Nonperturbative studies of higher-dimensional fuzzy-spheres in the matrix model

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Contents

1 Introduction

Curved-space classical solution of the matrix model

The curved spacetime is a fundamental feature of the gravitational interaction.

It is an important question how we realize the curved-space background manifestly in terms of the large-N reduced model.

The IIB matrix model has only a flat noncommutative background, and we want to build a matrix model which describes the curved-space background manifestly.

To this end, we consider the matrix model on the homogeneous space:

A homogeneous space is realized as G/H:

- G = (a Lie group)
- H = (a closed subgroup of G)

There are many cases for such homogeneous spacetimes:

$S^2 = SU(2)/U(1)$, $S^2 \times S^2$, $S^4 = SO(5)/U(2)$, $CP^2 = SU(3)/U(2)$, $\cdots$

Throughout this talk, we scrutinize the homogeneous space $S^2 \times S^2$.
$\Rightarrow$ This gives rise to the 4-dimensional noncommutative gauge theory in the large-N limit.

As a toy model, we investigate the following 6-dimensional bosonic model:

$$S = N\mathrm{tr}\left(-\frac{1}{4}\sum_{\mu,\nu=1}^{6}[A_\mu, A_\nu] + \frac{2i}{3}\sum_{\mu,\nu,\rho=1}^{6} f_{\mu\nu\rho}A_\mu A_\nu A_\rho\right).$$

- This model is defined in the 6-dimensional Euclidean space.
- A_μ: 6-dimensional bosonic vector. Each component is the $N \times N$ hermitian matrix.
- The structure constant is denoted by

$$f_{\mu\nu\rho} = \begin{cases} \alpha_1 \epsilon_{\mu\nu\rho}; & (\mu, \nu, \rho = 1, 2, 3), \\ \alpha_2 \epsilon_{\mu\nu\rho}; & (\mu, \nu, \rho = 4, 5, 6), \\ 0; & (\text{otherwise}). \end{cases}$$

Its classical equation of motion

$$[A_\nu, [A_\mu, A_\nu]] - i\alpha f_{\mu\nu\rho}[A_\nu, A_\rho] = 0$$

accommodates the $S^2 \times S^2$ fuzzy sphere classical solution.

$$A_\mu^{(FS)} = \begin{cases} \alpha_1 (j_\mu^{(1)} \otimes 1_{m_1}) \otimes 1_{k_1}; & (\mu, \nu, \rho = 1, 2, 3), \\ \alpha_2 (1_{m_2} \otimes \tilde{j}_\mu^{(2)}) \otimes 1_{k_2}; & (\mu, \nu, \rho = 4, 5, 6). \end{cases}$$

$j_\mu^{(1)}$, $j_\mu^{(2)}$ are the n_1, n_2-dimensional representation of $SU(2)$.

The total size of the matrices are given by

$$N = n_1 m_1 k_1 = n_2 m_2 k_2.$$

In the following, we focus on the following case:

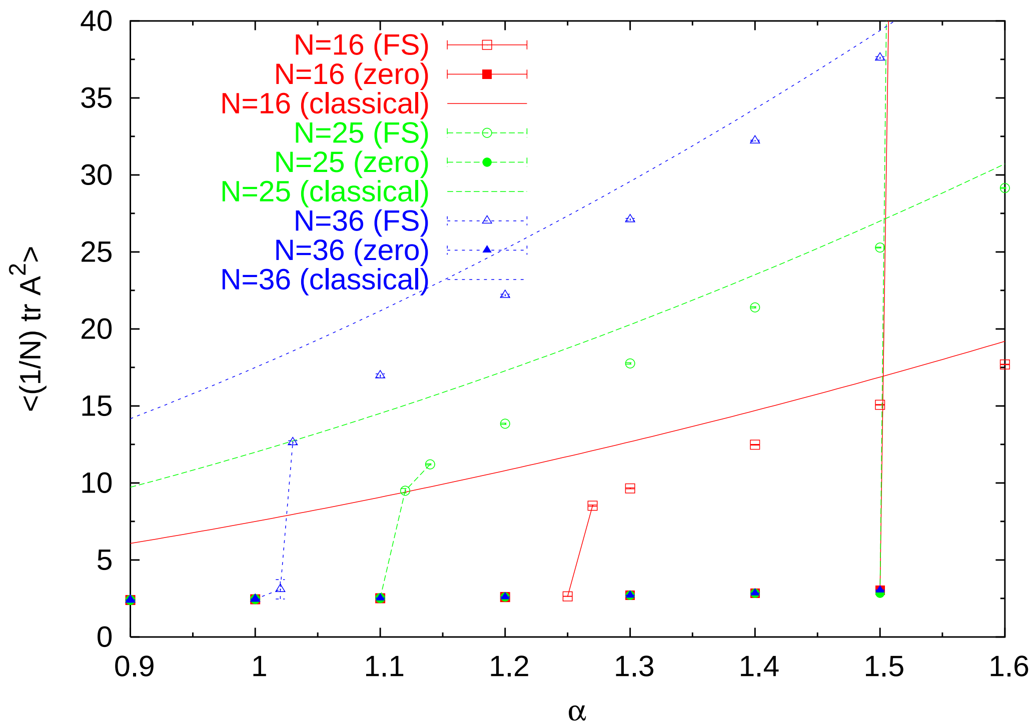
$$\alpha_1 = \alpha_2 (= \alpha), \ n_1 = n_2 = m_1 = m_2 (= n), \ k = 1.$$

2 The phase structure

We launch the simulation from the following two initial conditions for $N = 16, 25, 36$ $(n = 4, 5, 6)$:

$$A_\mu^{(0)} = \begin{cases} A_\mu^{(FS)} & \text{(fuzzy sphere start)}, \\ 0 & \text{(zero start)}. \end{cases}$$

We observe a first-order phase transition similar to the bosonic fuzzy S^2 case.



The lower (upper) critical point is found at

$$\alpha = \begin{cases} \alpha_{cr}^{(l)} \sim 2.5 N^{-\frac{1}{4}} & \text{(fuzzy sphere start)}. \\ \alpha_{cr}^{(u)} \sim 1.51 & \text{(zero start)}. \end{cases}$$

We have the following two phases:

- Yang-Mills phase: $\alpha < \alpha_{cr} \longrightarrow$ Large quantum effect.
- fuzzy sphere phase: $\alpha > \alpha_{cr} \longrightarrow$ The fuzzy $S^2 \times S^2$ is stable.

3 Lower critical point and the one-loop dominance

We launch the simulation from the fuzzy-sphere start $A_\mu^{(0)} = A_\mu^{(FS)}$ for $N = 16, 25, 36$ $(n = 4, 5, 6)$.

We plot the following quantities against $\tilde{\alpha} = \alpha N^{\frac{1}{4}}$. The vacuum expectation value of these quantities are given at one-loop by

$$\frac{1}{\sqrt{N}}\langle \frac{1}{N}\mathrm{tr}\sum_{\mu=1}^{6} A_\mu^2\rangle \simeq \underbrace{\frac{\tilde{\alpha}^2}{2k}}_{\text{classical}} \underbrace{-\frac{8}{\tilde{\alpha}^2}}_{\text{one-loop}},$$

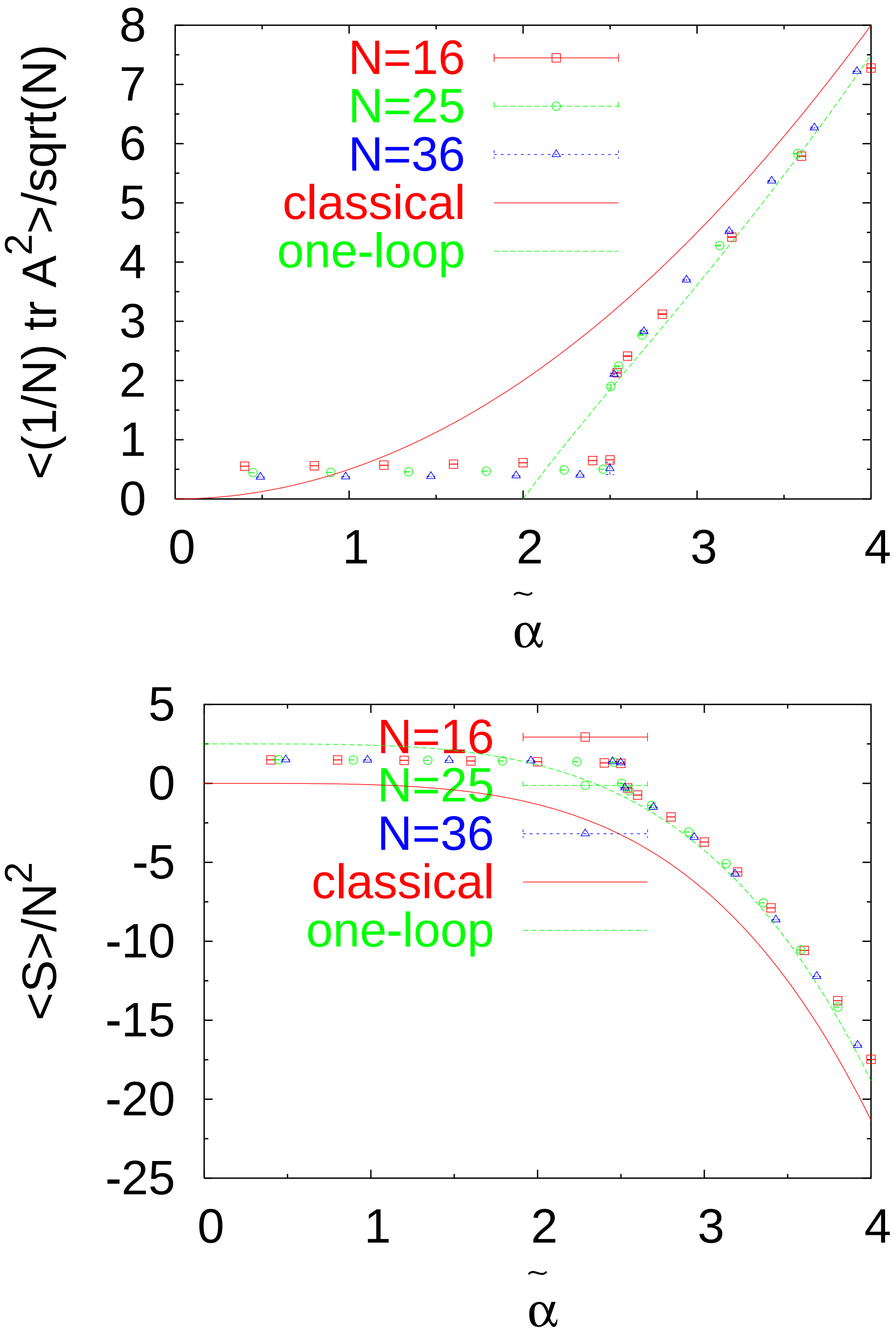
$$\frac{1}{N^2}\langle S\rangle = \underbrace{-\frac{\tilde{\alpha}^4}{12k}}_{\text{classical}} \underbrace{+\frac{D-1}{2}}_{\text{one-loop}},$$

$$\frac{1}{N^{\frac{1}{4}}}\langle M\rangle = \frac{1}{N^{\frac{1}{4}}}\langle \frac{2i}{3N}\sum_{\mu,\nu,\rho=1}^{6} f_{\mu\nu\rho}\mathrm{tr}A_\mu A_\nu A_\rho\rangle = \underbrace{-\frac{\tilde{\alpha}^3}{3k}}_{\text{classical}} \underbrace{+\frac{D-2}{\tilde{\alpha}}}_{\text{one-loop}},$$

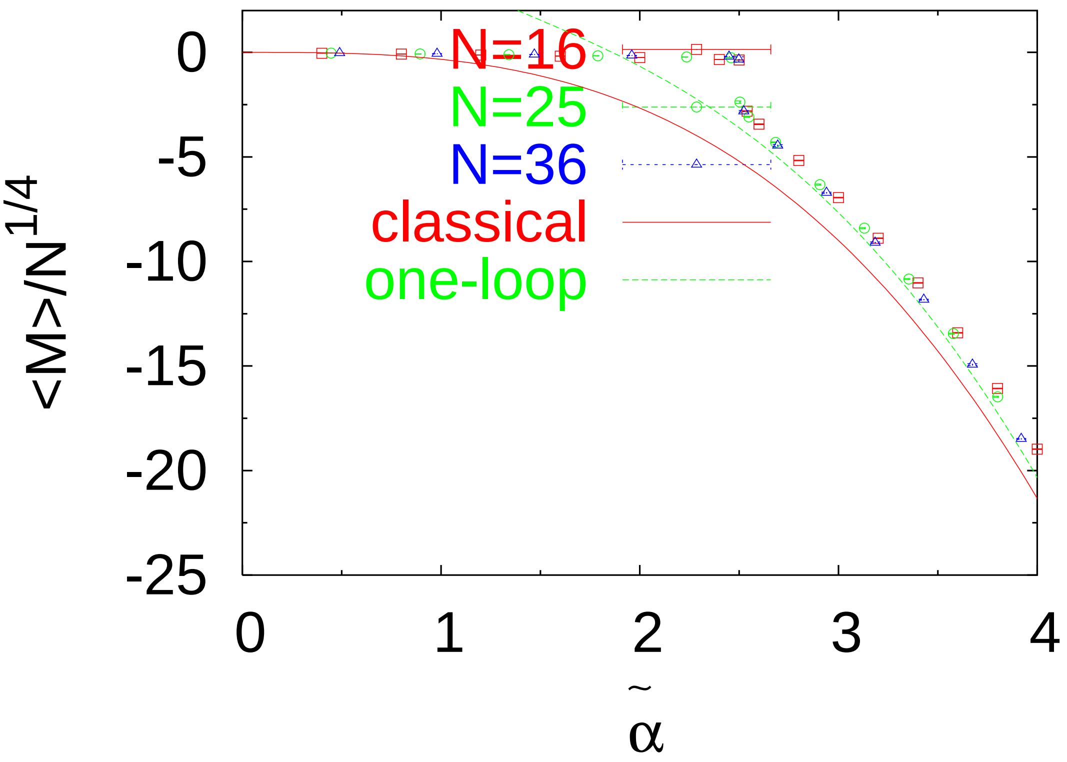
$$\langle \frac{1}{N}F_{\mu\nu}^2\rangle = \langle \frac{1}{N}(i[A_\mu, A_\nu])^2\rangle = \underbrace{\frac{\tilde{\alpha}^4}{k}}_{\text{classical}} \underbrace{+(-2D+6)}_{\text{one-loop}}.$$

$\boxed{\text{Results}}$

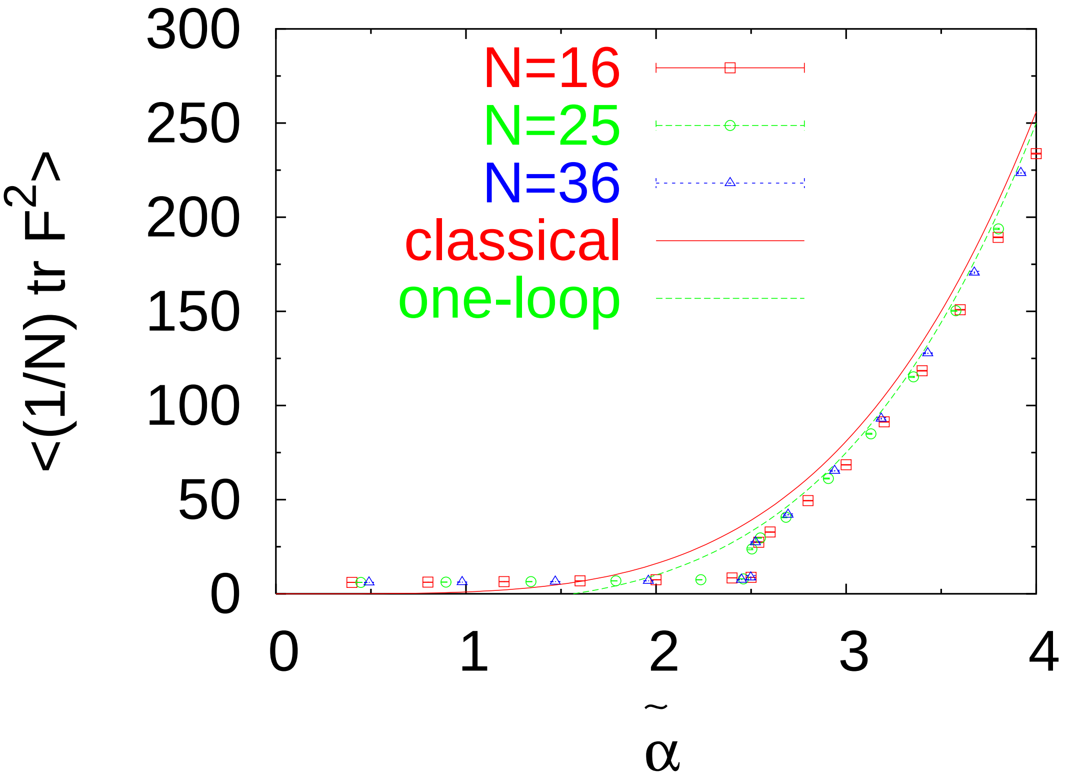
- **The critical point**: We have a first-order phase transition, with the critical point $\tilde{\alpha}_{cr} = \alpha_{cr}N^{\frac{1}{4}} \sim 2.5$.
- **One-loop dominance**: The one-loop effect is dominant at the fuzzy sphere phase.
  The finite-N effects are found to be $\mathcal{O}(\frac{1}{N})$.



<(1/N) tr A^2>/sqrt(N)
N=16
N=25
N=36
classical
one-loop
0
1
2
3
4
5
6
7
8
0
1
2
3
4
α̃
<S>/N^2
N=16
N=25
N=36
classical
one-loop
5
0
-5
-10
-15
-20
-25
0
1
2
3
4
α̃



N=16
N=25
N=36
classical
one-loop
<M>/N^{1/4}
0
-5
-10
-15
-20
-25
0
1
2
3
4
$\tilde{\alpha}$



N=16
N=25
N=36
classical
one-loop
<(1/N) tr F^2>
300
250
200
150
100
50
0
0
1
2
3
4
$\tilde{\alpha}$

4 Eigenvalue distribution of the Casimir

We launch the simulation from the fuzzy-sphere start $A_\mu^{(0)} = A_\mu^{(FS)}$.

We observe the eigenvalues of the Casimir

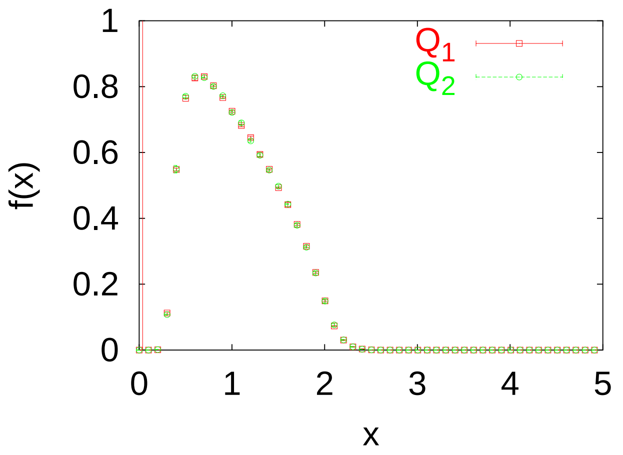
$$Q_1 = \sum_{\mu=1}^{3} A_\mu^2, \quad Q_2 = \sum_{\mu=4}^{6} A_\mu^2.$$

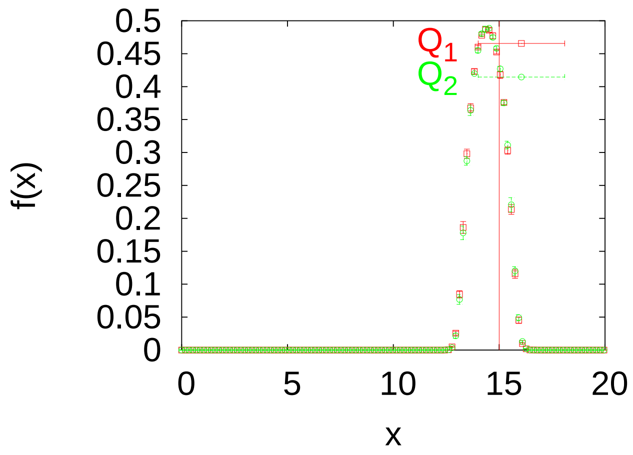
The eigenvalues are at the outset peaked at

$$Q_{1,2} = \frac{n^2 - 1}{4} 1_N.$$

The eigenvalue distribution is given for $N = 16$ $(n = 4)$:

- **$\alpha = 0.1$: in the Yang-Mills phase.**
- **$\alpha = 2.0$: in the fuzzy sphere phase.**





5 Conclusion

We have conducted the heat-bath algorithm of the Monte-Carlo simulation for the higher-dimensional manifolds.

In this talk, we have focused on the fuzzy $S^2 \times S^2$ case, which gives rise to the 4-dimensional noncommutative space in the large-N limit.

We have observed the phase structure similar to the fuzzy S^2 case:

- Yang-Mills phase: $\alpha < \alpha_{cr} \rightarrow$ Large quantum effect.
- fuzzy sphere phase: $\alpha > \alpha_{cr} \rightarrow$ The fuzzy sphere is stable.

Works in progress

- Analysis of the other higher-dimensional manifolds: $CP^2 = SU(3)/U(2)$, S^{2k}, $\cdots$.
- The extension to the supersymmetric system via the hybrid Monte Carlo simulation.
- The relation between the gauge group and the clustered eigenvalues.