Generalized factorization method for the overlap problem in a matrix model with complex action

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1 Introduction

Sign problem and overlap problem

Partition function with complex integrand:

$$Z = \int dA \exp\left(-S_0 + i\Gamma\right), \quad Z_0 = \int dA e^{-S_0}.$$

e.g. lattice QCD, large-N reduced models for superstring theory$\cdots$.

- **Sign problem:**

  Standard reweighting method

  $$\langle \mathcal{O} \rangle = \frac{\langle \mathcal{O} e^{i\Gamma} \rangle_0}{\langle e^{i\Gamma} \rangle_0}, \text{ where } \langle * \rangle_0 = (\text{ V.E.V. for the phase-quenched model } Z_0).$$

  (Number of configurations required) $\simeq e^{\mathrm{O}(N^2)}$.

- **Overlap problem:**

  Discrepancy of important configurations between phase-quenched partition function Z_0 and full partition function Z.

2 Factorization method

Method to sample important configurations for the full partition function.

We constrain the observables $\Sigma = \{\mathcal{O}_k | k = 1, 2, \cdots, n\}$.

Observables are normalized as $\tilde{\mathcal{O}}_k = \dfrac{\mathcal{O}_k}{\langle \mathcal{O}_k \rangle_0}$,

where $\langle \cdots \rangle_0$=(V.E.V. for the phase-quenched partition function Z_0).

Generalized distribution function $\rho(x_1, \cdots, x_n) = \left\langle \prod_{k=1}^{n} \delta(x_k - \tilde{\mathcal{O}}_k) \right\rangle$ factorizes as

$$\rho(x_1, \cdots, x_n) = \frac{1}{C} \rho^{(0)}(x_1, \cdots, x_n) \, w(x_1, \cdots, x_n) \ , \ \text{where}$$

$$\rho^{(0)}(x_1, \cdots, x_n) = \left\langle \prod_{k=1}^{n} \delta(x_k - \tilde{\mathcal{O}}_k) \right\rangle_0, \quad w(x_1, \cdots, x_n) \stackrel{\text{def}}{=} \langle e^{i\Gamma} \rangle_{x_1, \cdots, x_n}.$$

$\langle \cdots \rangle_{x_1, \cdots, x_n}$ = V.E.V. for partition function $Z_{x_1, \cdots, x_n} = \int dA e^{-S_0} \prod_{k=1}^{n} \delta(x_k - \tilde{\mathcal{O}}_k)$.

Evaluation of the observables $\langle \tilde{\mathcal{O}}_k \rangle$:

Peak of the distribution function $\rho(x_1, \cdots, x_n)$ at V = (system size) $\to +\infty$

$\Rightarrow$ solution of the saddle-point equation

$$\frac{d}{dx_k} \left\{ \lim_{V \to +\infty} \frac{1}{V} \log \rho^{(0)}(x_1, \cdots, x_n) \right\} = -\frac{d}{dx_k} \left\{ \lim_{V \to +\infty} \frac{1}{V} \log w(x_1, \cdots, x_n) \right\}$$

3 Simplified IKKT model

Simplified model with spontaneous rotational symmetry breakdown
Nishimura, hep-th/0108070.

$$\boxed{S = \underbrace{\frac{N}{2}\mathrm{tr}\, A_\mu^2}_{=S_b} \underbrace{-\bar{\psi}_\alpha^f (\Gamma_\mu)_{\alpha\beta} A_\mu \psi_\beta^f}_{=S_f}}$$

- A_μ: $N \times N$ hermitian matrices $(\mu = 1, \cdots, 4)$
  $\bar{\psi}_\alpha^f, \psi_\alpha^f$: N-dim vector $(\alpha = 1, 2, f = 1, \cdots, N_f)$
  $N_f =$ (number of flavors)
- SO(4) rotational symmetry.
- No supersymmetry.
- Partition function:
  $$Z = \int dA e^{-S_B} (\det \mathcal{D})^{N_f} = \int dA e^{-S_0} e^{i\Gamma},$$
  where
  $\mathcal{D} = \Gamma_\mu A_\mu = (2N \times 2N \text{ matrices}),$
  $S_0 = S_b - \log |\det \mathcal{D}|.$
  Fermion determinant $\det \mathcal{D}$ is complex.
- Gaussian Expansion Method (GEM) up to 9th order: Okubo, Nishimura and Sugino, hep-th/0412194.

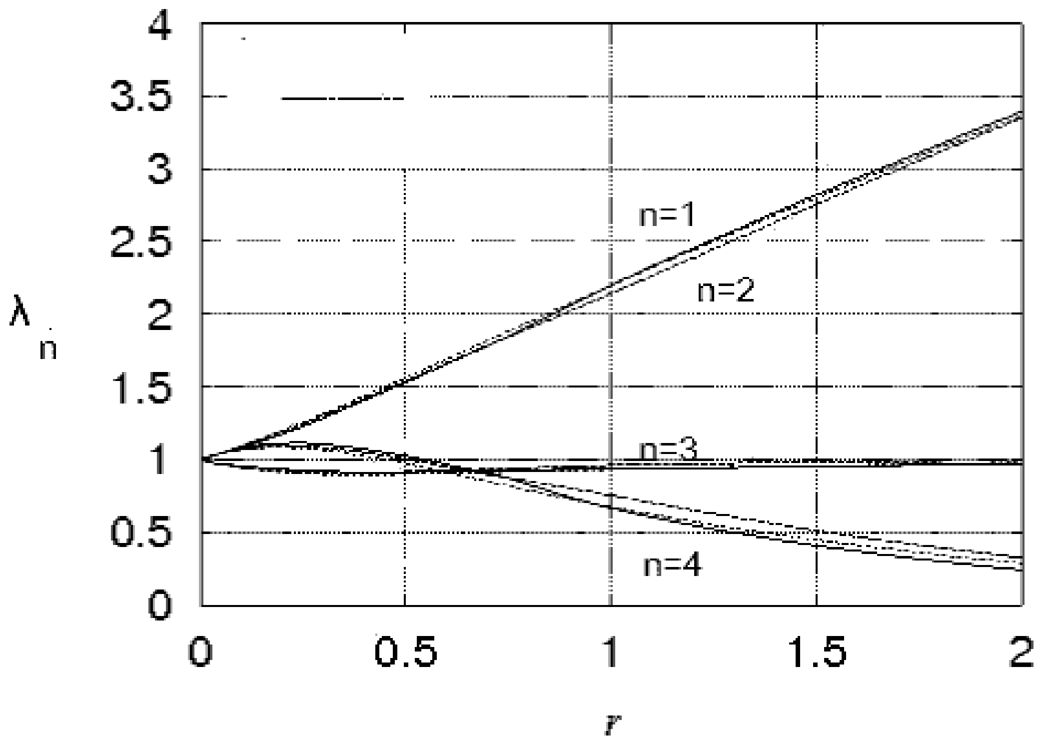
  Observable for probing dimensionality :
  $T_{\mu\nu} = \frac{1}{N}\mathrm{tr}\,(A_\mu A_\nu).$
  λ_n $(n = 1, 2, 3, 4)$: eigenvalues of $T_{\mu\nu}$
  $(\lambda_1 \geqq \lambda_2 \geqq \lambda_3 \geqq \lambda_4)$
  Spontaneous breakdown of SO(4) to SO(2) at finite $r \left(= \frac{N_f}{N}\right)$.
- Choice of the observables to constrain:
  $\Sigma = \{\mathcal{O}_k = \lambda_k | k = 1, 2, 3, 4\}$
  Partition function to simulate:
  $$Z_{x_1,x_2,x_3,x_4} = \int dA e^{-S_0} \prod_{n=1}^{4} \delta(x_n - \tilde{\lambda}_n)$$

SO(2) vacuum

Solutions which satisfy $x_1 = x_2 > 1 > x_3 > x_4$.

Result of GEM up to 9th order at large N:

Okubo, Nishimura and Sugino, hep-th/0412194.



SO(2) ansatz

$$\tilde{\lambda}_n = \frac{\lambda_n}{\langle \lambda_n \rangle_0}, \quad \langle \lambda_n \rangle_0 = 1 + \frac{r}{2} = 1.5 (\text{for } r = 1)$$

for $n = 1, 2, 3, 4$.

$\langle \tilde{\lambda}_{1,2} \rangle = 1.4$, $\langle \tilde{\lambda}_3 \rangle = 0.7$, $\langle \tilde{\lambda}_4 \rangle = 0.5$ $(r = 1)$.

Minimum of the free energy density $\mathcal{F}(x)$

$$\frac{\partial}{\partial x_n} \rho^{(0)}_{\mathrm{SO}(2)}(x_2, x_3, x_4) = -\frac{\partial}{\partial x_n} w_{\mathrm{SO}(2)}(x_2, x_3, x_4)$$

for $n = 2, 3, 4$,

where

$$\rho^{(0)}_{\mathrm{SO}(2)}(x_2, x_3, x_4) = \rho^{(0)}(x_2, x_2, x_3, x_4) \ ,$$

$$w_{\mathrm{SO}(2)}(x_2, x_3, x_4) = w(x_2, x_2, x_3, x_4)$$

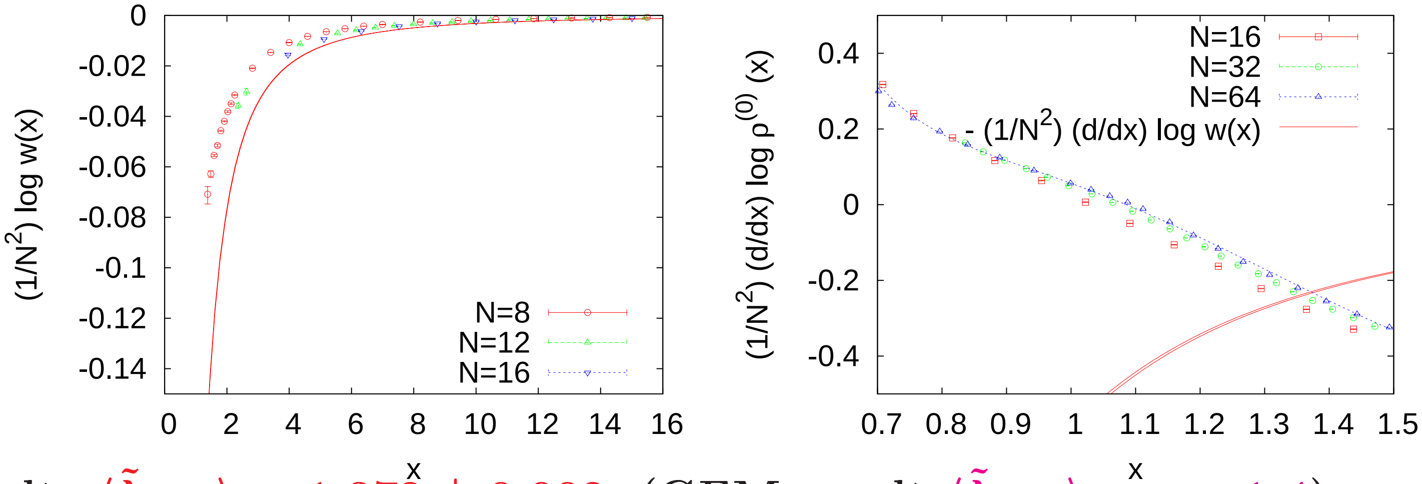
Calculation of $\langle\tilde{\lambda}_n\rangle$ at $r = 1$

Calculation of $\langle\tilde{\lambda}_{n=2}\rangle$ for fixed $\langle\tilde{\lambda}_{n=3}\rangle = 0.7$ and $\langle\tilde{\lambda}_{n=4}\rangle = 0.5$.

$$\frac{1}{N^2}\frac{d}{dx}\log\rho^{(0)}(x) = -\frac{\partial}{\partial x}\frac{1}{N^2}\log w(x),$$

where $\rho^{(0)}(x) = \rho^{(0)}_{\mathrm{SO}(2)}(x, y = 0.7, z = 0.5)$ and $w(x) = w_{\mathrm{SO}(2)}(x, y = 0.7, z = 0.5)$.

Scaling behavior of the phase: $\frac{1}{N^2}\log w(x) \sim -\tilde{d}_1 x^{-2} + \tilde{d}_2 x^{-2.5}$.



Numerical Result: $\langle\tilde{\lambda}_{n=2}\rangle = 1.372 \pm 0.002$, (GEM result $\langle\tilde{\lambda}_{n=2}\rangle_{\mathrm{GEM}} = 1.4$).

Similarly, $\langle\tilde{\lambda}_{n=3}\rangle = 0.648 \pm 0.004$, (GEM result $\langle\tilde{\lambda}_{n=3}\rangle_{\mathrm{GEM}} = 0.7$).

$\langle\tilde{\lambda}_{n=4}\rangle = 0.550 \pm 0.002$, (GEM result $\langle\tilde{\lambda}_{n=4}\rangle_{\mathrm{GEM}} = 0.5$).

Is there any more overlap problem?

Observables to constrain: $\Sigma = \{\mathcal{O}_k = \lambda_k | k = 1, 2, 3, 4\}$. **Is this enough?**

Partition function $Z_{\mathcal{O}} = \int dA e^{-S_0} \delta(x - \tilde{\mathcal{O}}) \prod_{n=1}^{4} \delta(x_n - \tilde{\lambda}_n)$

(here we constrain $\Sigma = \{\mathcal{O}, \lambda_1, \cdots, \lambda_4\}$**).**

Peak of the distribution function $\underbrace{\rho(x_1, x_2, x_3, x_4, x)}_{=\rho_{\mathcal{O}}(x)} \Rightarrow$ **solution of the saddle-point equation**

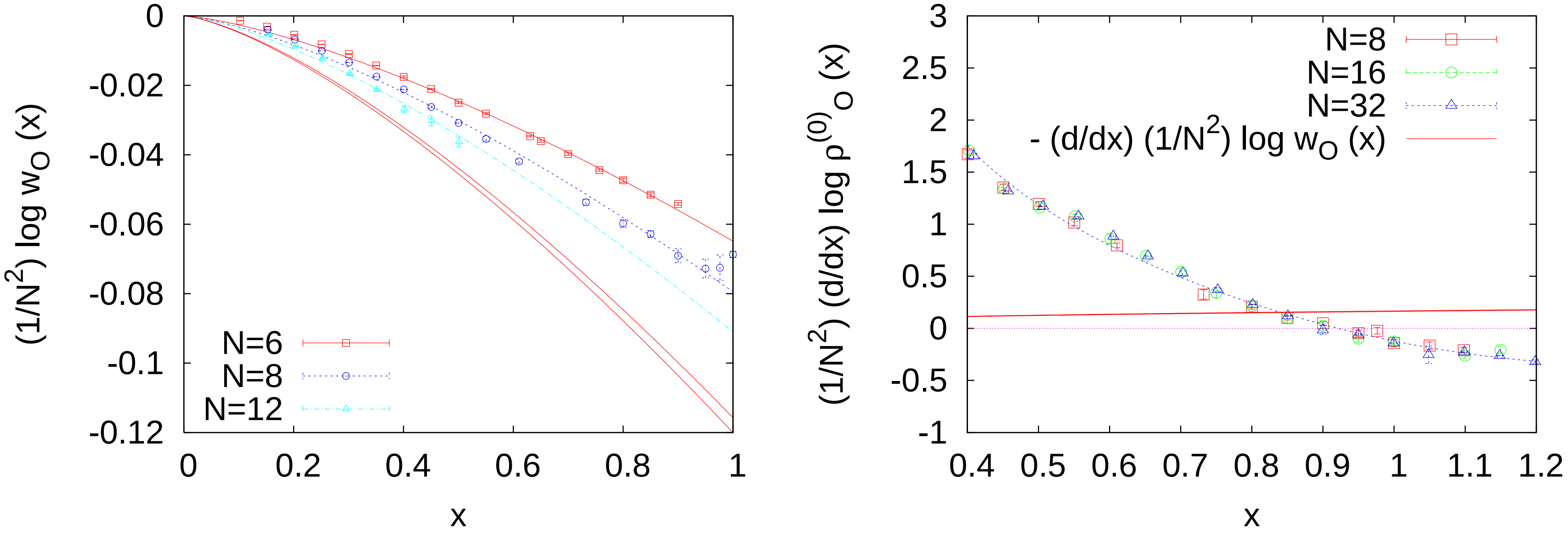
$$\frac{d}{dx} \frac{1}{N^2} \log \underbrace{\rho^{(0)}(x_1, x_2, x_3, x_4, x)}_{=\rho^{(0)}_{\mathcal{O}}(x)} = -\frac{d}{dx} \frac{1}{N^2} \log \underbrace{w(x_1, x_2, x_3, x_4, x)}_{=w_{\mathcal{O}}(x)}.$$

Do the peaks of $\rho^{(0)}_{\mathcal{O}}(x)$ **and** $\rho_{\mathcal{O}}(x)$ **match?**

We consider $\mathcal{O} = -\frac{1}{N} \mathrm{tr} \, [A_\mu, A_\nu]^2$.

Simulation with $\tilde{\lambda}_n$ fixed at GEM result $\langle\tilde{\lambda}_{1,2}\rangle = 1.4$, $\langle\tilde{\lambda}_3\rangle = 0.7$, $\langle\tilde{\lambda}_4\rangle = 0.5$ $(r = 1)$.

- **Peak of $\rho^{(0)}_{\mathcal{O}}(x)$: Solution of $\frac{1}{N^2}\frac{d}{dx}\log\rho^{(0)}_{\mathcal{O}}(x) = 0 \Rightarrow x \simeq 0.92$.**
- **Peak of $\rho_{\mathcal{O}}(x)$: Solution of $\frac{1}{N^2}\frac{d}{dx}\log\rho^{(0)}_{\mathcal{O}}(x) = -\frac{1}{N^2}\frac{d}{dx}\log w_{\mathcal{O}}(x) \Rightarrow x \simeq 0.84$.**
  Systematic error is around 10%.



4 Conclusion

Factorization method as a practical approach to the sign problem.

Resolution of the overlap problem.

Monte Carlo simulation of simplified IKKT model

$\Rightarrow$ spontaneous breakdown of SO(4) rotational symmetry.

Good agreement with GEM result.

Future problems

Monte Carlo Simulation of the IKKT model **Anagnostopoulos, Aoyama, T. A., Hanada and Nishimura, in progress**

Effect of supersymmetry on dynamical generation of spacetime.