# *Matrix model with manifest general coordinate invariance*

*Takehiro Azuma* and Hikaru Kawai (Kyoto Univ.), *Phys.Lett.B538:393-405,2002*, [hep-th/0204078]

## 1. Constructive definition of superstring theory

There are a plethora of vacua in the perturbative superstring theory.
⇒ Now, we have come to the stage where we need a constrictive definition of superstring theory.

A series of large $N$ reduced models have been hitherto proposed as the candidates for the constructive definition.

### IIB matrix model

N.Ishibashi, H.Kawai, Y.Kitazawa and A.Tsuchiya, hep-th/9612115.

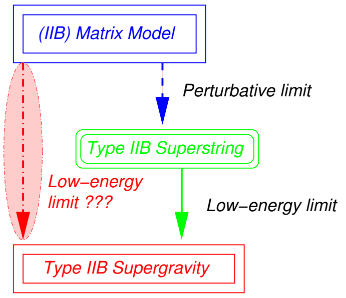
$$S = -\frac{1}{g^2} Tr_{N\times N}\left(\frac{1}{4}\sum_{a,b=0}^{9}[A_a, A_b]^2 - \frac{1}{2}\bar{\psi}\sum_{a=0}^{9}\Gamma^a[A_a,\psi]\right).$$

- $A_a$(10-dimensional vectors) and $\psi$(10-dimensional Majorana-Weyl (i.e. 16-component) spinors) are promoted to $N\times N$ Hermitian matrices.
- $SU(N)$ gauge symmetry and $SO(9,1)$ Lorentz symmetry.
- Dimensional reduction of $\mathcal{N}=1$ 10-dimensional SYM to 0 dimension.
- Matrix regularization of the Green-Schwarz action of type IIB superstring theory.
- $\mathcal{N}=2$ SUSY: This theory must contain spin-2 gravitons if it admits massless particles.
⇒ Therefore, the large $N$ matrices $A_a$, *per se*, represent the spacetime coordinate.

(Q)Is it possible to build a matrix model which describes the general coordinate invariance more manifestly?

Can a matrix model describe the physics in the curved space?

- How is the local Lorentz invariance realized in the matrix model?
- Does a matrix model reduce to the (type IIB) supergravity in the low-energy limit?




## 2. Matrix as a differential operator

A large $N$ matrix has the aspects of both spacetime coordinate and the differential operators:

- TEK model: $A_a \sim \partial_a + a_a(x)$ (covariant derivative)
- IIB matrix model: $A_a \sim X_a$ (spacetime coordinate)

IIB matrix model with noncommutative background

$$[\hat{p}_a, \hat{p}_b] = iB_{ab}, (B_{ab} = \text{real c-numbers})$$

interpolates these two pictures.
H. Aoki, N. Ishibashi, S. Iso, H. Kawai, Y. Kitazawa and T. Tada, hep-th/9908141

$Tr_{N\times N}\bar{\psi}\Gamma^a[A_a,\psi]$ reduces to the fermionic action $\int d^dx\bar{\psi}(x)i\Gamma^a(\partial_i\psi(x)+[a_i(x),\psi(x)])$ in the flat space in the low-energy limit.

### Attempts for a matrix model with local Lorentz invariance

The fermionic action in the curved space:

$$\begin{aligned}
S_F &= \int d^dx\, e(x)\bar{\psi}(x)\Gamma^a e_a{}^i(x)\left(i\partial_i\psi(x)+[A_i(x),\psi(x)]+\frac{i}{4}\Gamma^{bc}\omega_{ibc}(x)\psi(x)\right) \\
&\quad \Downarrow \text{(absorb the determinant } e(x) \text{ into the definition} \\
&\quad\quad \text{of } \Psi(x) \text{ as } \Psi(x)=e^{\frac{1}{2}}(x)\psi(x)\ ) \\
&= \int d^dx\left\{\bar{\Psi}(x)i\Gamma^a e_a{}^i(x)(\partial_i\Psi(x)+[A_i(x),\Psi(x)])\right. \\
&\quad \left.+\frac{i}{4}\bar{\Psi}(x)\Gamma^{a_1a_2a_3}e_{[a_1}{}^i(x)\omega_{ia_2a_3]}(x)\Psi(x)\right\}
\end{aligned}$$

- $a, b, c, \cdots$ : indices of the 10-dimensional Minkowskian spacetime.
- $i, j, k, \cdots$ : indices of the 10-dimensional curved spacetime.

The correspondence between the matrix model and the continuum limit:

$$\begin{aligned}
Tr_{N\times N} &\rightarrow \int d^dx, \\
\psi &\rightarrow \underbrace{\Psi(x)=e^{\frac{1}{2}}(x)\psi(x)}_{\text{spinor root density}}, \\
[A_a,\quad] &\rightarrow ie^{\frac{1}{2}}(x)e_a{}^i(x)(\partial_i+[A_i(x),\quad])e^{-\frac{1}{2}}(x), \\
\{A_{a_1a_2a_3},\psi\} &\rightarrow \underbrace{e_{[a_1}{}^i(x)\omega_{ia_2a_3]}(x)\psi(x)}_{\text{anti-commutator} \Leftrightarrow \text{product}}.
\end{aligned}$$

The rank-3 matrices correspond to the spin connection!

*Note the commutation relations of the (anti)-hermitian operators:*
$[h_1,h_2]\in\mathbf{A}$, $[h,a]\in\mathbf{H}$, $[a_1,a_2]\in\mathbf{A}$, $\{h_1,h_2\}\in\mathbf{H}$, $\{h,a\}\in\mathbf{A}$, $\{a_1,a_2\}\in\mathbf{H}$.

The corresponding matrix model is

$$\begin{aligned}
S_F &\Leftrightarrow \frac{1}{2}Tr\bar{\psi}\Gamma^a[A_a,\psi]+\frac{i}{2}\bar{\psi}\Gamma^{abc}\{A_{abc},\psi\} \\
&\stackrel{\star}{=} Tr(\bar{\psi}\Gamma^aA_a\psi+i\bar{\psi}\Gamma^{a_1a_2a_3}A_{a_1a_2a_3}\psi).
\end{aligned}$$

($\stackrel{\star}{=}$ holds only if $\psi$ is hermitian.)

### Local Lorentz transformation and the "gauged" model

$SO(9,1)$ and $U(N)$ symmetry is decoupled in IIB matrix model: The $SO(9,1)\times U(N)$ symmetry is a tensor product of the group. For $\zeta\in so(9,1)$ and $u\in u(N)$,

$$\exp(\zeta\otimes\mathbf{1}+\mathbf{1}\otimes u)=e^{\zeta}\otimes e^{u}.$$

The spacetime coordinate is embedded in the eigenvalues of the large $N$ matrices.
⇒ If we are to formulate a matrix model with local Lorentz invariance, the $so(9,1)$ Lorentz symmetry and the $u(N)$ gauge symmetry must be unified.

The gauge group must close with respect to the commutator

$$[a\otimes A, b\otimes B]=\frac{1}{2}\left([a,b]\otimes\{A,B\}+\{a,b\}\otimes[A,B]\right).$$

### Local Lorentz transformation of the matrix model

$$\delta\psi=\frac{1}{4}\Gamma^{a_1a_2}\varepsilon_{a_1a_2}\psi,$$

instead of $\delta\psi = \frac{1}{4}\Gamma^{a_1a_2}\{\varepsilon_{a_1a_2}, \psi\}$ at the cost of the hermiticity of $\psi$.

At this time, the product $A_a\psi$ does not directly correspond to the covariant derivative $(\partial_a\psi(x) + [A_a(x), \psi(x)])$.

The local Lorentz transformation of the action:

$$\delta S'_F = \frac{1}{4}Tr\bar{\psi}[\Gamma^a A_a + i\Gamma^{a_1a_2a_3}A_{a_1a_2a_3}, \Gamma^{b_1b_2}\varepsilon_{b_1b_2}]\psi.$$

However, this action does not close with respect to the local Lorentz transformation:

$$\begin{aligned}&[i\Gamma^{a_1a_2a_3}A_{a_1a_2a_3}, \Gamma^{b_1b_2}\varepsilon_{b_1b_2}]\\ =\ & \frac{i}{2}\underbrace{[\Gamma^{a_1a_2a_3}, \Gamma^{b_1b_2}]}_{\text{rank 3}}\{A_{a_1a_2a_3}, \varepsilon_{b_1b_2}\} + \frac{i}{2}\underbrace{\{\Gamma^{a_1a_2a_3}, \Gamma^{b_1b_2}\}}_{\text{rank 1, 5}}[A_{a_1a_2a_3}, \varepsilon_{b_1b_2}].\end{aligned}$$

We need the terms of all odd ranks in order to formulate a local Lorentz invariant matrix model.

The algebra of the local Lorentz transformation must include all the even-rank gamma matrices:

$$\begin{aligned}&[\Gamma^{a_1a_2}\varepsilon_{a_1a_2}, \Gamma^{b_1b_2}\varepsilon'_{b_1b_2}]\\ =\ & \frac{1}{2}\underbrace{[\Gamma^{a_1a_2}, \Gamma^{b_1b_2}]}_{\text{rank-2}}\{\varepsilon_{a_1a_2}, \varepsilon'_{b_1b_2}\} + \frac{1}{2}\underbrace{\{\Gamma^{a_1a_2}, \Gamma^{b_1b_2}\}}_{\text{rank-0, 4}}[\varepsilon_{a_1a_2}, \varepsilon'_{b_1b_2}].\end{aligned}$$

## 3. Matrix model related to type IIB supergravity

$$S = Tr_{N\times N}[tr_{32\times 32}V(m^2) + \bar{\psi}m\psi]$$

$m$ includes all odd-rank gamma matrices in 10 dimensions.

We want to identify $m$ with the Dirac operator.

$\Rightarrow$ We introduce $D = [(\text{length})^{-1}]$ as an extension of the Dirac operator.

$$\begin{aligned} m &= \tau^{\frac{1}{2}}D, \text{ where } \tau = [(\text{length})]^2, \\ D &= A_a\Gamma^a + \frac{i}{3!}A_{a_1a_2a_3}\Gamma^{a_1a_2a_3} - \frac{1}{5!}A_{a_1\cdots a_5}\Gamma^{a_1\cdots a_5} \\ &\quad - \frac{i}{7!}A_{a_1\cdots a_7}\Gamma^{a_1\cdots a_7} + \frac{1}{9!}A_{a_1\cdots a_9}\Gamma^{a_1\cdots a_9}.\end{aligned}$$

$\tau$ is not an $N$-dependent ultraviolet cut-off parameter, but a reference scale $(\sim l_s^2)$.

$A_{a_1\cdots a_{2n-1}} = \frac{i^{2n-1}}{32\times(2n-1)!}tr(D\Gamma_{a_1\cdots a_{2n-1}})$ are hermitian differential operators, so that $m$ satisfies $\Gamma^0 m^\dagger \Gamma^0 = m$.

$\Rightarrow$ They are expanded by the number of the derivatives:

$$\begin{aligned} A_{a_1\cdots a_{2n-1}} &= a_{a_1\cdots a_{2n-1}}(x) \\ &\quad + \sum_{k=1}^{\infty}\frac{i^k}{2}\{\partial_{i_1}\cdots\partial_{i_k}, \underbrace{a^{(i_1\cdots i_k)}{}_{a_1\cdots a_{2n-1}}(x)}_{[(\text{length})^{-1+k}]}\}.\end{aligned}$$

${a_a}^{(i)}(x)$ is identified with the vielbein ${e_a}^i(x)$ in the background metric.

$$\begin{aligned} D &= e^{\frac{1}{2}}(x)\left[ie_a{}^i(x)\Gamma^a\left(\partial_i + \frac{1}{4}\Gamma^{bc}\omega_{ibc}(x)\right)\right]e^{-\frac{1}{2}}(x) \\ &\quad + (\text{higher-rank terms}) + (\text{higher-derivative terms}).\end{aligned}$$

The potential $V(m^2)$ is generically $V(m^2) \sim \exp(-(m^2)^\alpha)$.

$\Rightarrow$ The damping factor is naturally included in the bosonic term.

$\Rightarrow$ The trace for the infinitely large $N$ matrices is finite.

$\psi$ is a Weyl fermion, but not Majorana.

We need to introduce a damping factor so that the trace should be finite.

$$\psi = (\chi(x) + \sum_{l=1}^{\infty}\underbrace{\chi^{(i_1\cdots i_l)}(x)}_{[(\text{length})]^l}\partial_{i_1}\cdots\partial_{i_l}) \times e^{-(\tau D^2)^\alpha}.$$

### Local Lorentz invariance

The action is invariant under the local Lorentz transformation:

$$\begin{aligned} \delta m &= [m, \varepsilon], \quad \delta\psi = \varepsilon\psi, \quad \delta\bar{\psi} = -\bar{\psi}\varepsilon, \text{ where} \\ \varepsilon &= -i\varepsilon_\emptyset + \frac{1}{2!}\Gamma^{a_1a_2}\varepsilon_{a_1a_2} + \cdots + \frac{1}{10!}\Gamma^{a_1\cdots a_{10}}\varepsilon_{a_1\cdots a_{10}}.\end{aligned}$$

The invariance under the local Lorentz transformation:

$$\delta S = 2Tr[tr(V'_S(m^2)m[m,\varepsilon])] + Tr[tr(\bar{\psi}[m,\varepsilon]\psi)] = 0.$$

The cyclic property still holds true of the trace for the large $N$ matrices, if we assume that the coefficients damp rapidly at infinity:

$$\lim_{|x|\to\infty} a^{(i_1\cdots i_k)}{}_{a_1\cdots a_{2n-1}}(x) = \lim_{|x|\to\infty}\chi^{(i_1\cdots i_k)}(x) = 0.$$

### Heat kernel expansion

If this matrix model is to reduce to the type IIB supergravity in the low-energy limit,

$\Rightarrow$ the following fields should be massless, while the other fields should be massive and decoupled in the low-energy limit.

- The antisymmetric even-rank tensors $a'^{(i)}{}_{ia_1\cdots a_{2n}}(x)$.
- Dilatino $\chi(x)$, gravitino $\chi^{(i)}(x)$.

$V(m^2)$ is determined so that

- The fields $a'^{(i)}{}_{ia_1\cdots a_{2n}}(x)$ become massless (then, we surmise that the gravitino and the dilatino will be massless due to the supersymmetry).
- The Dirac operator in the flat space $m_0 = i\Gamma^a\partial_a$ becomes a classical solution (as a corollary, the cosmological constant cancels in the action).

$V(u)$ is chosen as, for example,

$$V_0(u) = \frac{\partial^{\frac{d}{2}-1}(e^{-u^{\frac{1}{4}}}\sin u^{\frac{1}{4}})}{\partial u^{\frac{d}{2}-1}}.$$

The model reduces to the Einstein gravity in the classical low-energy limit:

The linear term of the vielbein ${a_a}^{(a)}(x)$ vanishes, and hence the cross terms ${a_a}^{(a)}(x){a_b}^{(bi_1\cdots i_k)}(x)$ also vanish, due to the general coordinate invariance.

$$\begin{aligned} Tr[trV(m^2)] &= \int_0^\infty dsg(s)Tr[tr(e^{-s\tau D^2})] \\ &= \int_0^\infty dsg(s)\left\{\int d^dx\frac{32}{(2\pi\tau)^{\frac{d}{2}}}s\tau e(x)\frac{R(x)}{6} + \cdots\right\}.\end{aligned}$$

### $\mathcal{N} = 2$ SUSY

The SUSY transformation of the model:

$$\delta\psi = 2V'(m^2)\epsilon, \quad \delta\bar{\psi} = 2\bar{\epsilon}V'(m^2), \quad \delta m = \epsilon\bar{\psi} + \psi\bar{\epsilon}.$$

In the following, we assume that the Taylor expansion of $V(u)$ around $u = 0$ is possible.

Commutator of the SUSY transformation on shell:

$$\begin{aligned} &[\delta_\epsilon, \delta_\xi]m = 2[\xi\bar{\epsilon} - \epsilon\bar{\xi}, V'(m^2)], \\ &[\delta_\epsilon, \delta_\xi]\psi = 2\psi\left(\bar{\epsilon}m\frac{V'(m^2) - V'(0)}{m^2}\xi - \bar{\xi}m\frac{V'(m^2) - V'(0)}{m^2}\epsilon\right).\end{aligned}$$

In order to see the structure of the $\mathcal{N} = 2$ SUSY, we separate the SUSY parameters into the hermitian and the antihermitian parts as

$$\epsilon = \epsilon_1 + i\epsilon_2, \quad \xi = \xi_1 + i\xi_2,$$

$(\xi_1, \xi_2, \epsilon_1, \epsilon_2$ are Majorana-Weyl fermions.)

The translation of the bosons is attributed to the quartic term in the Taylor expansion of $V(m) = \sum_{k=1}^{\infty}\frac{a_{2k}}{2k}m^{2k}$.

$$\begin{aligned} &[\delta_\epsilon, \delta_\xi]a_a(x) \\ =\ & \frac{a_4}{8}(\bar{\xi}_1\Gamma^i\epsilon_1 + \bar{\xi}_2\Gamma^i\epsilon_2) \times (\underbrace{i(\partial_i a_a(x))}_{\text{translation}}\ \underbrace{-i(\partial_a a_i(x)) + [a_i(x), a_a(x)]}_{\text{gauge transformation}}) + \cdots,\end{aligned}$$

However, the commutator for the fermion does not constitute a translation:

$$[\delta_\epsilon, \delta_\xi]\chi(x) = (\text{non-linear terms of the fields}).$$

It is a future problem to surmount this difficulty and find a proper SUSY transformation.