R}(\epsilon_{ij}\epsilon_{kl} + \epsilon_{ik}\epsilon_{jl})X^l\frac{\partial^2 G}{\partial X^j \partial X^k}.
\end{aligned}
$$

The field strength of the gauge field $A_A(X)$ is (for $j = \frac{1}{2}$, $\alpha = \beta = 3R$):

$$
F_{AB}(X) = \frac{i}{R^2}\left([\mathcal{X}^A, \mathcal{X}^B]_\star - \frac{1}{3R}\epsilon_{ABCDE}\mathcal{X}_C \star \mathcal{X}_D \star \mathcal{X}_E\right),
$$

where $\mathcal{X}^A = X^A + RA^A$.

6 B-field background

In this section, we build the star product on the S^4 sphere, in relation to the B-fields.

We introduce the following $S^2 \times S^2$ parameterization of the S^4 sphere:

$$X^1 = R\cos\theta_1 \sin\theta_2 \cos\varphi_2, \;\; X^2 = R\cos\theta_1 \sin\theta_2 \sin\varphi_2,$$
$$X^3 = R\cos\theta_1 \cos\theta_2, \;\; X^4 = R\sin\theta_1 \cos\varphi_1, \;\; X^5 = R\sin\theta_1 \sin\varphi_1.$$

This gives a double cover of the S^4 sphere. In S^4, the following two points are identified:

$$\begin{aligned} P &= (\theta_1, \varphi_1, \theta_2, \varphi_2), \\ P' &= (\pi - \theta_1, \varphi_1, \pi - \theta_2, \varphi_2 + \pi). \end{aligned}$$

The B-field background which has the maximal symmetry of the two spheres $S^2 \times S^2$ is

$$B = \frac{n_1}{2} \sin\theta_1 d\theta_1 d\varphi_1 + \frac{n_2}{2} \sin\theta_2 d\theta_2 d\varphi_2.$$

This is rewritten in terms of X^A as

$$\begin{aligned} B &= \frac{n_2}{4D(X)^3} \epsilon_{abc} X^a dX^b \wedge dX^c + \frac{n_1}{4RD(X)} \epsilon_{ij} dX^i \wedge dX^j \\ &= \frac{1}{2} B_{AB} dX^A \wedge dX^B. \end{aligned}$$

B_{AB} has the tangential condition $B_{AB}X^A = 0$.

$$B_{ab} = \frac{n_2}{2D(X)^3} \epsilon_{abc} X^C, \;\; B_{ij} = \frac{n_1}{2RD(X)} \epsilon_{ij},$$
$$B_{ai} = -B_{ia} = \frac{n_1}{2RD(X)^3} \epsilon_{ij} X^j X^a.$$

They are singular at the equator $\mathcal{C}$, in which $D(X) = 0$.

The inverse matrix α^{AB}, such that $\alpha^{AB}B_{BC} = \delta_{AC} - X^A X^C/R^2$, is

$$\alpha^{ab} = -\frac{2D(X)}{n_2}\epsilon_{abc}X^c,\ \alpha^{ij} = -\frac{2D(X)^3}{n_1 R}\epsilon_{ij},$$
$$\alpha^{ai} = -\alpha^{ia} = -\frac{2D(X)}{n_1 R}\epsilon_{ij}X^j X^a.$$

This defines the Poisson bracket:

$$\{F(X), G(X)\}_{PB} = \frac{1}{2}\alpha^{AB}\partial_A F(X)\partial_B G(X).$$

Via α^{AB}, we define the noncommutative product as

$$\begin{aligned}
& F(X) \star' G(X) \\
= \ & F(X)G(X) + i\ \alpha^{AB}\ \partial_A F(X)\ \partial_B G(X) \\
& -\frac{1}{2}\alpha^{AB}\ \alpha^{CD}\ \partial_A\partial_C F(X)\ \partial_B\partial_D G(X) \\
& -\frac{1}{3}\ \alpha^{AB}\ \left(\partial_B\ \alpha^{CD}\right)\ \{\partial_A\partial_C F(X)\ \partial_D G(X) \\
& -\partial_C F(X)\ \partial_A\partial_D G(X)\ \} + \cdots .
\end{aligned}$$

7 Conclusion

In this paper, the authors built the noncommutative product for the fuzzy S^4 sphere.

To this end, they broke the symmetry $SO(5)$ to $SO(3)\times SO(2)$.

This has enabled us to apply the algebra of the fuzzy S^2 sphere.