Numerical Simulation of the large-N reduced model

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Prospects of computational physics via supercomputer,

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Contents

1 Introduction

What is superstring theory?

Promising candidate for the unification of all interaction.

First string boom (1980's)

Understanding of perturbative aspects of superstring theory.

- **The energy of gravity is free from divergence.**
- **Prospect for reproducing standard model ($E_8 \times E_8$ heterotic superstring theory).**

Second string boom (late 1990's)

Nonperturbative aspects of superstring theory.

- **Discovery of the D-brane $\Rightarrow$ T/S duality of string theory**
- **Proposal of matrix model as a constructive definition (nonperturbative formulation) of superstring theory**

Third string boom (???)

Completion of constructive definition of superstring theory.

IIB matrix model

N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, hep-th/9612115.

Promising candidate for the constructive definition of superstring theory.

$$S = N\left(-\frac{1}{4}\mathrm{tr}\,[A_\mu, A_\nu]^2 + \frac{1}{2}\mathrm{tr}\,\bar{\psi}\Gamma^\mu[A_\mu, \psi]\right).$$

- Dimensional reduction of $\mathcal{N}=1$ 10-dimensional Super-Yang-Mills (SYM) theory to 0 dimension.

  A_μ (10-dimensional vector) and ψ (10-dimensional Majorana-Weyl spinor) are $N \times N$ matrices .

- Matrix regularization of the Schild from of the Green-Schwarz action of the type IIB superstring theory.

- $\mathcal{N}=2$ supersymmetry:

  The theory must contain spin-2 graviton if it contains massless particles.

- Difficulty of Monte Carlo simulation :

  The determinant (from integrating out fermions) is complex $\Rightarrow$ Sign problem.

2 Heat bath algorithm of bosonic matrix model

(a) Simplest case: quadratic U(N) one-matrix model

$$S = \frac{N}{2} \mathrm{tr}\, \phi^2.$$

To analyze this model via the heat bath algorithm, we rewrite the matrix ϕ as

$$\phi_{ii} = \frac{a_i}{\sqrt{N}}, \begin{cases} \phi_{ij} = \frac{x_{ij}+iy_{ij}}{\sqrt{2N}} \\ \phi_{ji} = \frac{x_{ij}-iy_{ij}}{\sqrt{2N}}, \end{cases} \quad (\text{for } i < j).$$

The N^2 real quantities a_i, x_{ij}, y_{ij} comply with the independent normal Gaussian distribution.

$$S = \frac{1}{2}\sum_{i=1}^{N} a_i^2 + \frac{1}{2}\sum_{i<j}((x_{ij})^2 + (y_{ij})^2).$$

$$Z = \int \prod_{i=1}^{N} da_i \prod_{1 \leq i<j \leq N} dx_{ij} dy_{ij} \exp\left(-\frac{1}{2}\sum_{i=1}^{N} a_i^2 - \frac{1}{2}\sum_{1 \leq i<j \leq N}((x_{ij})^2 + (y_{ij})^2)\right).$$

a_i, x_{ij}, y_{ij} are updated by the Gaussian random number.

There is no need for thermalization or no autocorrelation.

Feynman rule of this model (use the Gaussian integral $\frac{1}{a} = \frac{\int_{-\infty}^{+\infty} dx x^2 e^{-ax^2/2}}{\int_{-\infty}^{+\infty} dx e^{-ax^2/2}}$):

$$\begin{aligned} \langle \phi_{ij}\phi_{kl} \rangle &= \begin{cases} \frac{1}{N}\langle a_i a_k \rangle = \frac{1}{N}\delta_{ik} & (i=j, k=l) \\ \frac{1}{2N}\langle (x_{ij}+iy_{ij})(x_{kl}+iy_{kl}) \rangle = \left\{ \begin{array}{ll} \frac{1}{2N}\langle x_{ij}x_{ij} - y_{ij}y_{ij} \rangle = 0, & (i=k, j=l) \\ \frac{1}{2N}\langle x_{ij}x_{ij} - y_{ij}y_{ji} \rangle = \frac{1}{N}, & (i=l, j=k) \end{array} \right\} & (i \neq j, k \neq l) \end{cases} \\ &= \frac{1}{N}\delta_{il}\delta_{jk} \end{aligned}$$

Some exact results:

$$\begin{aligned} \langle \frac{1}{N}\mathrm{tr}\,\phi^2 \rangle &= \frac{1}{N}\langle \phi_{ij}\phi_{ji} \rangle = \frac{1}{N} \times \frac{1}{N} \times N^2 = 1, \\ \langle \frac{1}{N}\mathrm{tr}\,\phi^4 \rangle &= \frac{1}{N}\langle \phi_{ij}\phi_{jk}\phi_{kl}\phi_{li} \rangle = 2 + \frac{1}{N^2}. \end{aligned}$$

(b) The bosonic IIB matrix model

T. Hotta, J. Nishimura and A. Tsuchiya, hep-th/9811220 .

$$S = -\frac{N}{4}\sum_{\mu,\nu=1}^{d} \mathrm{tr}\,[A_\mu, A_\nu]^2 = -\frac{N}{2}\sum_{1\leq\mu<\nu\leq d} \mathrm{tr}\,\{A_\mu, A_\nu\}^2 + 2N \sum_{1\leq\mu<\nu\leq d} \mathrm{tr}\,(A_\mu^2 A_\nu^2),$$

defined in the d-dimensional Euclidean space.

Auxiliary field $Q_{\mu\nu}$ (where $G_{\mu\nu} = \{A_\mu, A_\nu\}$):

$$\tilde{S} = \frac{N}{2}\sum_{1\leq\mu<\nu\leq d} \mathrm{tr}\,\left(Q_{\mu\nu}^2 - 2(Q_{\mu\nu}G_{\mu\nu}) + 4(A_\mu^2 A_\nu^2)\right) = \frac{N}{2}\sum_{1\leq\mu<\nu\leq d} \mathrm{tr}\,(Q_{\mu\nu} - G_{\mu\nu})^2 + S.$$

Update of $Q_{\mu\nu}$:

$$(Q_{\mu\nu})_{ii} = \frac{a_i}{\sqrt{N}} + (G_{\mu\nu})_{ii},\ (Q_{\mu\nu})_{ij} = \frac{x_{ij} + i y_{ij}}{\sqrt{2N}} + (G_{\mu\nu})_{ij},$$

Dependence of A_λ:

$$\tilde{S} = -N\mathrm{tr}\,(T_\lambda A_\lambda) + 2N\mathrm{tr}\,(S_\lambda A_\lambda^2) + \cdots, \text{ where}$$

$$S_\lambda = \sum_{\mu\neq\lambda}(A_\mu^2),\ T_\lambda = \sum_{\mu\neq\lambda}(A_\mu Q_{\lambda\mu} + Q_{\lambda\mu}A_\mu).$$

- **Dependence of $(A_\lambda)_{ii}$:**

$$\tilde{S} = 2N(S_\lambda)_{ii}(A_\mu)_{ii}^2 - 4Nh_i(A_\mu)_{ii}, \text{ where}$$
$$h_i = \frac{N}{4}[(T_\lambda)_{ii} - 2\sum_{j\neq i}((S_\lambda)_{ji}(A_\lambda)_{ij} + (S_\lambda)_{ij}(A_\lambda)_{ji})].$$

Update of $(A_\lambda)_{ii}$:

$$(A_\lambda)_{ii} = \frac{a_i}{\sqrt{4N(S_\lambda)_{ii}}} + \frac{h_i}{(S_\lambda)_{ii}}.$$

- **Dependence of $(A_\lambda)_{ij}$:**

$$\tilde{S} = 2Nc_{ij}|(A_\lambda)_{ij}|^2 - 2Nh_{ji}(A_\lambda)_{ij}, \text{ where}$$
$$c_{ij} = (S_\lambda)_{ii} + (S_\lambda)_{jj},$$
$$h_{ij} = \frac{1}{2}(T_\lambda)_{ij} - \sum_{k\neq i}(S_\lambda)_{ik}(A_\lambda)_{kj} - \sum_{k\neq j}(S_\lambda)_{kj}(A_\lambda)_{ik}.$$

Update of $(A_\lambda)_{ij}$:

$$(A_\lambda)_{ij} = \frac{x_{ij} + iy_{ij}}{\sqrt{4Nh_{ij}}} + \frac{h_{ij}}{c_{ij}}.$$

3 3d bosonic Yang-Mills-Chern-Simons(YMCS) model

S. Iso, Y. Kimura, K. Tanaka and K. Wakatsuki, hep-th/0101102, T. Azuma, S. Bal, K. Nagao and J. Nishimura, hep-th/0401038

Yang-Mills-Chern-Simons model $\Rightarrow$ toy model with fuzzy sphere solutions:

$$\boxed{S = N\mathrm{tr}\left(-\frac{1}{4}[A_\mu, A_\nu]^2 + \frac{2i\alpha}{3}\epsilon_{\mu\nu\rho}A_\mu A_\nu A_\rho\right).}$$

- Defined in the 3-dimensional Euclidean space: $(\mu, \nu, \rho = 1, 2, 3)$.
- Classical equation of motion: $[A_\nu, [A_\mu, A_\nu]] - i\alpha\epsilon_{\mu\nu\rho}[A_\nu, A_\rho] = 0$.
- Fuzzy S^2 classical solution $A_\mu = X_\mu = \alpha L^{(n)}_\mu \otimes 1_k$

  (where $[L^{(n)}_\mu, L^{(n)}_\nu] = i\epsilon_{\mu\nu\rho}L^{(n)}_\rho$, $N = nk$).

  $L^{(n)}_\mu$ = ($n \times n$ irreducible representation of the SU(2) Lie algebra).

  Casimir operator: $Q = A_1^2 + A_2^2 + A_3^2 = R^2 1_N$, where $R^2 = \alpha^2\frac{n^2-1}{4}$.
- Expansion around k coincident spheres $\Rightarrow$ Field theory with U(k) gauge group.

  Dynamical generation of gauge group.

Monte Carlo simulation of 3d YMCS model

Heat bath algorithm of the 3d YMCS model:

$$\tilde{S} = \sum_{1 \leq \mu < \nu \leq 3} \left(\frac{N}{2} \mathrm{tr}\, Q_{\mu\nu}^2 - N \mathrm{tr}\, (Q_{\mu\nu} G_{\mu\nu}) + 2N \mathrm{tr}\, (A_\mu^2 A_\nu^2) \right) + \frac{2i\alpha N}{3} \epsilon_{\mu\nu\rho} \mathrm{tr}\, A_\mu A_\nu A_\rho.$$

Update of $Q_{\mu\nu}$: parallel to $\alpha = 0$ case (in Sec. 2).

Dependence of A_λ:

$$\tilde{S} = -N \mathrm{tr}\, (T_\lambda A_\lambda) + 2N \mathrm{tr}\, (S_\lambda A_\lambda^2) + \cdots, \text{ where}$$

$$S_\lambda = \sum_{\mu \neq \lambda} A_\mu^2, \quad T_\lambda = \sum_{\mu \neq \lambda} (A_\mu Q_{\lambda\mu} + Q_{\lambda\mu} A_\mu) \underbrace{-2i\alpha \epsilon_{\lambda\mu\nu} A_\mu A_\nu}_{\text{the only difference!}}.$$

Update of A_λ: parallel to the $\alpha = 0$ case, except for T_λ.

Initial condition:

$$A_\mu^{(0)} = \begin{cases} X_\mu & \text{(fuzzy sphere start)}, \\ 0 & \text{(zero start)}. \end{cases}$$

Discontinuity:

$$\alpha = \begin{cases} \alpha_{\mathrm{cr}}^{(l)} = 2.1 \sqrt{\frac{k}{N}} & \text{(fuzzy sphere start)} \\ \alpha_{\mathrm{cr}}^{(u)} = 0.66 & \text{(zero start)}. \end{cases}$$

First-order phase transition:

- $\alpha < \alpha_{\rm cr}$: Yang-Mills phase

  Strong quantum effects.

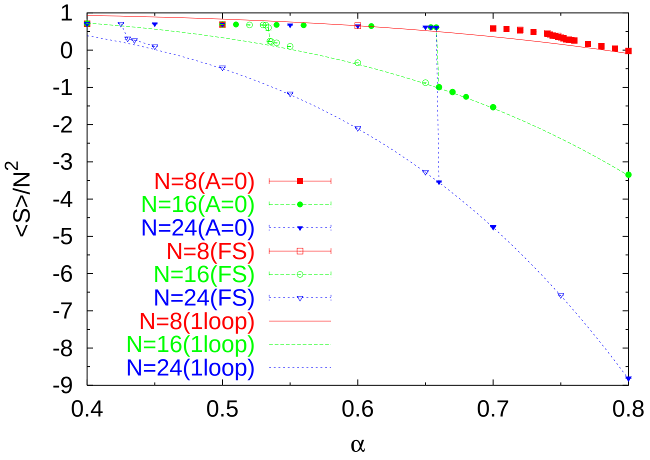
  Behavior like $\alpha = 0$ case.

  $\langle \frac{S}{N^2} \rangle \simeq O(1)$, $\langle \frac{1}{N} {\rm tr}\, A_\mu^2 \rangle \simeq O(1)$.

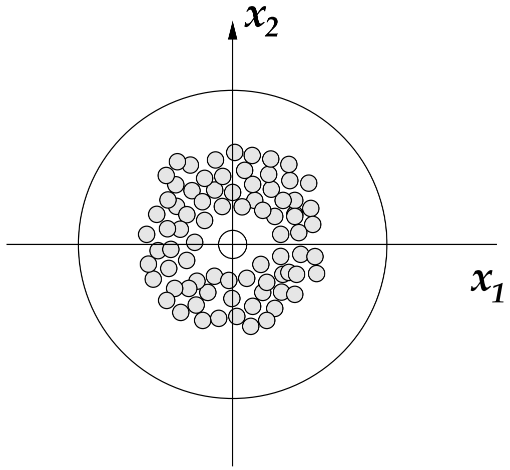
- $\alpha > \alpha_{\rm cr}$: fuzzy sphere phase

  Fuzzy sphere configuration is stable.

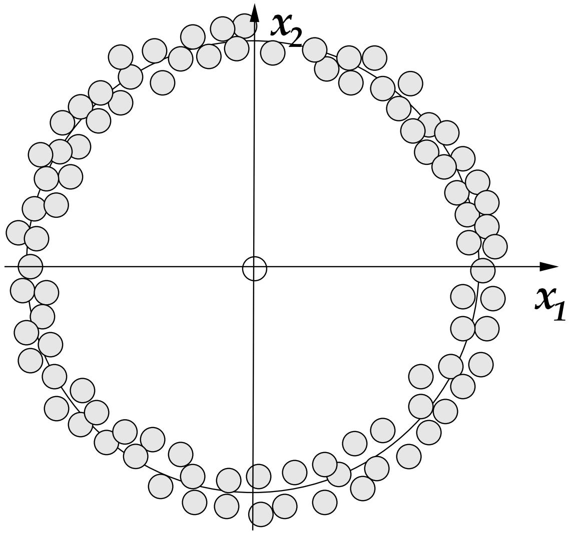
$k = 1$ result:



Yang–Mills phase



Fuzzy sphere phase



4 Simulation of supersymmetric matrix model

Simulation of IIB matrix model is difficult due to sign problem.

- **hybrid R (or hybrid Monte Carlo) simulation of the 4d supersymmetric model (fermion determinant is real positive, O($N^{5,(6)}$) CPU times).**

  J. Ambjorn, K. N. Anagnostopoulos, W. Bietenholz, T. Hotta and J. Nishimura, hep-th/0003208,
  K. N. Anagnostopoulos, T. Azuma, K. Nagao and J. Nishimura, hep-th/0506062.

- **hybrid Monte Carlo simulation of the one-loop effective action of the quenched 10d IIB matrix model, (O(N^3) CPU time).**

  J. Ambjorn, K. N. Anagnostopoulos, W. Bietenholz, T. Hotta and J. Nishimura, hep-th/0005147.

Complex action plays a key role in spontaneous breakdown of Lorentz symmetry:

J. Nishimura and G. Vernizzi, hep-th/0003223.

- **Factorization method to simulate a complex action system.**

  K. N. Anagnostopoulos and J. Nishimura, hep-th/0108041 .

5 Conclusion

We have reviewed the basic techniques of the heat bath algorithm of the large-N reduced model.

We investigated the matrix model with the Chern-Simons term, to deepen the understanding of the fuzzy-sphere background.

Other related works

- **Extension to the four-dimensional manifolds: fuzzy S^4, CP^2, $S^2 \times S^2$.**
  $\Rightarrow$ dynamical generation of four-dimensional spacetime.

  T. Azuma, S. Bal, K. Nagao and J. Nishimura, hep-th/0405096,0405277,0506205
- **3d bosonic massive YMCS model $\Rightarrow$ nontrivial gauge group.**

  T. Azuma, S. Bal and J. Nishimura, hep-th/0504217