Simulating the four-dimensional fuzzy manifolds

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Takehiro Azuma

High-energy Accelerator Research Organization (KEK)

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collaborated with S. Bal, K. Nagao and J. Nishimura

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1 Introduction

Large-N reduced models are the most powerful candidates for the constructive definition of superstring theory.

$\boxed{\text{Matrix models on the homogeneous space}}$

Several alterations of the IIB matrix model have been proposed, to accommodate the curved-space background.

- The matrix model with the Chern-Simons term:
  (hep-th/0101102,0204256,0207115)

These matrix models accommodate the curved-space fuzzy-manifold classical solutions, based on the homogeneous space.

A homogeneous space is realized as G/H:

- $G =$ (a Lie group)
- $H =$ (a closed subgroup of G)

$$\mathrm{S}^2 = \mathrm{SU}(2)/\mathrm{U}(1),\ \mathrm{S}^2 \times \mathrm{S}^2,\ \mathrm{S}^4 = \mathrm{SO}(5)/\mathrm{U}(2),$$
$$\mathrm{CP}^2 = \mathrm{SU}(3)/\mathrm{U}(2), \cdots.$$

Such curved-space fuzzy-manifold solutions are interesting in the following senses:

- More manifest realization of the curved-space background:
  Essential for an eligible framework for gravity.
- We may get insight into the dynamical generation of the gauge group.

2 The model and its classical solutions

Here, we scrutinize the bosonic matrix model that accommodates the four-dimensional fuzzy manifold.
In the following, we focus on the fuzzy CP^2 manifold.

$$\boxed{S = N\mathrm{tr}\left(-\frac{1}{4}\sum_{\mu,\nu=1}^{8}[A_\mu, A_\nu]^2 + \frac{2i\alpha}{3}\sum_{\mu,\nu,\rho=1}^{8} f_{\mu\nu\rho}A_\mu A_\nu A_\rho\right).}$$

- Defined in the 8-dimensional Euclidean space: $(\mu, \nu, \cdots = 1, \cdots, 8)$
- A_μ are promoted to the $N \times N$ hermitian matrices.
- $f_{\mu\nu\rho}$ are the structure constant of the SU(3).

$$f_{123} = 1, \; f_{458} = f_{678} = \frac{\sqrt{3}}{2},$$
$$f_{147} = f_{246} = f_{257} = f_{345} = f_{516} = f_{637} = \frac{1}{2}.$$

Its equation of motion

$$[A_\nu, [A_\mu, A_\nu]] - i\alpha f_{\mu\nu\rho}[A_\nu, A_\rho] = 0$$

accommodates the following two classical solutions:

(a)fuzzy S^2 sphere

$$A_\mu^{(\mathrm{S}^2)} = \begin{cases} \alpha L_\mu^{(N)}, & (\mu = 1, 2, 3), \\ 0, & (\text{otherwise}). \end{cases}$$

The Casimir $Q = \sum_{\mu=1}^{8} A_\mu^2$ is given by

$$Q = \rho_{\mathrm{S}^2}^2 1_N = \alpha^2 \frac{N^2 - 1}{4} 1_N.$$

(b)fuzzy CP^2 space

The fuzzy CP^2 space is realized by the $(m, 0)$ representation of the SU(3) Lie algebra:

$$A_\mu^{(\mathrm{CP}^2)} = \alpha T_\mu^{(m,0)}.$$

This corresponds to the SU(3)/U(2) homogeneous space.

This space is realized by the symmetric tensor product of the fundamental representation of the SU(3) Lie algebra t_μ:

$$T_\mu^{(m,0)} = \underbrace{(t_\mu \otimes 1_3 \otimes \cdots \otimes 1_3)_{\mathrm{sym}} + (1_3 \otimes t_\mu \otimes \cdots \otimes 1_3)_{\mathrm{sym}} + \cdots + (1_3 \otimes \cdots \otimes 1_3 \otimes t_\mu)_{\mathrm{sym}}}_{m\text{-fold}}, \quad \text{where}$$

$$t_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad t_2 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad t_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$t_4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad t_5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad t_6 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$t_7 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad t_8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Here $_{\mathrm{sym}}$ denotes the symmetric tensor product.

For the orthonormal states $|i\rangle$ and $|j\rangle$, the matrix element of A is $(A)_{ij} = \langle i|A|j\rangle$.

The usual tensor product is

$$\langle i_1, i_2|A \otimes B|j_1, j_2\rangle = \langle i_1|A|j_1\rangle\langle i_2|B|j_2\rangle.$$

The two-fold symmetric tensor product is

$$\begin{aligned} &{}_{\mathrm{sym}}\langle i_1, i_2|A \otimes B|j_1, j_2\rangle_{\mathrm{sym}}, \quad \text{where} \\ &|j_1, j_2\rangle_{\mathrm{sym}} = \begin{cases} |j_1\rangle|j_2\rangle, & (\text{for } j_1 = j_2), \\ \frac{1}{\sqrt{2}}(|j_1\rangle|j_2\rangle + |j_2\rangle|j_1\rangle), & (\text{for } j_1 \neq j_2). \end{cases} \end{aligned}$$

The symmetric state for the three products:

$$\begin{aligned} |1,1,1\rangle_{\mathrm{sym}} &= |1\rangle|1\rangle|1\rangle, \\ |1,1,2\rangle_{\mathrm{sym}} &= \frac{1}{\sqrt{3}}(|1\rangle|1\rangle|2\rangle + |1\rangle|2\rangle|1\rangle + |2\rangle|1\rangle|1\rangle), \\ |1,2,3\rangle_{\mathrm{sym}} &= \frac{1}{\sqrt{6}}(|1\rangle|2\rangle|3\rangle + (\text{the other 5 permutations})). \end{aligned}$$

For the 3×3 matrices A and B

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix},$$

the symmetrized product is defined as

$$\begin{aligned} &(A \otimes B)_{\mathrm{sym}} \\ &= \left(\begin{array}{c|cccccc} & |j_1=1,j_2=1\rangle_{\mathrm{sym}} & |1,2\rangle_{\mathrm{sym}} & |1,3\rangle_{\mathrm{sym}} & |2,2\rangle_{\mathrm{sym}} & |2,3\rangle_{\mathrm{sym}} & |3,3\rangle_{\mathrm{sym}} \\ \hline {}_{\mathrm{sym}}\langle i_1=1,i_2=1| & a_{11}b_{11} & \frac{a_{12}b_{11}+a_{11}b_{12}}{\sqrt{2}} & \frac{a_{13}b_{11}+a_{11}b_{13}}{\sqrt{2}} & a_{12}b_{12} & \frac{a_{12}b_{13}+a_{13}b_{12}}{\sqrt{2}} & a_{13}b_{13} \\ {}_{\mathrm{sym}}\langle 1,2| & C_{12,11} & C_{12,12} & C_{12,13} & C_{12,22} & C_{12,23} & C_{12,33} \\ {}_{\mathrm{sym}}\langle 1,3| & C_{13,11} & C_{13,12} & C_{13,13} & C_{13,22} & C_{13,23} & C_{13,33} \\ {}_{\mathrm{sym}}\langle 2,2| & a_{21}b_{21} & \frac{a_{22}b_{21}+a_{21}b_{22}}{\sqrt{2}} & \frac{a_{23}b_{21}+a_{21}b_{23}}{\sqrt{2}} & a_{22}b_{22} & \frac{a_{22}b_{23}+a_{23}b_{22}}{\sqrt{2}} & a_{23}b_{23} \\ {}_{\mathrm{sym}}\langle 2,3| & C_{23,11} & C_{23,12} & C_{23,13} & C_{23,22} & C_{23,23} & C_{23,33} \\ {}_{\mathrm{sym}}\langle 3,3| & a_{31}b_{31} & \frac{a_{32}b_{31}+a_{31}b_{32}}{\sqrt{2}} & \frac{a_{33}b_{31}+a_{31}b_{33}}{\sqrt{2}} & a_{32}b_{32} & \frac{a_{32}b_{33}+a_{33}b_{32}}{\sqrt{2}} & a_{33}b_{33} \end{array}\right), \end{aligned}$$

$$\begin{aligned} &C_{12,11} = \tfrac{a_{21}b_{11}+a_{11}b_{21}}{\sqrt{2}}, \quad C_{12,12} = \tfrac{a_{22}b_{11}+a_{21}b_{12}+a_{12}b_{21}+a_{11}b_{22}}{(\sqrt{2})^2}, \quad C_{12,13} = \tfrac{a_{23}b_{11}+a_{21}b_{13}+a_{13}b_{21}+a_{11}b_{23}}{(\sqrt{2})^2}, \\ &C_{12,22} = \tfrac{a_{22}b_{12}+a_{12}b_{22}}{\sqrt{2}}, \quad C_{12,23} = \tfrac{a_{22}b_{13}+a_{23}b_{12}+a_{12}b_{23}+a_{13}b_{22}}{(\sqrt{2})^2}, \quad C_{12,33} = \tfrac{a_{23}b_{13}+a_{13}b_{23}}{\sqrt{2}}, \\ &C_{13,11} = \tfrac{a_{31}b_{11}+a_{11}b_{31}}{\sqrt{2}}, \quad C_{13,12} = \tfrac{a_{32}b_{11}+a_{31}b_{12}+a_{12}b_{31}+a_{11}b_{32}}{(\sqrt{2})^2}, \quad C_{13,13} = \tfrac{a_{33}b_{11}+a_{31}b_{13}+a_{13}b_{31}+a_{11}b_{33}}{(\sqrt{2})^2}, \\ &C_{13,22} = \tfrac{a_{32}b_{12}+a_{12}b_{32}}{\sqrt{2}}, \quad C_{13,23} = \tfrac{a_{32}b_{13}+a_{33}b_{12}+a_{12}b_{33}+a_{13}b_{32}}{(\sqrt{2})^2}, \quad C_{13,33} = \tfrac{a_{33}b_{13}+a_{13}b_{33}}{\sqrt{2}}, \\ &C_{23,11} = \tfrac{a_{31}b_{21}+a_{21}b_{31}}{\sqrt{2}}, \quad C_{23,12} = \tfrac{a_{32}b_{21}+a_{31}b_{22}+a_{22}b_{31}+a_{21}b_{32}}{(\sqrt{2})^2}, \quad C_{23,13} = \tfrac{a_{33}b_{21}+a_{31}b_{23}+a_{23}b_{31}+a_{21}b_{33}}{(\sqrt{2})^2}, \\ &C_{23,22} = \tfrac{a_{32}b_{22}+a_{22}b_{32}}{\sqrt{2}}, \quad C_{23,23} = \tfrac{a_{32}b_{23}+a_{33}b_{22}+a_{22}b_{33}+a_{23}b_{32}}{(\sqrt{2})^2}, \quad C_{23,33} = \tfrac{a_{33}b_{23}+a_{23}b_{33}}{\sqrt{2}}. \end{aligned}$$

Using this definition, we derive the following formula:

$$\sum_{\mu=1}^{8}(t_\mu \otimes t_\mu) = \frac{1}{12}\begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 6 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 6 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix} \xrightarrow{\text{sym}} \frac{1}{12}\begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix}$$
$$= \frac{1}{3}(1_3 \otimes 1_3)_{\text{sym}}.$$

The Casimir is thus given by

$$\begin{aligned} Q &= \rho^2_{\mathrm{CP}^2} 1_N = \alpha^2 \sum_{\mu=1}^{8} T_\mu^{(m,0)} T_\mu^{(m,0)} \\ &= \alpha^2 \underbrace{(t_\mu^2 \otimes 1_3 \otimes \cdots \otimes 1_3) + \cdots + (1_3 \otimes \cdots \otimes 1_3 \otimes t_\mu^2)}_{m \text{ terms}} \\ &+ \alpha^2 \underbrace{(t_\mu \otimes t_\mu \otimes 1_3 \otimes \cdots \otimes 1_3) + \cdots}_{m(m-1) \text{ terms}} \\ &= \alpha^2 \left(\frac{4m}{3} + \frac{1}{3}m(m-1)\right) 1_N = \alpha^2 \frac{m(m+3)}{3} 1_N, \end{aligned}$$

where we have used $\sum_{\mu=1}^{8}(t_\mu \otimes t_\mu)_{\text{sym}} = \frac{1}{3}(1_3 \otimes 1_3)_{\text{sym}}$ and $\sum_{\mu=1}^{8} t_\mu^2 = \frac{4}{3}1_3$.

The matrix size of this representation is

$$N = \frac{(m+1)(m+2)}{3}, \ (\text{for } m = 1, 2, 3, \cdots).$$

Thus, this representation is realized for a limited size of the matrices $N = 3, 6, 10, 15, 21, \cdots$.

3 Monte Carlo simulation of the matrix model

We analyze the matrix model through the heat-bath algorithm of the Monte Carlo simulation.

In this sense, our approach is nonperturbative.

Heat bath algorithm of the matrix model

(a) Warm-up: quadratic U(N) one-matrix model

We start with the simplest case – quadratic U(N) one-matrix model:

$$S = \frac{N}{2} \mathrm{tr}\, \phi^2.$$

We analyze this model via the heat bath algorithm. To this end, we rewrite the U(N) matrix ϕ as

$$\phi_{ii} = \frac{a_i}{\sqrt{N}}, \quad \begin{cases} \phi_{ij} = \frac{x_{ij} + i y_{ij}}{\sqrt{2N}} \\ \phi_{ji} = \frac{x_{ij} - i y_{ij}}{\sqrt{2N}}, \end{cases} \quad (\text{for } i < j).$$

The N^2 real quantities a_i, x_{ij}, y_{ij} comply with the independent normal Gaussian distribution.

$$S = \frac{1}{2} \sum_{i=1}^{N} a_i^2 + \frac{1}{2} \sum_{i<j} ((x_{ij})^2 + (y_{ij})^2).$$

$$Z = \int \prod_{i=1}^{N} da_i \prod_{1 \leq i < j \leq N} dx_{ij} dy_{ij} \exp\left(-\frac{1}{2} \sum_{i=1}^{N} a_i^2 - \frac{1}{2} \sum_{1 \leq i < j \leq N} ((x_{ij})^2 + (y_{ij})^2) \right).$$

a_i, x_{ij}, y_{ij} are updated by the normal Gaussian random number.

Generation of the uniform random number

We use the congruence method.

- We give the random seed z_1, such as $z_1 = \mathsf{time}()$.
- We solve the recursion formula

$$z_{k+1} = az_k + c \pmod{2^{31} - 1}.$$

  The choice $(a, c) = (5^{11}, 0)$ is known to give a good pseudo-random number.

- The sequence $\{\frac{z_k}{2^{31}-1}\}$ gives a uniform pseudo-random number $[0{:}1]$.

Generation of the Gaussian random number

- We take two uniform random numbers $x, y \in [0 : 1]$.
- We introduce the quantity $r = \sqrt{-a^2 \log x^2}$. This complies with the probability distribution

$$P(r)dr = P(x)\frac{dx}{dr}dr = \frac{2r}{a^2}\exp\left(-\frac{r^2}{a^2}\right)dr.$$

- We next introduce the quantities

$$X = r\cos(2\pi y), \quad Y = r\sin(2\pi y).$$

  They comply with the probability distribution

$$P(r)drdy \propto \exp\left(-\frac{1}{a^2}(X^2 + Y^2)\right)dXdY.$$

(b) The bosonic IIB matrix model

T. Hotta, J. Nishimura and A. Tsuchiya hep-th/9811220.

We investigate the D-dimensional bosonic IIB matrix model via the the heat bath algorithm:

$$S = -\frac{N}{4}\sum_{\mu,\nu=1}^{D} \mathrm{tr}\,[A_\mu, A_\nu]^2 = -\frac{N}{2}\sum_{1\leq\mu<\nu\leq D} \mathrm{tr}\,\{A_\mu, A_\nu\}^2 + 2N\sum_{\mu<\nu}\mathrm{tr}\,(A_\mu^2 A_\nu^2).$$

This action is equivalent to $\tilde{S}$, after integrating out $Q_{\mu\nu}$ (where $G_{\mu\nu} = \{A_\mu, A_\nu\}$):

$$\begin{aligned}\tilde{S} &= N\sum_{\mu<\nu}\left(\frac{1}{2}\mathrm{tr}\,Q_{\mu\nu}^2 - \mathrm{tr}\,(Q_{\mu\nu}G_{\mu\nu}) + 2\mathrm{tr}\,(A_\mu^2 A_\nu^2)\right)\\ &= \frac{N}{2}\sum_{\mu<\nu}\mathrm{tr}\,(Q_{\mu\nu} - G_{\mu\nu})^2 + S.\end{aligned}$$

Then, $Q_{\mu\nu}$ is updated as

$$(Q_{\mu\nu})_{ii} = \frac{a_i}{\sqrt{N}} + (G_{\mu\nu})_{ii},\ (Q_{\mu\nu})_{ij} = \frac{x_{ij} + iy_{ij}}{\sqrt{2N}} + (G_{\mu\nu})_{ij},$$

We next update A_λ. We extract the dependence of A_λ.

$$\tilde{S} = -N\mathrm{tr}\,(T_\lambda A_\lambda) + 2N\mathrm{tr}\,(S_\lambda A_\lambda^2) + \cdots,\ \text{where}$$
$$S_\lambda = \sum_{\mu\neq\lambda}(A_\mu^2),\ T_\lambda = \sum_{\mu\neq\lambda}(A_\mu Q_{\lambda\mu} + Q_{\lambda\mu}A_\mu).$$

- The diagonal part A_λ is updated by extracting the dependence of $(A_\lambda)_{ii}$:

$$\tilde{S} = 2N(S_\lambda)_{ii}(A_\lambda)_{ii}^2 - 4Nh_i(A_\lambda)_{ii},\ \text{where}$$
$$h_i = \frac{N}{4}[(T_\lambda)_{ii} - 2\sum_{j\neq i}((S_\lambda)_{ji}(A_\lambda)_{ij} + (S_\lambda)_{ij}(A_\lambda)_{ji})].$$

Then, $(A_\lambda)_{ii}$ is updated as

$$(A_\lambda)_{ii} = \frac{a_i}{\sqrt{4N(S_\lambda)_{ii}}} + \frac{h_i}{(S_\lambda)_{ii}}.$$

- The other components $(A_\lambda)_{ij}$ are updated likewise by extracting their dependence:

$$\tilde{S} = 2Nc_{ij}|(A_\lambda)_{ij}|^2 - 2Nh_{ji}(A_\lambda)_{ij}, \text{ where}$$
$$c_{ij} = (S_\lambda)_{ii} + (S_\lambda)_{jj},$$
$$h_{ij} = \frac{1}{2}(T_\lambda)_{ij} - \sum_{k\neq i}(S_\lambda)_{ik}(A_\lambda)_{kj} - \sum_{k\neq j}(S_\lambda)_{kj}(A_\lambda)_{ik}.$$

Then, $(A_\lambda)_{ij}$ are updated as

$$(A_\lambda)_{ij} = \frac{x_{ij} + iy_{ij}}{\sqrt{4Nh_{ij}}} + \frac{h_{ij}}{c_{ij}}.$$

(c) Addition of the Chern-Simons term

The Chern-Simons term is *linear* with respect to *each* A_μ.

The algorithm is similar for the following actions:

$$\mathrm{S}^2 \quad : S_{\mathrm{S}^2} = N\mathrm{tr}\left(-\frac{1}{4}\sum_{\mu,\nu=1}^{3}[A_\mu, A_\nu]^2 + \frac{2i\alpha}{3}\epsilon_{\mu\nu\rho}A_\mu A_\nu A_\rho\right),$$
$$\mathrm{S}^4 \quad : S_{\mathrm{S}^4} = N\mathrm{tr}\left(-\frac{1}{4}\sum_{\mu,\nu=1}^{5}[A_\mu, A_\nu]^2 - \frac{\lambda}{5}\epsilon_{\mu_1\cdots\mu_5}A_{\mu_1}A_{\mu_2}A_{\mu_3}A_{\mu_4}A_{\mu_5}\right),$$
$$\mathrm{CP}^2 \quad : S_{\mathrm{CP}^2} = N\mathrm{tr}\left(-\frac{1}{4}\sum_{\mu,\nu=1}^{8}[A_\mu, A_\nu]^2 + \frac{2i\alpha}{3}\sum_{\mu,\nu=1}^{8}f_{\mu\nu\rho}A_\mu A_\nu A_\rho\right),$$
$$\mathrm{S}^2\times\mathrm{S}^2 \quad : S_{\mathrm{S}^2\times\mathrm{S}^2} = N\mathrm{tr}\left(-\frac{1}{4}\sum_{\mu,\nu=1}^{6}[A_\mu, A_\nu]^2 + \frac{2i}{3}f^{(\mathrm{S}^2\times\mathrm{S}^2)}_{\mu\nu\rho}A_\mu A_\nu A_\rho\right),$$

where the structure constant for the $\mathrm{S}^2\times\mathrm{S}^2$ model is

$$f^{(\mathrm{S}^2\times\mathrm{S}^2)}_{\mu\nu\rho} = \begin{cases} \alpha_1\epsilon_{\mu\nu\rho}, & (\text{for } \mu,\nu,\rho = 1,2,3), \\ \alpha_2\epsilon_{\mu\nu\rho}, & (\text{for } \mu,\nu,\rho = 4,5,6), \\ 0, & (\text{otherwise}). \end{cases}$$

We have only to replace T_ρ as

$$T_\rho = \sum_{\mu\neq\lambda}(A_\mu Q_{\lambda\mu} + Q_{\lambda\mu}A_\mu) + T^{(\mathrm{CS})}_\rho, \text{ where}$$
$$T^{(\mathrm{CS})}_\rho = \begin{cases} -2i\alpha\sum_{\mu,\nu=1}^{3}\epsilon_{\rho\mu\nu}A_\mu A_\nu, & (\text{for } S^2), \\ -2i\alpha\sum_{\mu,\nu=1}^{8}f_{\rho\mu\nu}A_\mu A_\nu, & (\text{for } CP^2), \\ -2i\sum_{\mu,\nu=1}^{6}f^{(S^2\times S^2)}_{\rho\mu\nu}A_\mu A_\nu, & (\text{for } S^2\times S^2), \\ \lambda\sum_{\mu_1\cdots\mu_4=1}^{5}\epsilon_{\rho\mu_1\cdots\mu_4}A_{\mu_1}A_{\mu_2}A_{\mu_3}A_{\mu_4}, & (\text{for } S^4). \end{cases}$$

4 Fuzzy CP^2 classical solution

We start from the fuzzy CP^2 initial condition:

$$A_\mu^{(0)} = A_\mu^{(\mathrm{CP}^2)}.$$

To see the behavior of this solution, we discuss the following observables:

- The action S.
- The spacetime extent $\frac{1}{N}\mathrm{tr}\,\sum_{\mu=1}^{8} A_\mu^2$.

Here, we introduce the rescaled parameter

$$\bar{\alpha} = \alpha N^{\frac{1}{4}}.$$

$\boxed{\text{first-order phase transition}}$

We have a first-order phase transition, at the critical point

$$\bar{\alpha} = \bar{\alpha}_{\mathrm{cr}}^{(\mathrm{CP}^2)} (= \alpha_{\mathrm{cr}}^{(\mathrm{CP}^2)} N^{\frac{1}{4}} \simeq 2.3).$$

- $\alpha < \alpha_{\mathrm{cr}}^{(\mathrm{CP}^2)}$: the effect of the Chern-Simons term is negated, and we see the following behavior typical of the pure Yang-Mills model:

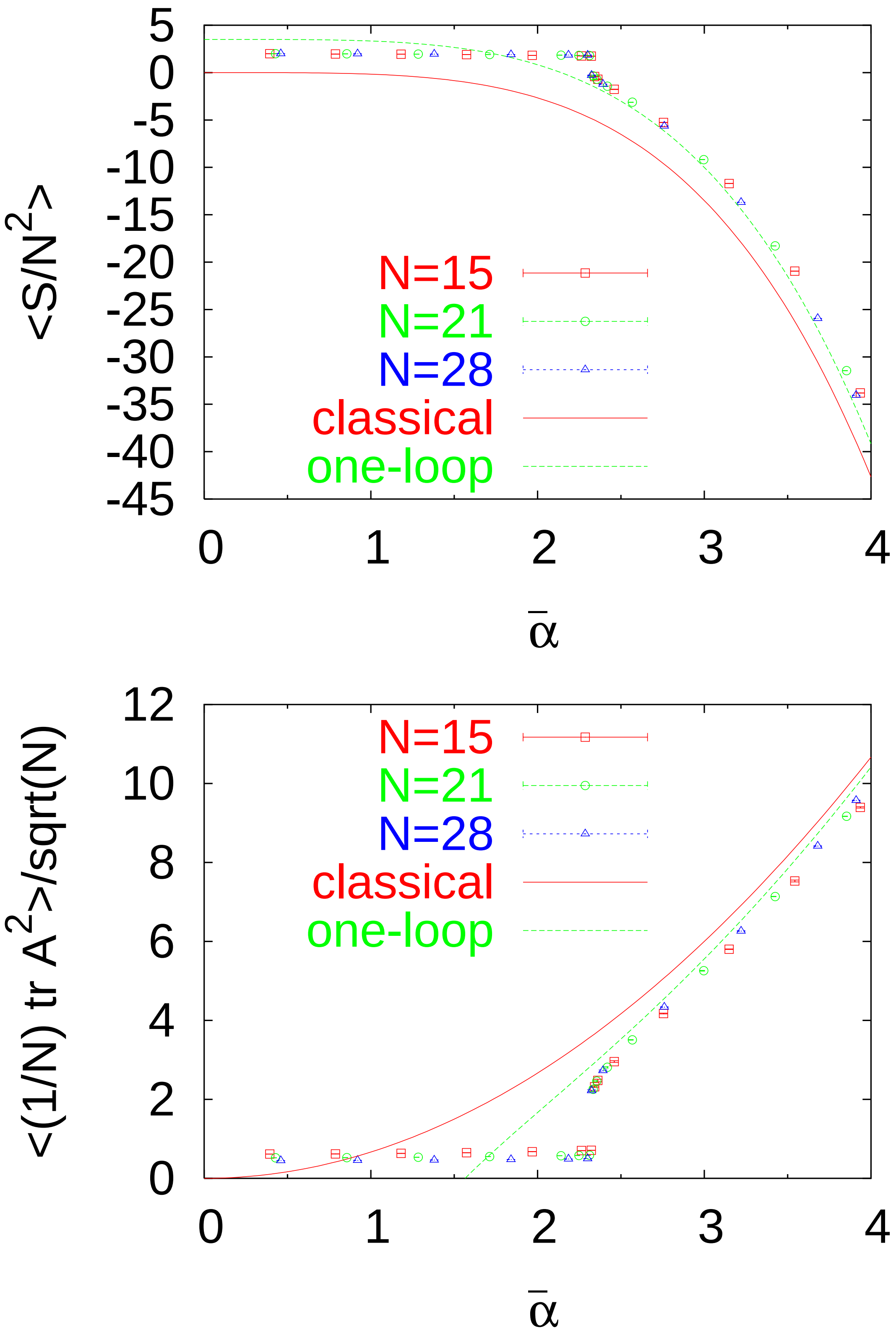
$$\frac{1}{N^2}\langle S \rangle \simeq \mathrm{O}(1), \ \langle \frac{1}{N}\mathrm{tr} A_\mu^2 \rangle \simeq \mathrm{O}(1).$$

- $\alpha > \alpha_{\mathrm{cr}}^{(\mathrm{CP}^2)}$: the fuzzy CP^2 is metastable.

$\boxed{\text{one-loop dominance}}$

The numerical results are close to the one-loop result at $\alpha > \alpha_{\mathrm{cr}}^{(\mathrm{CP}^2)}$:

$$\frac{1}{N^2}\langle S \rangle \simeq -\frac{\bar{\alpha}^4}{6} + \frac{7}{2}, \quad \frac{1}{\sqrt{N}}\langle \frac{1}{N}\mathrm{tr} \sum_{\mu=1}^{8} A_\mu^2 \rangle \simeq \frac{2\bar{\alpha}^2}{3} - \frac{4}{\bar{\alpha}^2}.$$

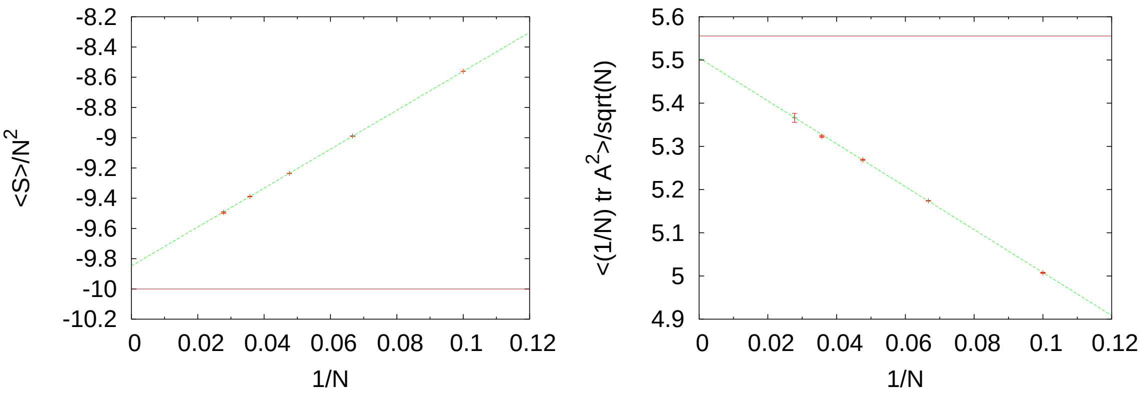


<S/N^2>
N=15
N=21
N=28
classical
one-loop
<(1/N) tr A^2>/sqrt(N)
$\bar{\alpha}$

finite-N effect

We extrapolate the finite-N effect, by plotting these observables against $\frac{1}{N}$:

- $N = 10, 15, 21, 28, 36$ $(m = 3, 4, 5, 6, 7)$.
- $\bar{\alpha} = 3.0$ is fixed.



- The finite-N effects are of the order $O(\frac{1}{N})$.
- We have a deviation from the one-loop calculation at large N.

Since the deviation is rather small, we nevertheless regard this system as retaining the "one-loop dominance".

In fact, the three-dimensional model with fuzzy S^2 classical solution (scrutinized in hep-th/0401038) also has the same deviation.

The critical point $\bar{\alpha}_{\rm cr}^{(\rm CP^2)} \simeq 2.3$ is consistent with the one-loop calculation.

We start with the one-loop effective action around $A_\mu = \beta T_\mu^{(m,0)}$ at large N.

$$W_{\rm CP^2} \simeq N^2 \left(\frac{2}{3k} \left(\frac{3\bar{\beta}^4}{4} - \bar{\alpha}\bar{\beta}^3 \right) + 6 \log \bar{\beta} + ({\rm const.}) \right).$$

This has a minimum at $\frac{\partial W_{\rm CP2}}{\partial \bar{\beta}} = 0$, namely

$$f(\bar{\beta}) = (\bar{\beta}^4 - \bar{\alpha}\bar{\beta}^3) + 3 = 0.$$

$f(\bar{\beta})$ has a minimum at $\bar{\beta}_{\rm min} = \frac{3}{4}\bar{\alpha}$.

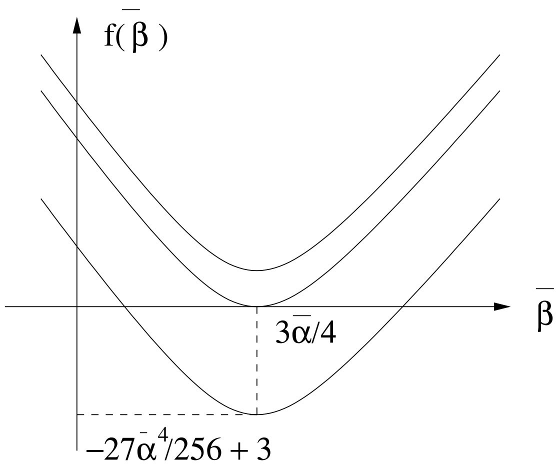
At this critical point, we have

$$f(\bar{\beta}_{\rm min}) \simeq -\frac{1}{3}(\frac{3}{4})^4 \bar{\alpha}^4 + 3 = 0.$$

Then, the critical point is determined as

$$\bar{\alpha}_{\rm cr}^{(\rm CP^2)} = \frac{4}{\sqrt{3}} \simeq 2.3094011 \cdots.$$

This is consistent with the numerical observation.



5 Fuzzy S^2 classical solution

We next start the simulation from the fuzzy S^2 initial condition:

$$A_\mu^{(0)} = A_\mu^{(\mathrm{S}^2)}.$$

We plot the observables against the rescaled parameter

$$\tilde{\alpha} = \alpha N^{\frac{1}{2}}.$$

first-order phase transition

We have a first-order phase transition, at the critical point

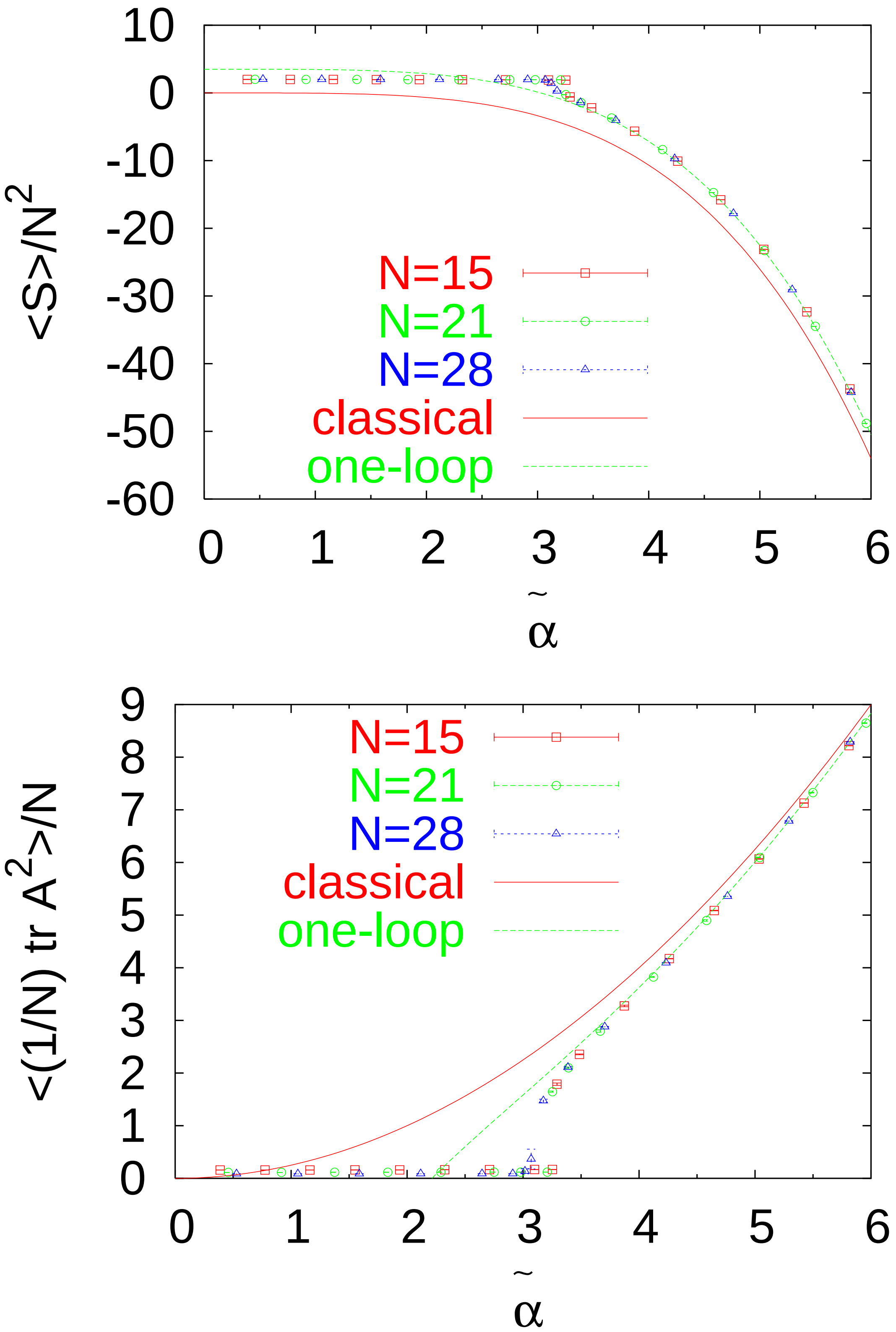
$$\tilde{\alpha} = \tilde{\alpha}_{\mathrm{cr}}^{(\mathrm{S}^2)} (= \alpha_{\mathrm{cr}}^{(\mathrm{S}^2)} N^{\frac{1}{2}} \simeq 3.2).$$

- $\alpha < \alpha_{\mathrm{cr}}^{(\mathrm{S}^2)}$: The behavior is similar to the pure Yang-Mills model.
- $\alpha > \alpha_{\mathrm{cr}}^{(\mathrm{S}^2)}$: the fuzzy S^2 is stable.

one-loop dominance

The numerical results are close to the one-loop result at $\alpha > \alpha_{\mathrm{cr}}^{(\mathrm{S}^2)}$:

$$\begin{aligned} \frac{1}{N^2}\langle S \rangle &\simeq -\frac{\tilde{\alpha}^4}{24} + \frac{7}{2}, \\ \frac{1}{N}\langle \frac{1}{N} \mathrm{tr} \sum_{\mu=1}^{8} A_\mu^2 \rangle &\simeq \frac{\tilde{\alpha}^2}{4} - \frac{6}{\tilde{\alpha}^2}. \end{aligned}$$

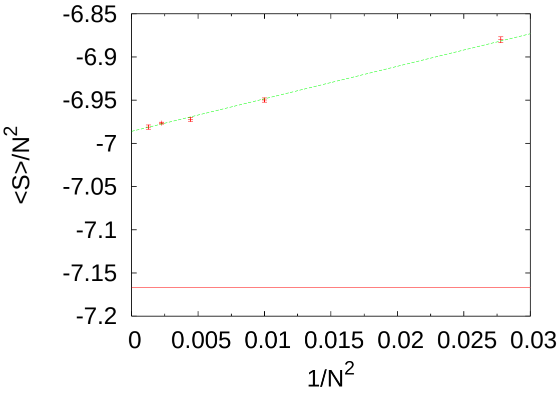


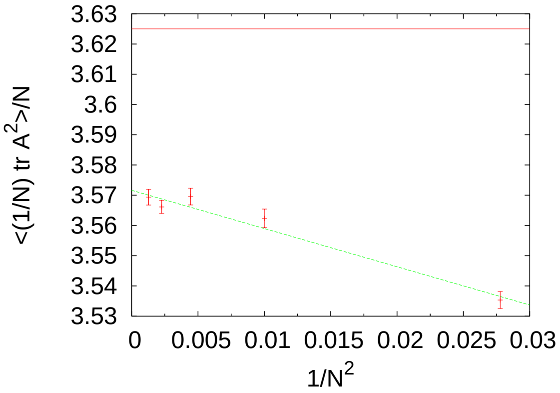
10
0
-10
-20
-30
-40
-50
-60
$<S>/N^2$
N=15
N=21
N=28
classical
one-loop
0
1
2
3
4
5
6
$\tilde{\alpha}$
9
8
7
6
5
4
3
2
1
0
$<(1/N)\,\mathrm{tr}\,A^2>/N$
N=15
N=21
N=28
classical
one-loop
0
1
2
3
4
5
6
$\tilde{\alpha}$

$\boxed{\text{finite-}N \text{ effect}}$

We extrapolate the finite-N effect, by plotting these observables against $\frac{1}{N^2}$:

- $N = 6, 10, 15, 21, 28$.
- $\tilde{\alpha} = 4.0$ is fixed.





For the fuzzy S^2 classical solution, we likewise see the nonperturbative deviation from the one loop at large N.

The critical point is derived from the one-loop effective action as

$$\tilde{\alpha}_{\mathrm{cr}}^{(\mathrm{S}^2)} = \sqrt{\frac{32}{3}} \simeq 3.2659863 \cdots .$$

6 Dynamical generation of the gauge group

We discuss the k $(k \geq 2)$ coincident fuzzy manifolds (multi fuzzy CP^2), to see the dynamical generation of the gauge group.

The expansion around the k coincide fuzzy manifolds gives rise to the $\mathrm{U}(k)$ gauge group.

fuzzy CP^2 space

We define the k coincident fuzzy CP^2 manifolds as

$$A_\mu^{(k,\mathrm{CP}^2)} = \alpha T_\mu^{(m,0)} \otimes 1_k$$

The size of the matrix is $N = \frac{k(m+1)(m+2)}{2}$.

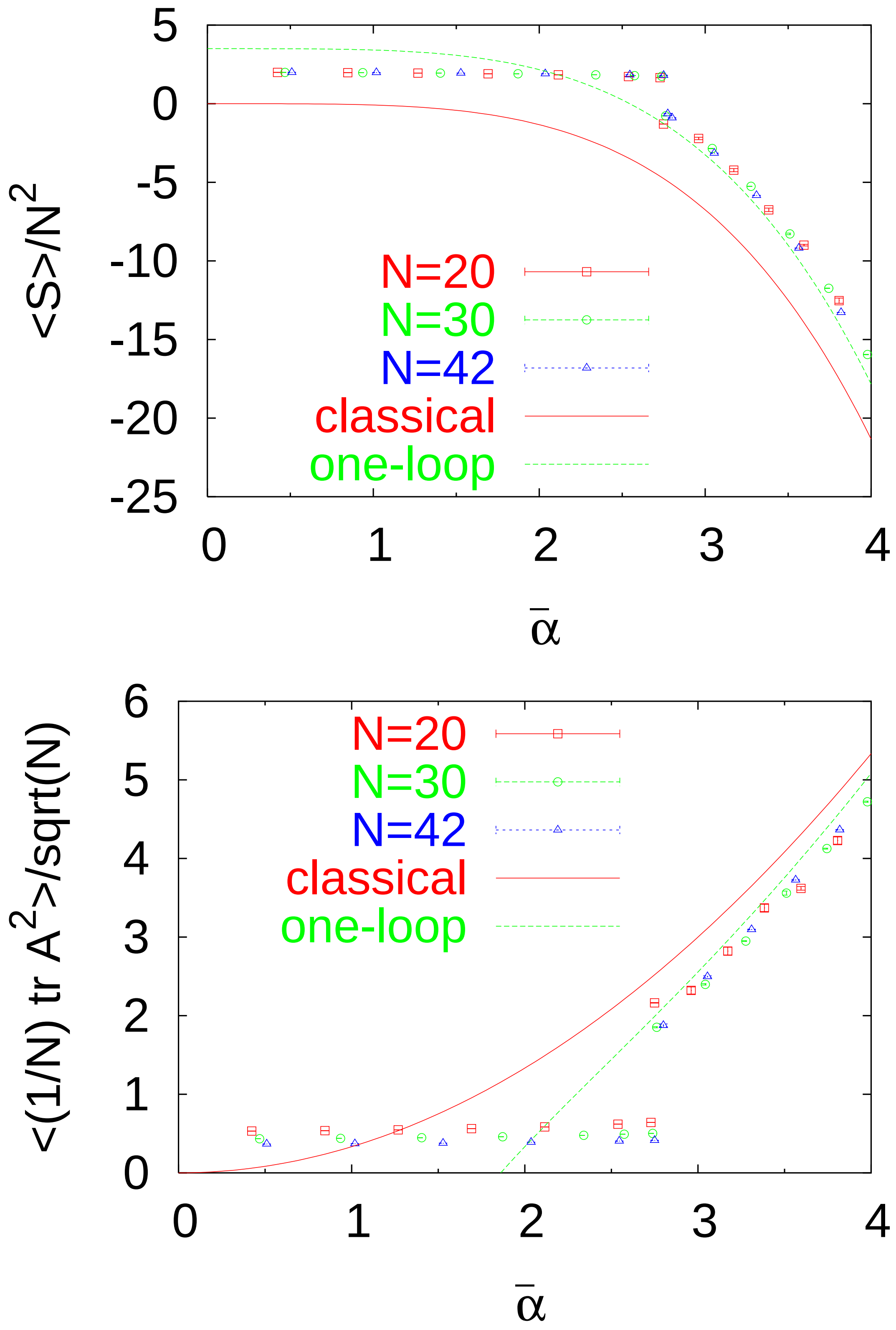
We launch a simulation for $k = 2, m = 3, 4, 5$ $(N = 20, 30, 42)$, starting from $A_\mu^{(0)} = A_\mu^{(k=2,\mathrm{CP}^2)}$.

Before this multi fuzzy CP^2's decay, the system has the first-order phase transition at

$$\bar{\alpha}_{\mathrm{cr}}^{(k=2,\mathrm{CP}^2)} \simeq 2.7.$$

At $\alpha > \alpha_{\mathrm{cr}}^{(k=2,\mathrm{CP}^2)}$, the system retains the one-loop dominance, in which the observables are close to the one-loop results.

$$\begin{aligned} \frac{1}{N^2}\langle S\rangle &\simeq -\frac{\bar{\alpha}^4}{6k} + \frac{7}{2}. \\ \frac{1}{\sqrt{N}}\langle \frac{1}{N}\mathrm{tr}\, A_\mu^2\rangle &\simeq \frac{2\bar{\alpha}^2}{3k} - \frac{4}{\bar{\alpha}^2}. \end{aligned}$$



<S>/N^2
5
0
-5
-10
-15
-20
-25
0
1
2
3
4
ᾱ
N=20
N=30
N=42
classical
one-loop
<(1/N) tr A^2>/sqrt(N)
6
5
4
3
2
1
0
0
1
2
3
4
ᾱ
N=20
N=30
N=42
classical
one-loop

The critical point agrees with the one-loop calculation

$$\bar{\alpha}_{\rm cr}^{(k,{\rm CP}^2)} = \frac{4}{\sqrt{3}} k^{\frac{1}{4}} \simeq 2.3094011 k^{\frac{1}{4}}.$$

We discuss the stability of the multi fuzzy CP^2 from the one-loop effective action.

This classical solution retains the one-loop dominance.

Thus, we discuss the stability of the k coincident fuzzy CP^2's via the one-loop effective action:

$$W_{k,{\rm CP}^2} = N^2 \left(-\frac{\bar{\alpha}^4}{6k} + 6 \log \bar{\alpha} + 3 \log \frac{N^{\frac{3}{2}}}{k} \right).$$

When the single fuzzy CP^2 ($k = 1$) is more stable than the the multi ($k \geq 2$) fuzzy CP^2's, we obtain

$$W_{k=1,{\rm CP}^2} < W_{k,{\rm CP}^2} \Rightarrow \bar{\alpha} > \bar{\alpha}_{k,{\rm CP}^2} = \left(\frac{18}{1-\frac{1}{k}} \log k \right)^{\frac{1}{4}}.$$

Since we always have $\bar{\alpha}_{k,{\rm CP}^2} < \bar{\alpha}_{\rm cr}^{(k,{\rm CP}^2)}$ when the fuzzy CP^2 is stable, we have

$$\bar{\alpha}_{k,{\rm CP}^2} < \bar{\alpha}_{\rm cr}^{(k,{\rm CP}^2)} < \bar{\alpha}.$$

Therefore, the single ($k = 1$) multi fuzzy CP^2 is always the most stable.

This leads to the dynamical generation of the U(1) gauge group.

fuzzy S^2 space

We likewise discuss the k coincident fuzzy S^2 spaces (multi fuzzy S^2):

$$A_\mu^{(k,S^2)} = \begin{cases} \alpha L_\mu^{(n)} \otimes 1_k, & (\text{for } \mu = 1,2,3), \\ 0, & (\text{otherwise}). \end{cases}$$

The size of the matrices is $N = nk$.
This system likewise retains the one-loop dominance.

The critical point is calculated as

$$\tilde{\alpha}_{\text{cr}}^{(k,S^2)} = \sqrt{\frac{32k}{3}} \simeq 3.2659863\sqrt{k}.$$

The multi fuzzy S^2 retains metastability for $\tilde{\alpha} > \tilde{\alpha}_{\text{cr}}^{(k,S^2)}$.
The one-loop effective action at large N is

$$W_{k,S^2} \simeq N^2 \left(-\frac{\tilde{\alpha}^4}{24k^2} + 6\log\tilde{\alpha} + 6\log\frac{N}{k} \right).$$

If the single ($k = 1$) fuzzy S^2 is more stable than the multi ($k \geq 2$) fuzzy S^2's, we have $W_{k=1,S^2} < W_{k,S^2}$:

$$\tilde{\alpha} > \tilde{\alpha}_{k,S^2} = \left(\frac{144\log k}{1 - \frac{1}{k^2}} \right)^{\frac{1}{4}}.$$

Since $\tilde{\alpha}_{k,S^2} < \tilde{\alpha}_{\text{cr}}^{(k,S^2)}$, we have in the fuzzy S^2 phase

$$\tilde{\alpha}_{k,S^2} < \tilde{\alpha}_{\text{cr}}^{(k,S^2)} < \tilde{\alpha}.$$

Therefore, the single ($k = 1$) multi fuzzy S^2 is always the most stable, which leads to the dynamical generation of the U(1) gauge group.

7 Fuzzy CP^2 versus S^2 — which is the true vacuum?

We determine which is the true vacuum, according to the one-loop dominance.

The one-loop effective action around the fuzzy CP^2 and S^2 is

$$
\begin{aligned}
W_{CP^2} &= -\frac{m(m+3)}{12}\alpha^4 N^2 + 3\sum_{c=1}^{m}(c+1)^3 \log[N\alpha^2 c(c+2)] \\
&\simeq N^2\left(-\frac{\alpha^4 N}{6} + 6\log\alpha + 6\log N\right), \\
W_{S^2} &= -\frac{1}{24}\alpha^4 N^2 (N^2-1) + 3\sum_{l=1}^{N-1}(2l+1)\log[N\alpha^2 l(l+1)] \\
&\simeq N^2\left(-\frac{\alpha^4 N^2}{24} + 6\log\alpha + 9\log N\right).
\end{aligned}
$$

The difference is calculated (at large N) as

$$
\Delta = W_{S^2} - W_{CP^2} = N^2\left\{\alpha^4\left(-\frac{N^2}{24} + \frac{N}{6}\right) + 3\log N\right\}.
$$

- The classical effect is $O(N^4)$.
- Whereas, the one-loop quantum effect is $O(N^2 \log N)$.

Therefore, $\Delta < 0$, namely $W_{S^2} < W_{CP^2}$.

The fuzzy S^2 is the true vacuum, and the fuzzy CP^2 is a metastable state.

Nevertheless, the fuzzy CP^2 state retains a very strong metastability.

We start from the initial condition $A_\mu^{(0)} = A_\mu^{(CP^2)}$, for $N = 10 (m = 3), \alpha = 1.4$ (in which the fuzzy CP^2 is metastable).

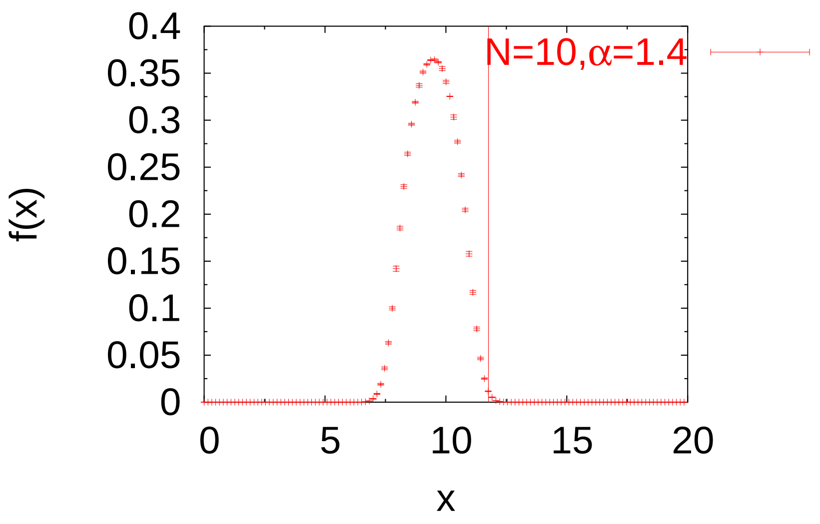
The initial fuzzy CP^2 state endures the 5×10^7 sweeps.

The eigenvalue distribution $f(x)$ of the Casimir Q is defined as

$$f(x) = \frac{1}{N} \sum_{j=1}^{N} \langle \delta(x - \lambda_j) \rangle,$$

where $\{\lambda_j\}$ = (eigenvalues of Q).

Measured after the 5×10^7 sweep, $f(x)$ is plotted below:



Here, the radius of the fuzzy CP^2 space is

$$\rho^2_{CP^2} = \alpha^2 \frac{m(m+3)}{3} = 11.76.$$

8 Conclusion

In this talk, we have discussed the bosonic matrix model that incorporates the four-dimensional fuzzy CP^2 space.

- The true vacuum of this matrix model is not the fuzzy CP^2 but the fuzzy S^2.
- The fuzzy CP^2 is realized as a metastable state.
- Both of these solutions have the one-loop dominance, with a small deviation at large N.
- The k $(k \geq 2)$ coincident fuzzy spaces are always unstable both for the fuzzy CP^2 and S^2.
  This leads to the dynamical generation of the U(1) gauge group.

Future works:

- The investigation of the supersymmetric system:
  We expect that the four-dimensional fuzzy manifold might be the true vacuum due to the supersymmetry.